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PAPER

Travelling wavefronts for the Belousov–Zhabotinsky system with non-local delayed interaction

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Abstract

This article offers an advanced and novel investigation into the intricate propagation dynamics of the Belousov–Zhabotinsky system with non-local delayed interaction, which exhibits dynamical transition structure from bistable to monostable. We first solved the enduring open problem concerning the existence, uniqueness and the speed sign of the bistable travelling waves. In the monostable case, we developed and derived new results for the minimal wave speed selection, which, as an application, further improved the existing investigations on pushed and pulled wavefronts. Our results can provide new estimate to the minimal speed as well as to the determinacy of the transition parameters. Moreover, these results can be directly applied to standard localised models and delayed reaction diffusion models by choosing appropriate kernel functions.

1. Introduction

KPP–Fisher reaction diffusion equations $u_t = \Delta u + f(u)$ are fundamental in the modelling of population growths [10, 29], by assuming either the reaction term as the monostable nonlinearity (e.g. $f(u) = u(1-u)$) or otherwise the bistable topological structure (e.g. $f(u) = u(a-u)(1-u)$, $0 < a < 1$). A natural and interesting question is: can a system exhibit rich dynamics by combining both monostable and bistable topological properties with just a switch of a impacting parameter? If so, how does the spatial spreading or invasion behave in view of the parameter? How to determine and estimate the moving speeds of the pushed waves, pulled waves or bistable waves?

To answer the above questions, we consider the following Belousov–Zhabotinsky (BZ for short) reaction-diffusion system

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t,x) - r(\bar{K} * v)(t,x)), \\ v_t(t,x) = \Delta v(t,x) - bu(t,x)v(t,x), \qquad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

which builds upon J.D. Murray's investigations of bio-reaction in [28, 29]. The variables u and v represent the bromous acid and bromide ion concentrations, respectively. Here Δ is the Laplace operator, and r and b are positive reaction parameters. The term $\bar{K} * v$ denote the convolution of the component v with a nonnegative normalised kernel $\bar{K}(s,y)$, $s \geq 0$, $y \in \mathbb{R}^n$, which satisfies the following conditions:

(K1) $\int_0^\infty \int_{\mathbb{R}^n} \bar{K}(s,y)dyds = 1$,

(K2) There exists $\tilde{\delta}_0 \in (0,\infty)$ such that $\int_0^\infty \int_{\mathbb{R}^n} \bar{K}(s,y)e^{\rho s}dyds < \infty$ for $\rho \in [0,\tilde{\delta}_0)$.

Let $u = \phi$ and $v = 1 - \psi$. System (1.1) becomes

$$\begin{cases} \phi_t(t,x) = \Delta\phi(t,x) + \phi(t,x)(1 - r - \phi(t,x) + r(\bar{K} * \psi)(t,x)), \\ \psi_t(t,x) = \Delta\psi(t,x) + b\phi(t,x)(1 - \psi(t,x)). \end{cases} \tag{1.2}$$

Note that system (1.2) has infinite number of constant equilibria

$$e_\beta = (1, 1), \quad e_a = (0, a), \tag{1.3}$$

where a is an arbitrary constant. We may only consider $0 \le a \le 1$ so that $(0, a)$ are bounded by e_β. If $a = 0$, then $e_0 = (0, 0)$. The kinetic system of model (1.2) is

$$\begin{cases} \phi_t = \phi(1 - r - \phi + r(\bar{K} * \psi)), \\ \psi_t = b\phi(1 - \psi), \end{cases} \tag{1.4}$$

where $\bar{K} * \psi = \int_0^{+\infty} \int_{\mathbb{R}^n} \bar{K}(s, y)\psi(t - s)dyds$. It can be noted that the conventional definition of monostability/bistability in [34] requires a clarification to system (1.2) that possesses a continuum of non-negative equilibria. The stability analysis of (1.4) around its equilibria reveals that e_β is stable while e_0 is unstable when $r < 1$. On the other hand, for $r > 1$, e_β remains stable, but e_0 becomes neutrally-stable (non-asymptotically stable). Hence, the value $r = 1$ serves as a critical point of stability transition of e_0 for (1.2), with the system exhibiting its monostability characteristics for $r \in (0, 1)$, and its bistability properties for $r > 1$. The wave propagation dynamics in the critical case $r = 1$ behaves similar to the monostable case as we can see later. Therefore, in this article, we divide our analysis into two nonlinearities:

H1 (bistable): $r > 1$;

H2 (monostable): $r \le 1$.

Travelling wave solutions of (1.2) are assumed to have the form $(\phi, \psi) = (\Phi, \Psi)(\nu \cdot x - ct)$, $||\nu|| = 1$, and satisfy the wave profile system

$$\begin{cases} \Phi''(z) + c\Phi'(z) + \Phi(z)(1 - r - \Phi(z) + r(K \star \Psi)(z)) = 0, \\ \Psi''(z) + c\Psi'(z) + b\Phi(z)(1 - \Psi(z)) = 0, \end{cases} \tag{1.5}$$

where

$$(K \star \Psi)(z) = \int_0^{+\infty} \int_{\mathbb{R}} K(s, \eta_1)\Psi(z + cs - \eta_1)d\eta_1 ds.$$

Here we have established a new coordinate-system $(\eta_1, \eta_2, \cdots, \eta_n)$ with $\eta_1 = \nu \cdot y$ and

$$K(s, \eta_1) = \int_{\mathbb{R}^{n-1}} \bar{K}(s, \eta_1, \eta_2, \cdots, \eta_n)d\eta_2 \cdots d\eta_n.$$

As such, the conditions (K1) and (K2) imply the following statements

1. $\int_0^\infty \int_{\mathbb{R}} K(s, y)dyds = 1$,
2. There exists $\tilde{\delta}_0 \in (0, \infty)$ such that $\int_0^\infty \int_{\mathbb{R}} K(s, y)e^{\rho s}dyds < \infty$ for $\rho \in [0, \tilde{\delta}_0)$

are true.

Since $(K \star \Psi)(z)$ depends on c, we may re-write it as $(K \star_c \Psi)(z)$ to denote $(K \star \Psi)(z)$. The boundary condition of (1.5) at infinity is

$$(\Phi, \Psi)(-\infty) = e_\beta \text{ and } (\Phi, \Psi)(\infty) = e_0. \tag{1.6}$$

1.1. Bistable waves and the speed sign in the bistable case H1: $r > 1$

Travelling waves are important phenomena and moved patterns in biological modelling and evolution. Following Murray's initial works, many researchers delved into the study of system (1.1), with the problem of travelling waves emerging as a matter of fundamental interest. The primary focus of related research thus far has been on the localised case (e.g., [16, 28, 29, 33, 43]), i.e. $(\bar{K} * v)(t, x) = v(t, x)$, and the cases involving delays (e.g., [19, 31, 42, 44]), $(\bar{K} * v)(t, x) = v(t - h, x)$. However, there has been limited research of system (1.1) with non-local delayed interaction. Especially in the bistable case H1, (in fact, r can vary between 5 and 50 in real experiments), a conjecture about the existence of bistable travelling wave was raised by Hasík et al. [13]. To the best of our knowledge, no related contributions have been made for this, and in fact this problem has remained open for long time. In this article, we are able to solve the problem and will prove that there exists a unique (up to translation) bistable travelling wave of system (1.5)-(1.6), namely the following two theorems.

Theorem 1.1. ***(Existence and the speed sign)*** *Assume that H1 and (K1) hold. There exists a monotone bistable travelling wave $(c, \Phi(z), \Psi(z))$, $z = \nu\cdot x - ct$, for any given ν with $||\nu|| = 1$, of system (1.5)-(1.6), where c is positive and unique. Moreover, $\Phi'(z) < 0$ and $\Psi'(z) < 0$ for $z \in \mathbb{R}$.*

Theorem 1.2. ***(Uniqueness)*** *Assume that H1, (K1) and (K2) hold. If $(\Phi^*(\nu \cdot x - c^*t), \Psi^*(\nu \cdot x - c^*t))$ is a bistable travelling wave of system (1.2) connecting e_β to e_0, then there exists $z^* \in \mathbb{R}$ such that $(\Phi^*(\nu \cdot x - c^*t), \Psi^*(\nu \cdot x - c^*t)) = (\Phi(\nu \cdot x - ct + z^*), \Psi(\nu \cdot x - ct + z^*))$ and $c^* = c$, where $(\Phi(\nu \cdot x - ct), \Psi(\nu \cdot x - ct))$ is the wave solution in Theorem 1.1. The bistable wave speed c is non-increasing in r and is non-decreasing in b.*

The presence of continuum equilibria, combined with the non-local delayed reaction term, incorporates significant complexity when analysing model (1.1). This specific property of equilibrium e_0 hinders the application of Fang and Zhao's abstract theory [9] of bistable travelling waves for monotone semiflows to our system, as e_0 is not strongly stable in their settings, nor the intermediate equilibria are un-ordered. Furthermore, [9] only consider a finite delay. Here, however, the delay in system (1.1) is infinitely distributed. Methods developed in other important studies (see, for example, [6, 18]) remain unapplicable because they only allow a limited number of steady states. Kanel [16] established the existence of the bistable travelling wave in the non-delayed local case (where $(\bar{K} * v)(t, x) = v(t, x)$ and $r > 1$) by using a shooting method in the phase plane. However, this method cannot be applied to non-local or delayed versions of system (1.1). Recently, Hasík et al. [13] proved the existence of the bistable travelling wave in the delayed local case (where $(\bar{K} * v)(t, x) = v(t - h, x)$ and $r > 1$) by constructing a perturbed system, which can approximate the original system. Application of the method in this article to the more complex system (1.1) with non-local delayed reaction is still in question, definitely in great challenge.

In order to address the challenges mentioned above, we shall develop novel approaches to rigourously show the existence of travelling wave solutions to the system exhibiting bistable nonlinearity. Our methodology utilises a combination of advanced analysis on partial differential equations (parabolic and elliptic) and dynamical techniques, enabling its application to nonlocal problems and high-dimensional phase systems. The novelty of our result is that we not only prove the existence, uniqueness and the positive speed sign of the bistable wave, but also provide the monotonicity of bistable wave speed c with respect to parameters r and b. The bistable wave speed c is non-increasing in r and is non-decreasing in b.

1.2. Monostable waves and speed selection in the case H2: $r \le 1$

In the H2 case, the solution to (1.5)-(1.6) is usually not unique, as can be seen in the classical KPP–Fisher model [10]. Denote $c_{\min}$ as

$$c_{\min} = \inf\{c : (1.5) - (1.6) \text{ has a non-negative solution}\}.$$

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Standard linearisation near the equilibrium point e_0 shows $c_{\min} \geq c_0 = 2\sqrt{1-r}$. For convenience, we give the following definition on the minimal wave speed selection in the monostable case.

Definition 1.1. *If $c_{\min} = c_0$, then we say the minimal wave speed is linearly selected and the wave is a pulled wave; If $c_{\min} > c_0$, then the minimal wave speed is said to be nonlinearly selected and the corresponding wave is a pushed wave.*

Recently, Hasík et al. [12] established the existence of monostable travelling wave of system (1.1) with spatio-temporal interaction for $r \in (0, 1)$ and derived two limiting cases for the appearance of pushed wavefronts (nonlinear selection). These two sufficient conditions for nonlinear selection to system (1.1) in [12] are as follows:

1) if $b \to \infty$, then $c_{\min} \to 2$;
2) if the kernel function $K(s, y)$ in (1.5) is replaced by $K_a(s, y) = K(s, y + a)$ and $a \to \infty$, then $c_{\min} \to 2$.

In this article, we shall first give necessary and sufficient conditions for distinguishing between linear selection and nonlinear selection for all $r \leq 1$, including the existence of a family of travelling waves in the critical case $r = 1$ whose dynamics behaves like the monostable case, instead of the bistable case, as shown in the following theorem.

Theorem 1.3. *Assume that H2 and (K1) hold.*

(i). There is a finite number $c_{\min} \geq c_0 = 2\sqrt{1-r}$ such that monotone and positive travelling wave solutions of (1.5)-(1.6) exist if and only if $c \geq c_{\min}$. The minimal wave speed $c_{\min}$ of (1.5)-(1.6) is nonlinearly selected if and only if there exists a wave front (c_2, Φ, Ψ), $c_2 > c_0$ such that

$$\Phi(z) \sim A_2 e^{-\mu_2(c_2)z}, \text{ as } z \to \infty, \ z = \nu \cdot x - c_2 t, \ ||\nu|| = 1,$$

with $A_2 > 0$. Here, μ_2 is the larger root of $\mu^2 - c\mu + (1-r) = 0$. Furthermore, $c_2 = c_{\min}$.

(ii) For $r \in (0, 1)$, the minimal speed $c_{\min}$ is non-decreasing with respect to b. Then there exists a finite value $b^* > 0$ such that $c_{\min} = c_0$ for $b \in (0, b^*]$, and $c_{\min} \in (c_0, 2]$ for $b > b^*$; If $r = 1$, then $c_{\min} > c_0$ for all positive parameter b.

(iii) For fixed b, the minimal speed c_{min} is a non-increasing function of r in $(0, 1]$. Furthermore, there exists a unique turning point $r^* < 1$ for the minimal speed selection, that is, $c_{\min} = c_0$ for $r \in (0, r^*]$, and $c_{\min} \in (c_0, 2]$ for $r \in (r^*, 1]$.

Here, Theorem 1.3 is a new development of the abstract results of speed selection in [23] to the system (1.1) with infinite points of equilibria. It shows that the minimal wave speed is non-decreasing with respect to b, when r is fixed, and there exists a transition point, denoted as b^*, which serves as a turning point for speed selection. Similarly, there exists the monotonicity of the minimal wave speed with respect to r and the transition r^* for speed selection (when b is fixed), although the linear speed varies with r.

Without any assumptions on the existence of wave fronts, we further provide the following easy-to-apply theorem, which can be used to derive a series of explicit conditions on speed selections by constructing various upper or lower solutions.

Theorem 1.4. *(Linear, nonlinear selection and estimate of $c_{\min}$) Assume that H2 and (K1) hold. μ_1 and $\mu_2(\geq \mu_1)$ are two roots of $\mu^2 - c\mu + (1-r) = 0$.*

(i) For $c_1 > c_0$, if there exists a nonnegative and monotonic lower solution $(\underline{\Phi}, \underline{\Psi})(z)$ of system (1.5)-(1.6) such that $\underline{\Phi}(z) \sim \underline{A} e^{-\mu_2 z}$ as $z \to \infty$ with $\underline{A} > 0$, and $\limsup_{z\to-\infty} \underline{\Phi}(z) < 1$, then no travelling wave solution exists for $c \in [c_0, c_1)$ and there exists $c_{\min} > c_1$.

(ii) For $r \in (0, 1)$ and $c = c_0 + \varepsilon$ where ε is any small positive number, if there exists a nonnegative and monotonic upper solution $(\overline{\Phi}, \overline{\Psi})(z)$ of system (1.5)-(1.6) such that $\overline{\Phi}(z) \sim \overline{A} e^{-\mu_1(c)z}$ as $z \to \infty$ with $\overline{A} > 0$, and $\limsup_{z\to-\infty} \overline{\Phi}(z) \geq 1$, then we have $c_{\min} = c_0$.

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By application of this theorem, two explicit estimates about b^* are given by applying Theorem 1.4. The comparison of our new results to previous ones is given in the discussion section which shows the practicalness of this finding.

We also provide details of the decay rate of the minimal-speed travelling wave as b varies around b^*, which are substantially novel than our previous work [23]. These findings imply the transition wave behavies completely different from other pulled waves for $b < b^*$, as shown in the following theorem.

Theorem 1.5. *Assume that H2 and (K1) hold. Let b^* be the turning point for the minimal speed selection for fixed r, specifically $c_{\min} = c_0$ for $b \in (0, b^*]$, and $c_{\min} \in (c_0, 2]$ for $b > b^*$. The behaviour of the minimal-speed travelling wave solution $(\Phi(z), \Psi(z))$ of system (1.5)-(1.6) can be described as follows:*

(1) if $b \in (0, b^*)$, then $\Phi(z) \sim Aze^{-\mu_1(c_0)z}$ as $z \to \infty$, where $A > 0$;

(2) if $b = b^*$, then $\Phi(z) \sim Be^{-\mu_1(c_0)z}$ as $z \to \infty$, where $B > 0$;

(3) if $b > b^*$, then $\Phi(z) \sim Be^{-\mu_2(c_{\min})z}$ as $z \to \infty$, where $B > 0$.

Here, μ_1 and $\mu_2(\geq \mu_1)$ are two roots of $\mu^2 - c\mu + (1 - r) = 0$.

Remark 1.1. *The above Theorem 1.1 corresponds to Theorem 2.6, Theorem 1.2 corresponds to Theorems 2.8-2.9, Theorem 1.3 corresponds to Theorems 3.3, 3.4, 3.6 and 3.10, Theorem 1.4 corresponds to Theorem 3.5, and Theorem 1.5 corresponds to Theorem 3.9.*

Finally, we would like to provide readers with a few more references related to our study. In recent years, considerable efforts have been made in studying the existence of travelling waves and their speed sign for monotonic systems. See e.g., [2, 4, 5, 8910, 22, 24, 26, 27, 34, 36, 37, 39]. The problem of the minimal speed selection was widely discussed in many papers. See, e.g., [1, 3, 4, 14, 15, 20, 21, 23, 25, 30, 35, 40, 41].

The remainder of this paper is organised as follows. In Section 2, we prove the existence and uniqueness of the monotone bistable travelling wave of (1.5)-(1.6) for the case H1. In Section 3, we prove the existence of monostable travelling wave of (1.5)-(1.6) for $r = 1$. We also derive necessary and sufficient conditions for speed selection and give the decay rate of the minimal-speed travelling wave as $z \to \infty$, in terms of different value domains of b around of b^*, under the case H2. Section 4 will present numerical results, while Section 5 covers the conclusion and the further discussion.

2. Bistable wave in the case H1: existence, uniqueness and speed sign

In this section, we consider system (1.2) under the condition H1. Note that e_β is stable, and e_0 is non-asymptotically stable nor strongly stable. As mentioned in Section 1, the use of the monotonic semiflow theory in [9] is not applicable for proving the existence of the bistable monotone travelling wave solution in (1.5)-(1.6). The non-local delayed reaction term in system (1.5) prevents the utilisation of the shooting method. Here, we present a novel approach to establish the existence of wavefront, which could potentially be extended to other models featuring non-isolated equilibrium points and non-local delayed reaction terms.

Obviously, it can be seen from the second equation of (1.5) that this model has no positive and monotone travelling wave satisfying (1.6) with speed $c \leq 0$. The following lemma gives the existence of solutions to the Ψ-equation of the system (1.5)-(1.6) when Φ is given, which holds for both bistable and monostable cases.

Lemma 2.1. *Let $c > 0$ and $b > 0$, and suppose that $\Phi(z)$ is a positive non-increasing continuous function such that $\Phi(-\infty) = 1$ and $\int_0^\infty \Phi(z)dz$ is finite. Then there exists a unique monotone C^2-smooth solution $\Psi(z)$, $\Psi'(z) < 0$, to the boundary value problem*

$$\Psi''(z) + c\Psi'(z) + b\Phi(z)(1 - \Psi(z)) = 0, \quad \Psi(-\infty) = 1, \quad \Psi(\infty) = 0. \tag{2.1}$$

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This defines the operator $\mathcal{L}_{b,c}$, by $\Psi = \mathcal{L}_{b,c}\Phi$ on the respective functional sets. $\mathcal{L}_{b,c}$ commutes with the translation operator, $(\mathcal{L}_{b,c}\Phi(\cdot + h))(z) = (\mathcal{L}_{b,c}\Phi(\cdot))(z + h)$, and is monotonic with respect to b, c and Φ so that

(a) if $\Phi_1(z) \le \Phi_2(z)$, then $(\mathcal{L}_{b,c}(\Phi_1))(z) \le (\mathcal{L}_{b,c}(\Phi_2))(z)$, $z \in \mathbb{R}$;

(b) if $b_1 < b_2$, then $(\mathcal{L}_{b_1,c}(\Phi))(z) < (\mathcal{L}_{b_2,c}(\Phi))(z)$, $z \in \mathbb{R}$, for each Φ from the domain of $\mathcal{L}_{b,c}$. Moreover, we have $\mathcal{L}_{b,c}\Phi \to 0$ as $b \to 0$, and $\mathcal{L}_{b,c}\Phi \to 1$ as $b \to \infty$;

(c) if $c_1 < c_2$, then $(\mathcal{L}_{b,c_1}(\Phi))(z) > (\mathcal{L}_{b,c_2}(\Phi))(z)$, $z \in \mathbb{R}$, for each Φ from the domain of $\mathcal{L}_{b,c}$. Moreover, we have $\mathcal{L}_{b,c}\Phi \to 1$ as $c \to 0$.

Proof. We only prove result *c*) and the limiting results in *b*). The remaining parts can refer to [12, Lemma 15]. We let $c_1 < c_2$, and set $\Psi_j = \mathcal{L}_{b,c_j}\Phi$, $j = 1, 2$. Take $\sigma(z) = \Psi_1(z) - \Psi_2(z)$, $z \in \mathbb{R}$. Then $\sigma(z)$ satisfies

$$\sigma''(z) + c_1\sigma'(z) + (c_1 - c_2)\Psi_2'(z) - b\Phi(z)\sigma(z) = 0, \quad \sigma(-\infty) = 0, \quad \sigma(\infty) = 0.$$

If $\sigma(s) \le 0$ at some point s, then there exists a critical point s^* such that $\sigma(s^*) \le 0$, $\sigma'(s^*) = 0$, and $\sigma''(s^*) \ge 0$. It follows that

$$\sigma''(s^*) + c_1\sigma'(s^*) + (c_1 - c_2)\Psi_2'(s^*) - b\Phi(s^*)\sigma(s^*) > 0$$

since $\Psi_2'(s^*) < 0$. It is a contradiction. Therefore, $\Psi_1(z) \ge \Psi_2(z)$, $z \in \mathbb{R}$.

To prove $\mathcal{L}_{b,c}\Phi \to 1$ as $c \to 0$, we consider a decreasing sequence $c_n = \frac{1}{n}$, $n \in \mathbb{N}^+$. Then by the monotonicity of $\mathcal{L}_{b,c}\Phi$ with respect to c, we know $\Psi^n(z) := \mathcal{L}_{b,c_n}\Phi(z)$ is increasing with respect to n. Note that 1 is an upper bound of the sequence $\Psi^n(z)$, and 1 is the only positive and non-increasing solution to the following problem

$$\Psi''(z) + b\Phi(z)(1 - \Psi(z)) = 0, \quad \Psi(-\infty) = 1.$$

Thus, we have $\Psi^n(z) \to 1$ as $n \to \infty$, that is, $\Psi = \mathcal{L}_{b,c}\Phi \to 1$ as $c \to 0$.

By the similar arguments to the proof of $\mathcal{L}_{b,c}\Phi \to 1$ as $c \to 0$ and the monotonicity of $\mathcal{L}_{b,c}\Phi$ with respect to b (refer to [12, Lemma 15]), we can derive $\mathcal{L}_{b,c}\Phi \to 0$ as $b \to 0$ and $\mathcal{L}_{b,c}\Phi \to 1$ as $b \to \infty$. □

2.1. Bistable waves of an auxiliary equation

In this subsection, we shall prove the existence of the bistable travelling wave of an auxiliary equation we derive. According to Lemma 2.1, system (1.5)-(1.6) can be transformed into the following equation

$$\begin{cases} \Phi''(z) + c\Phi'(z) + \Phi(z)(1 - r - \Phi(z) + r(K \star_c \mathcal{L}_{b,c}\Phi)(z)) = 0 \\ \Phi(-\infty) = 1, \quad \Phi(\infty) = 0. \end{cases} \tag{2.2}$$

Consider the wave speed $c > 0$ in $\mathcal{L}_{b,c}\Phi$ of (2.2) as a parameter. We would like to construct an auxiliary parabolic partial differential equation as follows

$$\phi_t(t,x) = \phi_{xx}(t,x) + \phi(t,x)(1 - r - \phi(t,x) + r(K * \mathcal{L}_{b,c}\phi)(t,x)), \; x \in \mathbb{R}. \tag{2.3}$$

Clearly, (2.3) has two constant equilibria $\phi = 0$ and $\phi = 1$. Due to the condition that $\int_0^\infty \Phi(z)dz$ is finite in Lemma 2.1, the initial function for (2.3) needs to be restricted as integrable at ∞. Since the model is monotone, the well-posedness of this problem has no problem and the solution still belongs to be integrable at ∞ for any future time t. To investigate the travelling wave to (2.3), we define

$$\phi(t,x) = \Phi(z), \quad z = x - C(c)t,$$

where $C(c)$ represents the wave speed dependent on c. Substituting this into (2.3), we obtain

$$\Phi''(z) + C(c)\Phi'(z) + \Phi(z)(1 - r - \Phi(z) + r(K \star_{C(c)} \mathcal{L}_{b,c}\Phi)(z)) = 0 \tag{2.4}$$

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subject to the boundary conditions

$$\Phi(-\infty) = 1, \quad \Phi(\infty) = 0. \tag{2.5}$$

We aim to prove the existence of wavefront $(C(c), \Phi)$, with $C(c)$ being non-increasing function of c.

Let $X = BC(\mathbb{R}, \mathbb{R})$ be the space of all bounded, continuous and integrable-at-positive-∞ functions from $\mathbb{R}$ to $\mathbb{R}$. Define $X_+ = \{\varphi \in X : \varphi(x) \geq 0, \forall x \in \mathbb{R}\}$. Let $\mathcal{C} = C(M, X)$ be the space of all functions from M to X with $|\cdot|_{\mathcal{C}} = ||\cdot||_\infty$, where $M = (-\infty, 0]$. Let $\mathcal{C}_+ = C(M, X_+)$. Define $\mathcal{M} \subset \mathcal{C}$ to be the set of all continuous and non-increasing functions. $\mathcal{M}_\beta = \{\varphi \in \mathcal{C} : 0 \leq \varphi \leq \beta\}$, where $\beta = \mathbf{1}$ in $\mathcal{C}$.

Denote $\{Q_t\}_{t\geq 0}$ as the solution semiflow associated with (2.3) on $\mathcal{C}_+$. That is,

$$Q_t[\varphi](\theta, x) = \phi(t + \theta, x, \varphi), \ \forall \varphi \in \mathcal{C}, \ x \in \mathbb{R}, \ t \geq 0,$$

where $\phi(t + \theta, x, \varphi)$ is the unique solution of (2.3) satisfying $\phi(\theta, \cdot, \varphi) = \varphi(\theta)$. Let E be the set of all fixed points of Q_t for $t > 0$ restricted to $\mathbb{R}$. Then $E = \{0, \beta\}$. We use Q to denote Q_1, and Q^n is the n-th iteration of Q.

For $\theta \in M$, we assume that $\underline{\phi}$ and $\bar{\phi}$ are non-increasing functions in $\mathcal{M}_\beta$ satisfying

$$\underline{\phi}(\theta, z) = \underline{\phi}(z) = \begin{cases} (1-\tilde{\delta})(1-e^{\tilde{\mu}z}), & z < 0, \\ 0, & z \geq 0, \end{cases} \quad \bar{\phi}(\theta, z) = \bar{\phi}(z) = \begin{cases} 1, & z \leq 0, \\ e^{-\sqrt{r-1}z}, & z > 0, \end{cases} \tag{2.6}$$

where $\tilde{\delta} > 0$ is small, and $\tilde{\mu}$ is a positive constant. Clearly, $\underline{\phi} \leq \bar{\phi}$.

Lemma 2.2. *Assume that H1 and (K1) hold. There exist positive numbers $\tilde{C} > 0$ and $\tilde{\mu} > 0$ such that for any $C \geq \tilde{C}$, we have*

$$Q[\underline{\phi}](\theta, x) \geq \underline{\phi}(\theta, x + C) \ \text{ and } \ Q[\bar{\phi}](\theta, x) \leq \bar{\phi}(\theta, x - C) \text{ for all } x \in \mathbb{R}. \tag{2.7}$$

Proof. Let $\tilde{z}_1 = x - Ct$. When $\tilde{z}_1 \leq 0$, we have $\bar{\phi}(\tilde{z}_1) = 1$, and then

$$\begin{aligned} &\bar{\phi}'' + C\bar{\phi}' + \bar{\phi}\{1 - r - \bar{\phi} + r(K \star_C \mathcal{L}_{b,c}\bar{\phi})\} \\ &\leq 1 - r - 1 + r = 0. \end{aligned} \tag{2.8}$$

When $\tilde{z}_1 > 0$, we obtain $\bar{\phi}(\tilde{z}_1) = e^{-\sqrt{r-1}\tilde{z}_1}$. For any $C \geq \tilde{C}_1$, where $\tilde{C}_1$ satisfies $-\sqrt{r-1}\tilde{C}_1 + r = 0$, it gives

$$\begin{aligned} &\bar{\phi}'' + C\bar{\phi}' + \bar{\phi}\{1 - r - \bar{\phi} + r(K \star_C \mathcal{L}_{b,c}\bar{\phi})\} \\ &= e^{-\sqrt{r-1}\tilde{z}_1}\{-\sqrt{r-1}C - e^{-\sqrt{r-1}\tilde{z}_1} + r(K \star_C \mathcal{L}_{b,c}\bar{\phi})\} \\ &\leq e^{-\sqrt{r-1}\tilde{z}_1}\{-\sqrt{r-1}C + r\} \leq 0. \end{aligned} \tag{2.9}$$

Hence, we conclude that $Q[\bar{\phi}](\theta, x) \leq \bar{\phi}(\theta, x - C)$. In order to prove $Q[\underline{\phi}](\theta, x) \geq \underline{\phi}(\theta, x + C)$, we let $\tilde{z}_2 = x + Ct$. Choose sufficiently small $\tilde{\epsilon} > 0$ and $\tilde{\epsilon}_1 > 0$ such that

$$\tilde{\delta} - r(\tilde{\epsilon} + \tilde{\epsilon}_1) > 0. \tag{2.10}$$

Since $\int_0^\infty \int_{\mathbb{R}} K(s, y) dy ds = 1$, there exist a large enough $\tilde{M}_1 > 0$ such that

$$\int_0^\infty \int_{-\tilde{M}_1}^\infty K(s, y) dy ds > 1 - \tilde{\epsilon}. \tag{2.11}$$

Note that we can still derive the existence of Ψ in (2.1) when we choose $\Phi(z) = \underline{\phi}(z)$ (Here, $\Phi(-\infty) = 1 - \tilde{\delta}$). There exists $M_0(c) < 0$ so that

$$\mathcal{L}_{b,c}\underline{\phi}(\tilde{z}_2) > 1 - \tilde{\epsilon}_1 \quad \text{when } \tilde{z}_2 < M_0(c). \tag{2.12}$$

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For $\tilde{z}_2 < M_0(c) - \tilde{M}_1 < 0$, if $C \geq \tilde{\mu}$, then by (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
&\underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\{1 - r - \underline{\phi} + r(K \star_{-C} \mathcal{L}_{b,c}\underline{\phi})\} \\
&= \underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\left\{1 - r - \underline{\phi} + r\int_0^\infty \int_{-\infty}^{-\tilde{M}_1} K(s,y)\mathcal{L}_{b,c}\underline{\phi}(\tilde{z}_2 - Cs - y)dyds \right.\\
&\left. + r\int_0^\infty \int_{-\tilde{M}_1}^{\infty} K(s,y)\mathcal{L}_{b,c}\underline{\phi}(\tilde{z}_2 - Cs - y)dyds\right\} \\
&\geq \underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\left\{1 - r - \underline{\phi} + r(1-\tilde{\epsilon}_1)\int_0^\infty \int_{-\tilde{M}_1}^{\infty} K(s,y)dyds\right\} \\
&\geq \underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\{1 - r - \underline{\phi} + r(1-\tilde{\epsilon})(1-\tilde{\epsilon}_1)\} \\
&= (1-\tilde{\delta})\left\{\tilde{\mu}e^{\tilde{\mu}\tilde{z}_2}(C - \tilde{\mu}) + (1 - e^{\tilde{\mu}\tilde{z}_2})[\tilde{\delta} - r(\tilde{\epsilon} + \tilde{\epsilon}_1) + r\tilde{\epsilon}\tilde{\epsilon}_1 + (1-\tilde{\delta})e^{\tilde{\mu}\tilde{z}_2}]\right\} \geq 0.
\end{aligned} \tag{2.13}$$

Take $\tilde{\eta} := \frac{\phi(M_0(c)-\tilde{M}_1)}{1-\tilde{\delta}} \in (0,1)$. For $\tilde{z}_2 \in [M_0(c) - \tilde{M}_1, 0)$, we have $\underline{\phi}(\tilde{z}_2) \in (0, \tilde{\eta}(1-\tilde{\delta})]$ and $e^{\tilde{\mu}\tilde{z}_2} \in [1-\tilde{\eta}, 1)$. If $C \geq \frac{\tilde{\mu}^2 + \tilde{\eta}[\tilde{\eta}(1-\tilde{\delta})+r-1]}{\tilde{\mu}(1-\tilde{\eta})}$, then

$$\begin{aligned}
&\underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\{1 - r - \underline{\phi} + r(K \star_{-C} \mathcal{L}_{b,c}\underline{\phi})\} \\
&\geq \underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\{1 - r - \underline{\phi}\} \\
&\geq -(1-\tilde{\delta})\tilde{\mu}^2 e^{\tilde{\mu}\tilde{z}_2} + (1-\tilde{\delta})C\tilde{\mu}e^{\tilde{\mu}\tilde{z}_2} + \tilde{\eta}(1-\tilde{\delta})[1 - r - \tilde{\eta}(1-\tilde{\delta})] \\
&\geq (1-\tilde{\delta})\left\{-\tilde{\mu}^2 + C\tilde{\mu}(1-\tilde{\eta}) + \tilde{\eta}[1 - r - \tilde{\eta}(1-\tilde{\delta})]\right\} \geq 0.
\end{aligned} \tag{2.14}$$

For $\tilde{z}_2 \geq 0$, we have $\underline{\phi}(\tilde{z}_2) = 0$, and thus

$$\underline{\phi}'' - C\underline{\phi}' + \underline{\phi}\{1 - r - \underline{\phi} + r(K \star_{-C} \mathcal{L}_{b,c}\underline{\phi})\} = 0. \tag{2.15}$$

Therefore, $Q[\underline{\phi}](\theta, x) \geq \underline{\phi}(\theta, x + C)$ for all $C \geq \tilde{C}_2 := \max\{\frac{\tilde{\mu}^2 + \tilde{\eta}[\tilde{\eta}(1-\tilde{\delta})+r-1]}{\tilde{\mu}(1-\tilde{\eta})}, \tilde{\mu}\}$. Let $\tilde{C} = \max\{\tilde{C}_1, \tilde{C}_2\}$. Thus, the proof is complete. □

According to this lemma, when C is large, the functions $\bar{\phi}(\theta, x - Ct)$ and $\underline{\phi}(\theta, x + Ct)$ serve as upper and lower solutions, respectively, to the auxiliary equation (2.3). For each value of n, a shift is applied to both the upper and lower solutions, resulting in the definitions of

$$\underline{\phi}_n(\theta, x) = \underline{\phi}(\theta, x + n + \tilde{C}) \quad \text{and} \quad \bar{\phi}_n(\theta, x) = \bar{\phi}(\theta, x - (n + \tilde{C})).$$

It is evident that due to the translation invariance property, both $\underline{\phi}_n(\theta, x)$ and $\bar{\phi}_n(\theta, x)$ still serve as upper and lower solutions, respectively, for Q. Let $\kappa_n := (n + \tilde{C})/n$. Define $A_\xi[\phi](x) = \phi(\xi(x))$ for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Similar to the proof of [9, Lemma 3.3], we have the following lemma.

Lemma 2.3. *Assume that H1 and (K1) hold. For each $n \in \mathbb{N}$, $G_n := Q \circ A_{\kappa_n}$ has a fixed point ϕ_n in $\mathcal{M}_\beta$ such that $\phi_n(\theta, x)$ is nonincreasing in x and $\underline{\phi}_n \leq \phi_n \leq \bar{\phi}_n$.*

Theorem 2.4. *Assume that H1 and (K1) hold. There exists a $C \in \mathbb{R}$ such that $\{Q^n\}_{n\geq 1}$ admits a nonincreasing traveling wave connecting β to 0 with speed C.*

Proof. Choose $\tilde{M} > 0$ such that $e^{-\sqrt{r-1}\tilde{M}} < \frac{1}{2}(1-\tilde{\delta})$. It follows

$$\begin{aligned}
e^{-\sqrt{r-1}\tilde{M}} &= \bar{\phi}(\theta, \tilde{M}) = \bar{\phi}_n(\theta, \tilde{M} + n + \tilde{C}) \geq \phi_n(\theta, \tilde{M} + n + \tilde{C}), \\
\frac{1}{2}(1-\tilde{\delta}) &= \underline{\phi}(\theta, -\frac{1}{\tilde{\mu}}\ln 2) = \underline{\phi}_n(\theta, -\frac{1}{\tilde{\mu}}\ln 2 - (n + \tilde{C})) \leq \phi_n(\theta, -\frac{1}{\tilde{\mu}}\ln 2 - (n + \tilde{C})).
\end{aligned}$$

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Define $b_n := \sup\{x : \phi_n(\theta,x) \in [\frac{1}{2}(1-\tilde{\delta}),1]\}$. In other word, b_n is a point so that $\phi_n(\theta,b_n) = \frac{1}{2}(1-\tilde{\delta})$. Then $-\frac{1}{\mu}\ln 2 - (n+\tilde{C}) \le b_n \le \tilde{M} + n + \tilde{C}$. Let $\phi_{+,n}(\theta,x) := \phi_n(\theta,x+b_n)$. Then we have

$$\phi_{+,n} = \phi_n(\theta,\cdot+b_n) = G_n[\phi_n](\theta,\cdot+b_n) = Q[\phi_n(\theta,\kappa_n(\cdot+b_n))] \in Q[C_\beta].$$

By the solution of the integral form of system (2.3), we have

$$\phi_{+,n}(\theta,x) = \int_{\mathbb{R}} \bar{G}(x-y,1)\phi_n(\theta,\kappa_n(y+b_n))dy + \int_0^1\int_{\mathbb{R}} \bar{G}(x-y,1-s)\bar{f}(\phi(1,y))dyds,$$

where $\bar{f}(\phi) = \phi(1-r-\phi+r(K*\mathcal{L}_{b,c}\phi))$ is the reaction term of (2.3) and $\bar{G}$ is the fundamental solution of $u_t = u_{xx}$. By calculation, we have $|(\phi_{+,n})_x(\theta,x)| < \frac{1}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}}$. Therefore, $\phi_{+,n}$ is equicontinuous and uniformly bounded. According to the Ascoli–Arzela theorem, there exist a subsequence (denoted by n), a non-increasing function ϕ_+, and ξ_+ such that

$$\lim_{n\to\infty}\frac{b_n}{n} = \xi_+, \quad \lim_{n\to\infty}\phi_{+,n} = \phi_+.$$

Define $C_+ := \tilde{C}\xi_+$. Observe that

$$\lim_{n\to\infty}\kappa_n(x+b_n) - b_n = \lim_{n\to\infty}(x+\tilde{C}\cdot\frac{x+b_n}{n}) = x + C_+$$

holds uniformly for x in any bounded subset of $\mathbb{R}$. For any $x\in\mathbb{R}$, we have

$$\begin{aligned}\phi_+(\theta,x-C_+) &= \lim_{n\to\infty}\phi_{+,n}(\theta,x-C_+) = \lim_{n\to\infty}\phi_n(\theta,x-C_++b_n) = \lim_{n\to\infty}G_n[\phi_n](\theta,x-C_++b_n)\\ &= \lim_{n\to\infty}Q[\phi_n(\theta,\kappa_n(\cdot+b_n))](x-C_+) = \lim_{n\to\infty}Q[\phi_{+,n}(\theta,\kappa_n(\cdot+b_n)-b_n)](x-C_+) \\ &= Q[\phi_+](\theta,x).\end{aligned} \tag{2.16}$$

Consequently, $\phi_+(\theta,-\infty)$ and $\phi_+(\theta,\infty)$ are fixed points of Q. Observing that $\frac{1}{2}(1-\tilde{\delta}) \le \lim_{n\to\infty}\phi_n(\theta,b_n) = \phi_+(\theta,0) < 1$. It follows that $\phi_+(\theta,-\infty) = 1$ and $\phi_+(\theta,\infty) = 0$. Hence, the proof is complete. □

The solution to equation (2.3) exhibits continuity with respect to its parameter c, which implies that C_+ is a continuous function of c. By substituting $C_+ = C(c)$ back into equation (2.3), we shall present the following theorem.

Theorem 2.5. *Assume that H1 and (K1) hold. There exist $C(c)\in\mathbb{R}$ and $\Phi\in\mathcal{M}_\beta$ connecting β to 0 such that $Q_t[\Phi](\theta,x) = \Phi(\theta,x-C(c)t)$ for all $x\in\mathbb{R}$, $\theta\in(-\infty,0]$, that is, (2.3) admits a non-increasing travelling wave $\Phi(\theta,x-C(c)t)$ connecting β to 0. Furthermore, this bistable travelling wave $\Phi(\theta,z)$ is strictly decreasing in z with $\Phi_z(\theta,z)<0$. The wave speed $C(c)$ is non-increasing with respect to c, which means $C(c_1)\ge C(c_2)$ if $0<c_1\le c_2$.*

Proof. The existence of Φ in (2.3) connecting β to 0 can be guaranteed by Theorem 2.4. We now prove $\Phi_z(\theta,z)<0$ by a contradiction argument. Assume $\Phi_z(\theta,z_0)=0$ for some $z_0\in\mathbb{R}$. The strong maximum principle implies that $\Phi_z(\theta,z)\equiv 0$ for $z\in\mathbb{R}$. It contradicts the boundary condition (2.5). Next, we will show the monotonicity of $C(c)$ with respect to c. Let $0<c_1\le c_2$. By taking the maximum value, we can set a same $\tilde{C}$ in Lemma 2.2 for both $c=c_1$ and $c=c_2$. According to Lemma 2.1 c), we conclude that the reaction term in (2.4) is non-increasing with respect to c. From Lemma 2.3, we have $\phi_n(c_1,\theta,\tilde{x}) \ge \phi_n(c_2,\theta,\tilde{x})$ (here $\phi_n(c,\theta,\tilde{x})$ implies that ϕ_n is dependent on c). Based on the proof of Theorem 2.4, we know that $b_n(c_1)\ge b_n(c_2)$ (Here $b_n(c)$ indicates that b_n is dependent on c). Consequently, $C(c_1)\ge C(c_2)$. Therefore, the proof is complete. □

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2.2. Bistable waves of the original system (1.5)-(1.6)

To establish the existence and uniqueness of a bistable monotone travelling wave for (1.5)-(1.6) under H1, we first crucially show that the equation $C(c) = c$ possesses a unique positive root. This leads us to the following theorem.

Theorem 2.6. *Assume that H1 and (K1) hold. There exists a monotone bistable travelling wave $(c, \Phi(z), \Psi(z))$, $z = \nu{\cdot}x - ct$, $||\nu|| = 1$, of system (1.5)-(1.6), where c is positive and unique. Moreover, $\Phi'(z) < 0$ and $\Psi'(z) < 0$ for $z \in \mathbb{R}$.*

Proof. Theorem 2.5 gives the existence of the monotone bistable travelling wave solution of (2.4)-(2.5). We also know that $C(c)$ is a non-increasing function of c. To return to (1.5)-(1.6) for the existence of waves, we need to show that $C(c) = c$ has a unique positive root. To this end, it suffices to show $C(0_+) > 0$. Let

$$f(\Phi) = \Phi(1 - r - \Phi + r(K \star_C \mathcal{L}_{b,c}\Phi)).$$

Then (2.4) can be written as

$$\Phi'' + C(c,f)\Phi' + f(\Phi) = 0. \tag{2.17}$$

Here we just use $C(c,f)$ to indicate that the speed is dependent on the reaction term.

By Lemma 2.1 c), there exist a sufficiently small $\delta > 0$ such that $\mathcal{L}_{b,c}\Phi(z) > 1 - \delta$ for sufficiently small $c < \delta$ and $\Phi > 0$. Then, for small $\hat{\varepsilon} > 0$, we have

$$f(\Phi) \geq g(\Phi) := \begin{cases} \Phi(1 - r\delta - \Phi), & \Phi > \hat{\varepsilon}, \\ \Phi(1 - r - \Phi), & \Phi < \hat{\varepsilon}. \end{cases} \tag{2.18}$$

Consider

$$\Phi'' + C(c, g)\Phi' + g(\Phi) = 0. \tag{2.19}$$

By choosing $\tilde{\delta}$ in (2.6) such that $\tilde{\delta} - r\delta > 0$, we can derive the existence of $\Phi(z)$ in (2.19) connecting $1 - r\delta$ to 0 according to the similar arguments in Lemmas 2.2 and 2.3, and Theorem 2.4. We multiply the left and right sides of (2.19) by Φ' and integrate both sides from $-\infty$ to ∞ with respect to z. It follows that the wave speed $C(c, g)$ satisfies

$$C(c, g) = \frac{\int_0^{1-r\delta} g(\Phi)d\Phi}{\int_{-\infty}^{\infty} (\Phi')^2 dz}.$$

Since $\hat{\varepsilon}$ and δ are small enough, we have

$$\int_0^{1-r\delta} g(\Phi)d\Phi = \frac{1}{6}(1 - r\delta)^3 - \frac{1}{2}r(1 - \delta)\hat{\varepsilon}^2 > 0.$$

It means $C(c, g) > 0$. Applying equations (2.17), (2.18), (2.19), and a similar proof as presented in Theorem 2.5 (concerning the monotonicity of $C(c)$ with respect to c), we can deduce that $C(c,f) \geq C(c, g) > 0$ when $c \to 0$. According to Theorem 2.5, the equation $C(c) = c$ has a unique positive solution. Consequently, system (2.2) has a monotonic solution $(c, \Phi(z))$, that is, there exists a monotone bistable travelling wave $(c, \Phi(z), \Psi(z))$ of system (1.5)-(1.6). Based on Lemma 2.1 and Theorem 2.5, we know $\Phi'(z) < 0$ and $\Psi'(z) < 0$ for $z \in \mathbb{R}$. It completes the proof. □

Our aim in the subsequent discussion is to establish the uniqueness of the above travelling wave profiles. To achieve this, we utilise the bistable travelling wave $(\Phi(z), \Psi(z))$ to create a pair of upper and lower solutions of (1.2). We introduce two non-increasing continuous functions

$$R_1(z) = \begin{cases} R_1^+, & z \leq -\eta_1 < 0, \\ e^{-2\mu z}, & z > -\eta_1, \end{cases} \qquad R_2(z) = \begin{cases} R_2^+, & z \leq -\eta_2 < 0, \\ e^{-\mu z}, & z > -\eta_2. \end{cases} \tag{2.20}$$

Here constants $R_i^+ > 1$, i = 1,2, and $\mu > 0$ is small to be determined. η_1 and η_2 are chosen so that the above two functions are continuous. We first give the following squeezing lemma, which is crucial in proving the uniqueness of the bistable travelling wave of (1.5)-(1.6).

Lemma 2.7. *Assume that H1, (K1) and (K2) hold. Define*

$$\phi^{\pm}(t,z) = \begin{cases} \Phi(z \pm \xi_0 \pm \sigma_1\delta(1-e^{-\rho t})) \pm \delta R_1(z \pm \xi_0 \pm \sigma_1\delta(1-e^{-\rho t}))e^{-\rho t}, & t \geq 0, \\ \Phi(z \pm \xi_0) \pm \delta R_1(z \pm \xi_0), & t < 0, \end{cases}$$

$$\psi^{\pm}(t,z) = \begin{cases} \Psi(z \pm \xi_0 \pm \sigma_1\delta(1-e^{-\rho t})) \pm \delta R_2(z \pm \xi_0 \pm \sigma_1\delta(1-e^{-\rho t}))e^{-\rho t}, & t \geq 0, \\ \Psi(z \pm \xi_0) \pm \delta R_2(z \pm \xi_0), & t < 0, \end{cases}$$

where $z = \nu\cdot x - ct$ and $||\nu|| = 1$. Then there exists $\sigma_1 < 0$, $\mu > 0$, $\rho > 0$ and $\delta_0 > 0$ such that for any $\xi_0 \in \mathbb{R}$, $\delta \in (0, \delta_0)$, the formulas $(\phi^{\pm}(t,z), \psi^{\pm}(t,z))$ are upper and lower solutions of (1.2), respectively. It further follows that $(\phi^-(t,z), \psi^-(t,z)) \leq (\phi(t,z), \psi(t,z)) \leq (\phi^+(t,z), \psi^+(t,z))$ for $(t,z) \in (0,\infty] \times \mathbb{R}$, if $(\phi^-(\theta,z), \psi^-(\theta,z)) \leq (\phi(\theta,z), \psi(\theta,z)) \leq (\phi^+(\theta,z), \psi^+(\theta,z))$ for $(\theta,z) \in (-\infty,0] \times \mathbb{R}$.

Proof. We only prove $(\phi^+(t,z), \psi^+(t,z))$ is an upper solution of (1.2), while the lower solution $(\phi^-(t,z), \psi^-(t,z))$ can be verified using a similar argument. Let $\xi = z + \xi_0 + \sigma_1\delta(1-e^{-\rho t})$. We choose δ, μ and $\rho < \min\{\mu^2, \tilde{\delta}_0\}$ that are small enough, where $\tilde{\delta}_0$ is defined in (K2). We take R_1^+ and R_2^+ such that

$$\frac{R_1^+}{R_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds} > r.$$

For small enough $\epsilon_1 > 0$, $\epsilon_3 > 0$ and $\epsilon_4 > 0$, we take sufficiently large $M_1 > \frac{1}{\mu^2}$ such that $\Phi(z), \Psi(z) < \epsilon_1$ for $\xi > \frac{M_1}{2}$ and

$$\int_0^{+\infty} \int_{\frac{1}{2}M_1}^{\infty} K(s,y)dyds < \epsilon_3, \quad \int_0^{+\infty} \int_{\frac{1}{2}M_1}^{\infty} K(s,y)e^{\rho s}dyds < \epsilon_4,$$

by using (K1) and (K2). Take $M_2 > 0$ and sufficiently small $\epsilon_2 > 0$ such that $\Phi(z), \Psi(z) > 1 - \epsilon_2$ for $\xi < -M_2$.

Let $M > \max\{\eta_1, \eta_2, M_1, M_2\}$. Set

$$\begin{aligned} L_1 &:= \max\{\max_{-M\leq\xi\leq M} \Phi'(\xi), \max_{-M\leq\xi\leq M} \Psi'(\xi)\}, \\ L_2 &:= \max\{\max_{\xi\in[-M,M]} R_i'(\xi), \max_{\xi\in[-M,M]} R_i''(\xi)\}, i = 1,2, \\ \sigma_1 &< \frac{L_2 + cL_2 + R_1^+\rho + R_2^+\rho + rR_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds + (1+b)R_1^+}{\rho L_1}. \end{aligned} \tag{2.21}$$

Substituting (ϕ^+, ψ^+) into (1.2), we get

$$\begin{aligned} &\phi_{zz}^+ + c\phi_z^+ - \phi_t^+ + \phi^+(1 - r - \phi^+ + r(K*\psi^+)) \\ &= \Phi'' + R_1''\delta e^{-\rho t} + c\Phi' - \sigma_1\rho\Phi'\delta e^{-\rho t} + cR_1'\delta e^{-\rho t} - \sigma_1\rho R_1'\delta^2 e^{-2\rho t} + R_1\rho\delta e^{-\rho t} \\ &+ (\Phi + R_1\delta e^{-\rho t})\Big\{1 - r - \Phi - R_1\delta e^{-\rho t} + r(K*\psi^+)\Big\} \\ &= R_1''\delta e^{-\rho t} - \sigma_1\rho\Phi'\delta e^{-\rho t} + cR_1'\delta e^{-\rho t} - \sigma_1\rho R_1'\delta^2 e^{-2\rho t} + R_1\rho\delta e^{-\rho t} \\ &+ \Phi\Big\{-R_1\delta e^{-\rho t} + J(\Psi) + J_1(R_2)\Big\} \\ &+ R_1\delta e^{-\rho t}\Big\{1 - r - \Phi - R_1\delta e^{-\rho t} + r(K*\psi^+)\Big\} \\ &:= I + II + III, \quad \forall t > 0, \end{aligned} \tag{2.22}$$

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where

$$K*\psi^+ = \int_0^t \int_{\mathbb{R}} K(s,y)[\Psi(\xi+cs-y+\sigma_1\delta e^{-\rho t}(1-e^{\rho s}))$$
$$+R_2(\xi+cs-y+\sigma_1\delta e^{-\rho t}(1-e^{\rho s}))\delta e^{-\rho(t-s)}]dyds$$
$$+\int_t^\infty \int_{\mathbb{R}} K(s,y)[\Psi(\xi+cs-y-\sigma_1\delta(1-e^{-\rho t}))+R_2(\xi+cs-y-\sigma_1\delta(1-e^{-\rho t}))\delta]dyds,$$

$$J(\Psi)=\int_0^t \int_{\mathbb{R}} K(s,y)[\Psi(\xi+cs-y+\sigma_1\delta e^{-\rho t}(1-e^{\rho s}))-\Psi(\xi+cs-y)]dyds$$
$$+\int_t^\infty \int_{\mathbb{R}} K(s,y)[\Psi(\xi+cs-y-\sigma_1\delta(1-e^{-\rho t}))-\Psi(\xi+cs-y)]dyds\le 0,$$

$$\begin{aligned}J_1(R_2)=&r\int_0^t \int_{\mathbb{R}} K(s,y)R_2(\xi+cs-y+\sigma_1\delta e^{-\rho t}(1-e^{\rho s}))\delta e^{-\rho(t-s)}dyds\\&+r\int_t^\infty \int_{\mathbb{R}} K(s,y)R_2(\xi+cs-y-\sigma_1\delta(1-e^{-\rho t}))\delta dyds\\&\le r\int_0^t \int_{\mathbb{R}} K(s,y)R_2(\xi+cs-y+\sigma_1\delta e^{-\rho t}(1-e^{\rho s}))\delta e^{-\rho(t-s)}dyds\\&+r\int_t^\infty \int_{\mathbb{R}} K(s,y)R_2(\xi+cs-y-\sigma_1\delta(1-e^{-\rho t}))\delta e^{-\rho(t-s)}dyds.\end{aligned}$$

and

$$\begin{aligned}&\psi^+_{zz}+c\psi^+_z-\psi^+_t+b\phi^+(1-\psi^+)\\&=\Psi''+R_2''\delta e^{-\rho t}+c\Psi'-\sigma_1\rho\Psi'\delta e^{-\rho t}+cR_2'\delta e^{-\rho t}-\sigma_1\rho R_2'\delta^2e^{-2\rho t}+R_2\rho\delta e^{-\rho t}\\&+b(\Phi+R_1\delta e^{-\rho t})(1-\Psi-R_2\delta e^{-\rho t})\\&=R_2''\delta e^{-\rho t}-\sigma_1\rho\Psi'\delta e^{-\rho t}+cR_2'\delta e^{-\rho t}-\sigma_1\rho R_2'\delta^2e^{-2\rho t}+R_2\rho\delta e^{-\rho t}\\&-b\Phi R_2\delta e^{-\rho t}+bR_1\delta e^{-\rho t}(1-\Psi-R_2\delta e^{-\rho t}),\quad \forall t>0.\end{aligned}\tag{2.23}$$

Here we use the fact (Φ,Ψ) is the bistable travelling wave of (1.2) and satisfies (1.5). In order to prove $(\phi^+(z,t),\psi^+(z,t))$ is an upper solution of (1.2), we consider three cases:

Case (i): $|\xi|\le M$. According to (2.21) and (2.22), we have

$$I\le \delta e^{-\rho t}[L_2-\sigma_1\rho L_1+cL_2+R_1^+\rho],$$

$$II\le rR_2^+\delta e^{-\rho t}\int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds,$$

and

$$III\le R_1^+\delta e^{-\rho t}[1-r+r(1+R_2^+\delta e^{-\rho t}\int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds)].$$

Then it gives

$$\begin{aligned}&\phi^+_{zz}+c\phi^+_z-\phi^+_t+\phi^+(1-r-\phi^++r(K*\psi^+))\\&\le \delta e^{-\rho t}[L_2-\sigma_1\rho L_1+cL_2+R_1^+\rho+rR_2^+\int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds+R_1^+\\&+rR_1^+R_2^+\delta\int_0^\infty \int_{\mathbb{R}} K(s,y)e^{\rho s}dyds]\le 0.\end{aligned}\tag{2.24}$$

The last inequality sign in (2.24) holds by (K2), (2.21) and δ is sufficiently small. By (2.21) and (2.23), we also have

$$\begin{aligned}&\psi^+_{zz}+c\psi^+_z-\psi^+_t+b\phi^+(1-\psi^+)\\&\le \delta e^{-\rho t}[L_2-\sigma_1\rho L_1+cL_2+R_2^+\rho+bR_1^+]\le 0.\end{aligned}\tag{2.25}$$

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Case (ii): $\xi < -M$. Note that $\xi < -M_2$, $R_1(\xi) = R_1^+$ and $R_2(\xi) = R_2^+$. According to (2.22) and the fact $\frac{R_1^+}{R_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds} > r$, we have

$$I \leq R_1^+ \rho \delta e^{-\rho t},$$

$$II \leq \Phi \delta e^{-\rho t}(-R_1^+ + rR_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds) \leq \delta e^{-\rho t}(1-\epsilon_2)(-R_1^+ + rR_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds),$$

and

$$III \leq R_1^+ \delta e^{-\rho t}[1 - r - (1-\epsilon_2) + r(1 + R_2^+ \delta e^{-\rho t} \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds)].$$

Then

$$\begin{aligned} &\phi_{zz}^+ + c\phi_z^+ - \phi_t^+ + \phi^+(1 - r - \phi^+ + r(K * \psi^+)) \\ &\leq \delta e^{-\rho t}[R_1^+ \rho - (1-\epsilon_2)(R_1^+ - rR_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds) \\ &+ R_1^+(\epsilon_2 + rR_2^+ \delta \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds)] \leq 0, \end{aligned} \tag{2.26}$$

since $\frac{R_1^+}{R_2^+ \int_0^\infty \int_{\mathbb{R}} K(s,y) e^{\rho s} dyds} > r$, and ρ, ϵ_2, δ are small enough. By (2.23), we have

$$\begin{aligned} &\psi_{zz}^+ + c\psi_z^+ - \psi_t^+ + b\phi^+(1 - \psi^+) \\ &\leq \delta e^{-\rho t}[R_2^+ \rho - bR_2^+(1-\epsilon_2) + bR_1^+(1 - (1-\epsilon_2) - R_2^+ \delta e^{-\rho t})] \\ &\leq \delta e^{-\rho t}[R_2^+ \rho - bR_2^+(1-\epsilon_2) + bR_1^+ \epsilon_2] \leq 0. \end{aligned} \tag{2.27}$$

The last inequality in (2.27) holds since ρ and ϵ_2 are small enough.

Case (iii): $\xi > M$. Note that $\xi > M_1$, $R_1(\xi) = e^{-2\mu\xi}$ and $R_2(\xi) = e^{-\mu\xi}$. We can calculate

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} K(s,y) \Psi(\xi + cs - y) dyds \\ &= \int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y) \Psi(\xi + cs - y) dyds + \int_0^\infty \int_{\frac{1}{2}M_1}^\infty K(s,y) \Psi(\xi + cs - y) dyds \\ &\leq \epsilon_1 \int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y) dyds + \int_0^\infty \int_{\frac{1}{2}M_1}^\infty K(s,y) dyds \\ &\leq \epsilon_1 + \epsilon_3 \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} K(s,y) R_2(\xi + cs - y) e^{\rho s} dyds \\ &= \int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y) R_2(\xi + cs - y) e^{\rho s} dyds + \int_0^\infty \int_{\frac{1}{2}M_1}^\infty K(s,y) R_2(\xi + cs - y) e^{\rho s} dyds \\ &\leq e^{-\frac{1}{2}\mu M_1} \int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y) e^{\rho s} dyds + R_2^+ \int_0^\infty \int_{\frac{1}{2}M_1}^\infty K(s,y) e^{\rho s} dyds \\ &\leq e^{-\frac{1}{2\mu}} \int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y) e^{\rho s} dyds + R_2^+ \epsilon_4. \end{aligned} \tag{2.29}$$

According to (2.22), we have

$$I \leq e^{-2\mu\xi} \delta e^{-\rho t}(4\mu^2 - 2c\mu + \rho),$$

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$$\begin{aligned} II &\le \Phi\left\{-e^{-2\mu\xi}\delta e^{-\rho t} + r\int_0^\infty \int_{\mathbb{R}} K(s,y)R_2(\xi+cs-y)\delta e^{-\rho(t-s)}dyds\right\} \\ &\le \Phi\delta e^{-\rho t}[-e^{-2\mu\xi} + r(e^{-\frac{1}{2\mu}}\int_0^\infty \int_{-\infty}^{\frac{1}{2}M_1} K(s,y)e^{\rho s}dyds + R_2^+\epsilon_4)], \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} III &\le e^{-2\mu\xi}\delta e^{-\rho t}\left\{1-r+r\int_0^\infty\int_{\mathbb{R}} K(s,y)\Psi(\xi+cs-y)dyds + rR_2^+\delta e^{-\rho t}\int_0^\infty\int_{-\infty}^\infty K(s,y)e^{\rho s}dyds\right\} \\ &\le e^{-2\mu\xi}\delta e^{-\rho t}\left\{1-r+r(\epsilon_1+\epsilon_3)+rR_2^+\delta\int_0^\infty\int_{-\infty}^\infty K(s,y)e^{\rho s}dyds\right\}. \end{aligned} \tag{2.31}$$

Then equation (2.22) becomes

$$\begin{aligned} &\phi^+_{zz} + c\phi^+_z - \phi^+_t + \phi^+(1-r-\phi^+ + r(K*\psi^+)) \\ &\le e^{-2\mu\xi}\delta e^{-\rho t}(4\mu^2 - 2c\mu+\rho) + \Phi\delta e^{-\rho t}[-e^{-2\mu\xi} + r(e^{-\frac{1}{2\mu}}\int_0^\infty\int_{-\infty}^{\frac{1}{2}M_1} K(s,y)e^{\rho s}dyds + R_2^+\epsilon_4)] \\ &+ e^{-2\mu\xi}\delta e^{-\rho t}\left\{1-r+r(\epsilon_1+\epsilon_3)+rR_2^+\delta\int_0^\infty\int_{-\infty}^\infty K(s,y)e^{\rho s}dyds\right\} \\ &\le e^{-2\mu\xi}\delta e^{-\rho t}\left\{4\mu^2-2c\mu+\rho+r(e^{-\frac{1}{2\mu}}\int_0^\infty\int_{-\infty}^{\frac{1}{2}M_1} K(s,y)e^{\rho s}dyds + R_2^+\epsilon_4)\right. \\ &\left. + 1-r+r(\epsilon_1+\epsilon_3)+rR_2^+\delta\int_0^\infty\int_{-\infty}^\infty K(s,y)e^{\rho s}dyds\right\} \le 0. \end{aligned} \tag{2.32}$$

The last inequality sign in (2.32) holds by (K2), $r>1$ and $\mu, \rho, \epsilon_1, \epsilon_3, \epsilon_4, \delta$ are small enough. It follows from (2.23) that

$$\begin{aligned} &\psi^+_{zz} + c\psi^+_z - \psi^+_t + b\phi^+(1-\psi^+) \\ &\le \delta e^{-\rho t}e^{-\mu\xi}[\mu^2 - c\mu+\rho] + be^{-2\mu\xi}\delta e^{-\rho t} \le \delta e^{-\rho t}e^{-\mu\xi}[\mu^2-c\mu+\mu^2+be^{-\frac{1}{\mu}}] \\ &\le \mu\delta e^{-\rho t}e^{-\mu\xi}[2\mu - c + b\frac{1}{\mu}e^{-\frac{1}{\mu}}] \le 0, \end{aligned} \tag{2.33}$$

since $\rho<\mu^2$, $\xi>M_1>\frac{1}{\mu^2}$, and μ, $\frac{1}{\mu}e^{-\frac{1}{\mu}}$ are sufficiently small.

Combining cases (i)-(iii), we know $(\phi^+(z,t), \psi^+(z,t))$ is an upper solution of (1.2). □

Subsequently, we shall establish the uniqueness of the bistable travelling wave of system (1.2) by using Lemma 2.7.

Theorem 2.8. *Assume that H1, (K1) and (K2) hold. If $(\Phi^*(\nu\cdot x-c^*t), \Psi^*(\nu\cdot x-c^*t))$ is a bistable travelling wave of system (1.2) connecting e_β to e_0, then there exists $z^*\in\mathbb{R}$ such that $(\Phi^*(\nu\cdot x-c^*t), \Psi^*(\nu\cdot x-c^*t)) = (\Phi(\nu\cdot x-ct+z^*), \Psi(\nu\cdot x-ct+z^*))$ and $c^*=c$, where $(\Phi(\nu\cdot x-ct), \Psi(\nu\cdot x-ct))$ is the solution in Theorem 2.6.*

Proof. By using a similar argument in [31, Lemma 13], we can conclude the bistable travelling wave of system (1.2) connecting e_β to e_0 decays exponentially as $z\to\infty$. Therefore, there exist $\xi_1\in\mathbb{R}$ and $h\gg 1$ such that, for $\theta\in(-\infty,0]$, we have

$$\begin{aligned} &\Phi(\nu\cdot x-c\theta+\xi_1)) - \delta R_1(\nu\cdot x-c\theta) \le \Phi^*(\nu\cdot x-c^*\theta) \le \Phi(\nu\cdot x-c\theta+\xi_1-h)+\delta R_1(\nu\cdot x-c\theta), \\ &\Psi(\nu\cdot x-c\theta+\xi_1) - \delta R_2(\nu\cdot x-c\theta) \le \Psi^*(\nu\cdot x-c^*\theta) \le \Psi(\nu\cdot x-c\theta+\xi_1-h)+\delta R_2(\nu\cdot x-c\theta), \end{aligned} \tag{2.34}$$

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thanks to the sufficiently small decay rate μ in R_1 and R_2. Then by Lemma 2.7 and the comparison theorem, we have

$$\begin{aligned}
&\Phi(\nu\cdot x-ct+\xi_1-\sigma_1\delta(1-e^{-\rho t}))-\delta R_1(\nu\cdot x-ct-\sigma_1\delta(1-e^{-\rho t}))e^{-\rho t}\le\Phi^*(\nu\cdot x-c^*t)\\
&\le\Phi(\nu\cdot x-ct+\xi_1-h+\sigma_1\delta(1-e^{-\rho t}))+\delta R_1(\nu\cdot x-ct+\sigma_1\delta(1-e^{-\rho t}))e^{-\rho t},\\
&\Psi(\nu\cdot x-ct+\xi_1-\sigma_1\delta(1-e^{-\rho t}))-\delta R_2(\nu\cdot x-ct-\sigma_1\delta(1-e^{-\rho t}))e^{-\rho t}\le\Psi^*(\nu\cdot x-c^*t)\\
&\le\Psi(\nu\cdot x-ct+\xi_1-h+\sigma_1\delta(1-e^{-\rho t}))+\delta R_2(\nu\cdot x-ct+\sigma_1\delta(1-e^{-\rho t}))e^{-\rho t},\quad t>0.
\end{aligned}\tag{2.35}$$

Fix $\bar{z}=\nu\cdot x-c^*t$ such that $\Phi^*(\bar{z})>0$. If $c^*>c$, then

$$\Phi^*(\bar{z})\le\Phi(\bar{z}+(c^*-c))t+\xi_1-h+\sigma_1\delta(1-e^{-\rho t}))+\delta R_1(z+\sigma_1\delta(1-e^{-\rho t}))e^{-\rho t}\to 0,\text{ as }t\to\infty.$$

It is in contradiction with the fact that $\Phi^*(\bar{z})>0$, implying that $c^*\le c$. Likewise, according to the left inequality in (2.35), we have $c^*\ge c$. Consequently, we can conclude that $c^*=c$.

By taking the limit as $t\to\infty$ in (2.35), we obtain the following inequalities:

$$\begin{aligned}
\Phi(z+\xi_1-\sigma_1\delta)\le\Phi^*(z)\le\Phi(z+\xi_1-h+\sigma_1\delta),\ \forall z\in\mathbb{R},\\
\Psi(z+\xi_1-\sigma_1\delta)\le\Psi^*(z)\le\Psi(z+\xi_1-h+\sigma_1\delta),\ \forall z\in\mathbb{R}.
\end{aligned}$$

Define

$$\begin{aligned}
\xi_*=\sup\{\xi:\ \Phi^*(\cdot)\le\Phi(\cdot+\xi),\ \Psi^*(\cdot)\le\Psi(\cdot+\xi)\},\\
\xi^*=\inf\{\xi:\ \Phi^*(\cdot)\ge\Phi(\cdot+\xi),\ \Psi^*(\cdot)\ge\Psi(\cdot+\xi)\}.
\end{aligned}$$

By employing a similar approach as the proof in [6, Theorem 2.1], we can prove $\xi_*=\xi^*$. Thus, the uniqueness of the wave profile, up to translation, is established. □

The following theorem shows the monotonicity of bistable wave speed c with respect to parameters r and b in the system (1.5)-(1.6).

Theorem 2.9. *Assume that H1, (K1) and (K2) hold. For system* (1.5)-(1.6)*, the bistable wave speed c is non-increasing in r when b is fixed; The bistable wave speed c is non-decreasing with respect to b for fixed r.*

Proof. We only prove that c is non-increasing in r. The monotonicity of c with respect to b can be proved by a similar method. Take $0<r_1\le r_2$. To the contrary, we assume $c_1<c_2$, where c_i is the bistable wave speed when $r=r_i$, $i=1,2$. Let $(\Phi_i,\Psi_i)(z)$, $i=1,2$, be the travelling wave with speed c_i, satisfying system (1.5)-(1.6). since

$$\begin{aligned}
\Phi_1(z)\sim 1-a_1e^{\zeta(c_1)z},\ \Phi_2(z)\sim 1-a_2e^{\zeta(c_2)z},\ \text{as }z\to-\infty,\\
\Phi_1(z)\sim a_3e^{-\xi(c_1,r_1)z},\ \Phi_2(z)\sim a_4e^{-\xi(c_2,r_2)z},\ \text{as }z\to\infty,
\end{aligned}$$

where $0<\zeta(c_2)<\zeta(c_1)$, $0<\xi(c_1,r_1)<\xi(c_2,r_2)$ and a_i, $i=1,2,3,4$, are positive constants, and then by translation if necessary, we have $\Phi_2(\nu\cdot x)\le\Phi_1(\nu\cdot x)$ for all $x\in\mathbb{R}^n$. By the Ψ equation in system (1.5), we know $(\Phi_2(\nu\cdot x),\Psi_2(\nu\cdot x))\le(\Phi_1(\nu\cdot x),\Psi_1(\nu\cdot x))$ for all $x\in\mathbb{R}^n$. Using the comparison principle, we have

$$\Phi_2(\nu\cdot x-c_2t)\le\Phi_1(\nu\cdot x-c_1t),\qquad\Psi_2(\nu\cdot x-c_2t)\le\Psi_1(\nu\cdot x-c_1t).$$

Fix $z^*=\nu\cdot x-c_2t$ such that $\Phi_2(z^*)>0$. It follows that

$$\Phi_2(\nu\cdot x-c_2t)\le\Phi_1(\nu\cdot x-c_1t)=\Phi_1(z^*+(c_2-c_1)t)\to 0,\ \text{as }t\to\infty,$$

which contradicts $\Phi_2(z^*)>0$. Therefore, $c_1\ge c_2$. □

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3. Travelling waves and speed selection in the case H2

In this section, we consider the monostable case H2, where e_β is stable, and e_0 is unstable. Linearising (1.5) at e_0, we derive the characteristic equation as

$$\mu^2 - c\mu + (1-r) = 0, \tag{3.1}$$

which yields two roots

$$\mu_1 = \mu_1(c) = \frac{c - \sqrt{c^2 - 4(1-r)}}{2}, \quad \mu_2 = \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1-r)}}{2}. \tag{3.2}$$

Denote $c_0 = 2\sqrt{1-r}$ as the linear speed so that μ_1 and μ_2 are real when $c \geq c_0$. Define $\bar{\mu} = \mu_1(c_0) = \mu_2(c_0) = \sqrt{1-r}$.

3.1. Existence of monostable travelling waves

Before giving the existence of monostable monotone travelling waves of system (1.5)-(1.6), we first construct a lower solution for $c > c_0$, which will be used in the proof of the subsequent theorem.

For any $c > c_0$, define a continuous function

$$\underline{\Phi}_0(z) = \begin{cases} e^{-\mu_1(c)z}(1 - Me^{-\delta_1 z}), & z \geq z_1, \\ 0, & z < z_1, \end{cases} \tag{3.3}$$

where $0 < \delta_1 \ll 1$, M is a positive constant to be determined, and $z_1 = \frac{\log M}{\delta_1}$.

Lemma 3.1. *Assume that $r \in (0, 1)$ and (K1) hold. If $c = c_0 + \delta_2$ and $\delta_2 > 0$, then $(\underline{\Phi}_0, \underline{\Psi}_0)(z)$ is a lower solution of system (1.5)-(1.6), where $\underline{\Phi}_0(z)$ is defined in (3.3), and $\underline{\Psi}_0(z)$ is the solution of the Ψ-equation derived from the system using $\Phi(z) = \underline{\Phi}_0(z)$.*

Proof. Substituting $\underline{\Phi}_0(z)$ into Φ-equation of (1.5), we obtain

$$\begin{aligned} &\underline{\Phi}_0'' + c\underline{\Phi}_0' + \underline{\Phi}_0(1 - r - \underline{\Phi}_0 + r(K \star_c \underline{\Psi}_0)) \\ &= e^{-\mu_1 z}[\mu_1^2 - c\mu_1 + (1-r)] - Me^{-(\mu_1+\delta_1)z}[(\mu_1+\delta_1)^2 - c(\mu_1+\delta_1) + 1 - r] \\ &\quad - e^{-2\mu_1 z}(1 - Me^{-\delta_1 z})^2 + r(K \star_c \underline{\Psi}_0)e^{-\mu_1 z}(1 - Me^{-\delta_1 z}). \end{aligned}$$

It is observed that the second term is positive. We can choose a sufficiently large M such that the sum of the second and the third terms is positive. The first term is 0, and the last term is positive. Therefore, the proof is complete. □

In order to prove the existence of monostable monotone travelling waves of system (1.5)-(1.6) for the degenerate case $r = 1$, we introduce the following theorem.

Theorem 3.2. *Assume H2 and (K1) hold. If there exists a wavefront (c_2, Φ, Ψ) of (1.5)-(1.6), $c_2 > c_0$ such that*

$$\Phi(z) \sim A_2 e^{-\mu_2(c_2)z}, \ \textit{as } z \to \infty, \ z = \nu \cdot x - c_2 t, \ ||\nu|| = 1,$$

with $A_2 > 0$, then $c_2 = c_{\min}$.

Proof. we assume that there exists $c_2 > c_0$ such that the travelling wave solution $(\Phi(z), \Psi(z))$ satisfies

$$\Phi(z) \sim A_2 e^{-\mu_2(c_2)z}, \ \text{as } z \to \infty, \ z = \nu \cdot x - c_2 t,$$

with $A_2 > 0$. In this case, we claim that (1.5)-(1.6) have no travelling wave solutions for any $c \in [c_0, c_2)$. To prove this by contradiction, assume that there exists $\hat{c} \in (c_0, c_2)$ for which the system (1.2) has a decreasing travelling wave solution $(\bar{\Phi}, \bar{\Psi})(\nu \cdot x - \hat{c}t)$ with initial conditions

$$\phi(0, x) = \bar{\Phi}(\nu \cdot x) \quad \text{and} \quad \psi(0, x) = \bar{\Psi}(\nu \cdot x).$$

Clearly, $(\bar{\Phi}, \bar{\Psi})(z)$ satisfies system (1.5) with $c = \hat{c}$. Since $\mu_2(c)$ is non-decreasing with respect to c, we get $\mu_1(\hat{c}) \le \mu_2(\hat{c}) \le \mu_2(c_2)$. Recall that $\Phi(z) \sim A_2 e^{-\mu_2(c_2)z}$ as $z \to \infty$. It follows that $\Phi(z) \le \bar{\Phi}(z)$ as $z \to \infty$. Note that $\Phi(z) \sim 1 - A_3 e^{\gamma z}$ as $z \to -\infty$ for positive A_3 and γ, and γ is non-increasing with respect to c. Therefore, $\Phi(z) \le \bar{\Phi}(z)$ as $z \to -\infty$. Making a translation if necessary, we can always ensure that $\Phi(z) \le \bar{\Phi}(z)$ for all z. Considering the second equation of (1.2), we then have $(\Phi, \Psi)(\nu \cdot x) \le (\bar{\Phi}, \bar{\Psi})(\nu \cdot x)$ for all $x \in \mathbb{R}^n$ (by shift if necessary). By comparison, it gives that

$$\Phi(\nu \cdot x - c_2 t) \le \bar{\Phi}(\nu \cdot x - \hat{c}t). \tag{3.4}$$

Fix $\hat{z} = \nu \cdot x - c_2 t$ such that $\Phi(\hat{z}) > 0$. Observe that

$$\bar{\Phi}(\nu \cdot x - \hat{c}t) = \bar{\Phi}(\hat{z} + (c_2 - \hat{c})t)) \to 0, \quad \text{as} \quad t \to \infty.$$

Thus, we obtain

$$\Phi(\hat{z}) \le \bar{\Phi}(\nu \cdot x - \hat{c}t) \to 0, \quad \text{as} \quad t \to \infty,$$

which contradicts the fact $\Phi(\hat{z}) > 0$. Therefore, there is no travelling wave solution of system (1.5)-(1.6) for any $\hat{c} \in (c_0, c_2)$. When $r \in (0, 1)$, if $\hat{c} = c_0$ in our assumption, then the travelling waves exist if and only if $\hat{c} \ge c_0$ [12, Theorem 5]. We can still derive a contradiction by the above argument. If $r = 1$, then $c_0 = 0$. The second equation of (1.5) and the boundary condition (1.6) mean $\hat{c} > c_0$. Therefore, the proof is complete. □

The existence of monostable monotone travelling waves of system (1.5)-(1.6) is established in the theorem below.

Theorem 3.3. *Assume that H2 and (K1) hold. There is a finite $c_{\min} \ge c_0$ such that monotone and positive travelling wave solutions of (1.5)-(1.6) exist if and only if $c \ge c_{\min}$. Here $c_{\min}$ is non-increasing with respect to r. Furthermore, when $r = 1$, the travelling wave solution $(\Phi, \Psi)(z)$ of (1.5)-(1.6) with speed $c > c_{\min}$ has the following algebraic decay behaviour*

$$\Phi(z) \sim \frac{2c^2}{b} z^{-2}, \quad \Psi(z) \sim 2cz^{-1}, \quad as \;\; z \to \infty.$$

Proof. When $r < 1$, the existence of the minimal wave speed $c_{\min}$, ensuring that monotone and positive travelling wave solutions of (1.5)-(1.6) exist if and only if $c \ge c_{\min}$, is guaranteed by [12, Theorem 5]. We claim that $c_{\min}$ is non-increasing with respect to r. Indeed, let $r_a < r_b < 1$. There exists the minimal-speed travelling wave solution $(c_{\min}^{r_a}, \Phi^{r_a}(z))$ satisfying system (2.2) with $r = r_a$ (here, $c_{\min}^{r_a}$ represents the minimal wave speed when $r = r_a$). Note that $(c_{\min}^{r_a}, \Phi^{r_a}(z))$ is an upper solution of the system with $r = r_b$. A lower solution is defined as in (3.3). Then there exists a travelling wave solution of the system for $r = r_b$ when $c = c_{\min}^{r_a}$. It follows that $c_{\min}^{r_b} \le c_{\min}^{r_a}$. Therefore, $c_{\min}^{r}$ is non-increasing with respect to r.

For the existence of monotone travelling wave solutions of (1.5)-(1.6) when $r = 1$, we divide the proof into four steps.

Step 1. In this step, we shall prove that when $r = 1$, there exists a minimal speed so that (1.5)-(1.6) has a monotone travelling wave solution. Take a decreasing real sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \bar{r} = 1$ and $r_0 = 2$. By Theorem 2.6, for each r_n, there exists monotone travelling wave $(c^{r_n}, \Phi^{r_n}(z))$ of (2.2) with $r = r_n$ (here, c^{r_n} represents the unique positive bistable wave speed when $r = r_n$), and $\Psi^{r_n}(z) = \mathcal{L}_{b,c^{r_n}} \Phi^{r_n}(z)$. By shifting, we can choose a large $L_1 > 0$ to fix $\Phi^{r_n}(L_1) = \epsilon_1$ for each n, where ϵ_1 is sufficiently small.

Claim 1: The bistable wave speeds c^{r_n} have a finite upper bound.

Let $\hat{r} = 0.5$. Assume that $\Phi^{\hat{r}}(z)$ is a travelling wave of (2.2) with speed $c = c_1 = 2c_{\min}^{\hat{r}}$ and $r = \hat{r}$, where $c_{\min}^{\hat{r}}$ represents the minimal wave speed when $r = \hat{r}$. We shall prove $c^{r_n} \le c_1$ for all $n \in \mathbb{N}^+$. To the contrary, assume there exists $j \in \mathbb{N}^+$ such that $c_2 := c^{r_j} > c_1$. By asymptotic analysis, we have

$$\begin{aligned} &\Phi^{\hat{r}}(z) \sim A_1 e^{-\frac{c_1 - \sqrt{c_1^2 - 4(1-\hat{r})}}{2} z}, \quad \Phi^{r_j}(z) \sim B_j e^{-\frac{c_2 + \sqrt{c_2^2 - 4(1-r_j)}}{2} z}, \text{ as } z \to \infty, \; A_1, B_j > 0, \\ &\Phi^{\hat{r}}(z) \sim 1 - \bar{A}_1 e^{\frac{-c_1 + \sqrt{c_1^2+4}}{2} z}, \quad \Phi^{r_j}(z) \sim 1 - \bar{B}_j e^{\frac{-c_2 + \sqrt{c_2^2+4}}{2} z}, \text{ as } z \to -\infty, \;\; \bar{A}_1, \bar{B}_j > 0. \end{aligned} \tag{3.5}$$

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Thus, by translation if necessary, we obtain $\Phi^{\hat{r}}(\nu\cdot x)\geq\Phi^{r_j}(\nu\cdot x)$ for $x\in\mathbb{R}^n$. A similar argument as in Theorem 2.9 leads to a contradiction. It means $c^{r_n}\leq c_1$ for all $n\in\mathbb{N}^+$.

Since c^{r_n} is non-increasing with respect to r_n as stated in Theorem 2.9, we know c^{r_n} is non-decreasing with respect to n. Together with Claim 1, we conclude that c^{r_n} has a limit when $n\to\infty$, denoted by $c^{\bar{r}}$. Let $\bar{\alpha}$ be large enough so that

$$\bar{F}(\Phi):=\bar{\alpha}\Phi+\Phi(1-r-\Phi+r(K\star_c\mathcal{L}_{b,c}\Phi))$$

is non-decreasing in Φ. Then we can express Φ-equation in (2.2) as

$$\Phi''+c\Phi'-\bar{\alpha}\Phi=-\bar{F}(\Phi). \tag{3.6}$$

Define $\bar{\beta}_1$ and $\bar{\beta}_2$ as

$$\bar{\beta}_1=\frac{-c-\sqrt{c^2+4\bar{\alpha}}}{2}<0,\quad \bar{\beta}_2=\frac{-c+\sqrt{c^2+4\bar{\alpha}}}{2}>0. \tag{3.7}$$

Then, the integral form of (3.6) is

$$\Phi(z)=\frac{1}{\bar{\beta}_2-\bar{\beta}_1}\left\{\int_{-\infty}^{z}e^{\bar{\beta}_1(z-t)}\bar{F}(\Phi)(t)dt+\int_{z}^{\infty}e^{\bar{\beta}_2(z-t)}\bar{F}(\Phi)(t)dt\right\}. \tag{3.8}$$

Note that $(c^{r_n},\Phi^{r_n}(z))$ satisfies the integral system (3.8). Then we have

$$\left|\frac{d\Phi^{r_n}(z)}{dz}\right|\leq\frac{\beta_2}{\beta_2-\beta_1}\left|\left\{\int_{-\infty}^{z}e^{\beta_1(z-t)}\bar{F}(\Phi)(t)dt+\int_{z}^{\infty}e^{\beta_2(z-t)}\bar{F}(\Phi)(t)dt\right\}\right|\leq\beta_2<\infty,$$

which means $(\Phi^{r_n}(z))$ is equicontinuous and uniformly bounded. According to the Ascoli–Arzela theorem, there exists a subsequence $\{r_{n_j}\}$ such that $(c^{r_{n_j}},\Phi^{r_{n_j}}(z))$ converges to a limit $(c^{\bar{r}},\Phi^{\bar{r}}(z))$ uniformly on any compact interval and pointwise on $\mathbb{R}$. Take $\bar{\lambda}:=\inf_{n\in\mathbb{N}^+}\{\frac{c^{r_n}+\sqrt{(c^{r_n})^2-4(1-r_n)}}{2}\}>0$. Since ϵ_1 is sufficiently small, and by the asymptotic behaviour as in (3.5), we have $\Phi^{r_n}(z)\leq\epsilon_1e^{-\bar{\lambda}(z-L_1)}$ when $z\geq L_1$. It follows that

$$\Phi^{\bar{r}}(z)\leq\epsilon_1e^{-\bar{\lambda}(z-L_1)},\ \text{when}\ \ z\geq L_1, \tag{3.9}$$

which means $\int_0^\infty\Phi^{\bar{r}}(z)dz$ is finite. Let $j\to\infty$. By the dominated converges theorem, we know $(c^{\bar{r}},\Phi^{\bar{r}}(z))$ satisfies the integral system (3.8) and $\Psi^{\bar{r}}(z)=\mathcal{L}_{b,c^{\bar{r}}}\Phi^{\bar{r}}(z)$. Using the monotone convergence theorem, we know $(c^{\bar{r}},\Phi^{\bar{r}}(\pm\infty))$ satisfy system (3.8). Therefore, $\Phi^{\bar{r}}(-\infty)=1$ and $\Phi^{\bar{r}}(\infty)=0$. By Lemma 2.1, we have $\Psi^{\bar{r}}(-\infty)=1$ and $\Psi^{\bar{r}}(\infty)=0$. Notice that $\mu_1=0$ when $r=\bar{r}=1$. Thus, inequality (3.9) shows that $\Phi^{\bar{r}}(z)$ decays exponentially to 0 with decay rate $\mu_2(c^{\bar{r}})$, as $z\to\infty$. It follows from Theorem 3.2 that $c^{\bar{r}}$ is the minimal speed when $r=1$, that is, $c^{\bar{r}}_{\min}=c^{\bar{r}}$.

Step 2. In this step, we aim to prove the existence of monostable monotone travelling waves of system (1.5)-(1.6) with $r=1$ when c is sufficiently large. By letting

$$\tilde{z}=cz,\quad \tilde{\Phi}(\tilde{z})=\Phi(z),\quad \tilde{\Psi}(\tilde{z})=\Psi(z),$$

system (1.5)-(1.6) can be transformed into

$$\begin{cases}\frac{1}{c^2}\tilde{\Phi}''(\tilde{z})+\tilde{\Phi}'(\tilde{z})+\tilde{\Phi}(\tilde{z})\left[-\tilde{\Phi}(\tilde{z})+\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{\Psi}(\tilde{z}+s-\frac{1}{c}y)dyds\right]=0,\\ \frac{1}{c^2}\tilde{\Psi}''(\tilde{z})+\tilde{\Psi}'(\tilde{z})+b\tilde{\Phi}(\tilde{z})(1-\tilde{\Psi}(\tilde{z}))=0,\\ (\tilde{\Phi},\tilde{\Psi})(-\infty)=e_\beta,\ \ (\tilde{\Phi},\tilde{\Psi})(+\infty)=e_0,\end{cases} \tag{3.10}$$

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when $r=1$. Let $c=\frac{1}{\varepsilon}$, where ε is small enough. We first consider

$$\begin{cases} \tilde{\Phi}'(\tilde{z})+\tilde{\Phi}(\tilde{z})\left[-\tilde{\Phi}(\tilde{z})+\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{\Psi}(\tilde{z}+s)dyds\right]=0,\\ \tilde{\Psi}'(\tilde{z})+b\tilde{\Phi}(\tilde{z})(1-\tilde{\Psi}(\tilde{z}))=0,\\ (\tilde{\Phi},\tilde{\Psi})(-\infty)=e_\beta,\quad (\tilde{\Phi},\tilde{\Psi})(+\infty)=e_0. \end{cases} \tag{3.11}$$

From the second equation in (3.11) and $(\tilde{\Phi},\tilde{\Psi})(+\infty)=e_0$, we have

$$\tilde{\Psi}(\tilde{z})=1-e^{-\int_{\tilde{z}}^\infty b\tilde{\Phi}(\tau)d\tau}.$$

Thus, system (3.11) becomes

$$\tilde{\Phi}'(\tilde{z})+\tilde{\Phi}(\tilde{z})\left[-\tilde{\Phi}(\tilde{z})+\int_0^\infty\int_{\mathbb{R}}K(s,y)(1-e^{-\int_{\tilde{z}+s}^\infty b\tilde{\Phi}(\tau)d\tau})dyds\right]=0 \tag{3.12}$$

with the boundary condition

$$\tilde{\Phi}(-\infty)=1,\quad \tilde{\Phi}(+\infty)=0. \tag{3.13}$$

The integral form of (3.12) is

$$\tilde{T}_1(\tilde{\Phi})=\int_{\tilde{z}}^\infty e^{4(\tilde{z}-\tau_1)}\tilde{\Phi}(\tau_1)\left[4-\tilde{\Phi}(\tau_1)+\int_0^\infty\int_{\mathbb{R}}K(s,y)(1-e^{-\int_{\tau_1+s}^\infty b\tilde{\Phi}(\tau)d\tau})dyds\right]d\tau_1.$$

By calculation, we can establish $\int_0^\infty \tilde{T}_1(\tilde{\Phi})(\tilde{z})d\tilde{z}$ is finite when $\tilde{\Phi}(\tilde{z})\in L^1[0,\infty)$. Note that $\tilde{T}_1$ has only two fixed points 0 and 1. Then $\tilde{T}_1:\mathcal{M}_\beta\cap L^1[0,\infty)\to\mathcal{M}_\beta\cap L^1[0,\infty)$, where $\mathcal{M}_\beta$ is defined in Section 2 with $\beta=1$, is a strictly monotone continuous operator. By the Dancer–Hess Lemma (see [7, Proposition 1], [45, p. 45]), there exists an entire orbit of $\tilde{T}_1$ connecting 1 to 0. Therefore, system (3.12)-(3.13) has a monotone solution, denoted by $\tilde{\Phi}_1(\tilde{z})$. From (3.12), $\tilde{\Phi}_1(\tilde{z})$ does not decay exponentially to 0 as $z\to\infty$. Consider $z\to\infty$. We let

$$\tilde{\Phi}_1(\tilde{z})=a_1\tilde{z}^{-\gamma}+o(\tilde{z}^{-\gamma}),\quad a_1,\gamma>0. \tag{3.14}$$

Substituting (3.14) into (3.12) yields

$$\begin{aligned} &-a_1\gamma\tilde{z}^{-(\gamma+1)}+a_1\tilde{z}^{-\gamma}\left[-a_1\tilde{z}^{-\gamma}+\int_0^\infty\int_{\mathbb{R}}k(s,y)(1-e^{-\int_{\tilde{z}+s}^\infty ba_1\tau^{-\gamma}d\tau})\right]+o(\tilde{z}^{-(\gamma+1)})\\ &=-a_1\gamma\tilde{z}^{-(\gamma+1)}-a_1^2\tilde{z}^{-2\gamma}+a_1\tilde{z}^{-\gamma}\int_0^\infty\int_{\mathbb{R}}k(s,y)\frac{ba_1}{\gamma-1}(\tilde{z}+s)^{-\gamma+1}dyds+o(\tilde{z}^{-2\gamma+1})+o(\tilde{z}^{-(\gamma+1)})\\ &=-a_1\gamma\tilde{z}^{-(\gamma+1)}-a_1^2\tilde{z}^{-2\gamma}+\frac{ba_1^2}{\gamma-1}\tilde{z}^{-2\gamma+1}\int_0^\infty\int_{\mathbb{R}}k(s,y)(\frac{\tilde{z}+s}{\tilde{z}})^{-\gamma+1}dyds+o(\tilde{z}^{-2\gamma+1})+o(\tilde{z}^{-(\gamma+1)})\\ &=-a_1\gamma\tilde{z}^{-(\gamma+1)}-a_1^2\tilde{z}^{-2\gamma}+\frac{ba_1^2}{\gamma-1}\tilde{z}^{-2\gamma+1}+o(\tilde{z}^{-2\gamma+1})+o(\tilde{z}^{-(\gamma+1)})=0. \end{aligned} \tag{3.15}$$

According to the leading terms in (3.15), we derive that $\gamma=2$ and $a_1=\frac{2}{b}$. Therefore, system (3.11) has a monotone solution $(\tilde{\Phi}_1,\tilde{\Psi}_1)(\tilde{z})$, where $\tilde{\Psi}_1(\tilde{z})=1-e^{-\int_{\tilde{z}}^\infty b\tilde{\Phi}_1(\tau)d\tau}$, with decay behaviour

$$\tilde{\Phi}_1(\tilde{z})=\frac{2}{b}\tilde{z}^{-2},\quad \tilde{\Psi}_1(\tilde{z})=2\tilde{z}^{-1},\quad \text{as } \tilde{z}\to\infty. \tag{3.16}$$

From (3.11), we know $\tilde{\Phi}_1''$ and $\tilde{\Psi}_1''$ are smooth. Next, we want to apply a perturbation argument to prove that solutions of (3.10) exist for large speed c by using $(\tilde{\Phi}_1,\tilde{\Psi}_1)$. Let $\tilde{\alpha}_1$ be large enough so that

$$F_3(\tilde{\Phi},\tilde{\Psi}):=\tilde{\alpha}_1\tilde{\Phi}+\tilde{\Phi}(\tilde{z})\left[-\tilde{\Phi}(\tilde{z})+\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{\Psi}(\tilde{z}+s-\frac{1}{c}y)dyds\right]$$

and

$$F_4(\tilde{\Phi},\tilde{\Psi}):=\tilde{\alpha}_1\tilde{\Psi}+b\tilde{\Phi}(\tilde{z})(1-\tilde{\Psi}(\tilde{z}))$$

are non-decreasing functions of $\tilde{\Phi}$ and $\tilde{\Psi}$. Then we can write system (3.10) as

$$\begin{aligned} &\frac{1}{c^2}\tilde{\Phi}''(\tilde{z})+\tilde{\Phi}'(\tilde{z})-\tilde{\alpha}_1\tilde{\Phi}=-F_3(\tilde{\Phi},\tilde{\Psi}),\\ &\frac{1}{c^2}\tilde{\Psi}''(\tilde{z})+\tilde{\Psi}'(\tilde{z})-\tilde{\alpha}_1\tilde{\Psi}=-F_4(\tilde{\Phi},\tilde{\Psi}). \end{aligned} \tag{3.17}$$

Define $\tilde{\beta}_1$ and $\tilde{\beta}_2$ as

$$\tilde{\beta}_1(c)=\frac{-c^2-c\sqrt{c^2+4\tilde{\alpha}_1}}{2}<0,\quad \tilde{\beta}_2(c)=\frac{-c^2+c\sqrt{c^2+4\tilde{\alpha}_1}}{2}>0. \tag{3.18}$$

The integral form of (3.17) is

$$\begin{aligned} \tilde{\Phi}(\tilde{z})&=\frac{c^2}{\tilde{\beta}_2-\tilde{\beta}_1}\left\{\int_{-\infty}^{\tilde{z}}e^{\tilde{\beta}_1(\tilde{z}-t)}F_3(\tilde{\Phi},\tilde{\Psi})(t)dt+\int_{\tilde{z}}^{\infty}e^{\tilde{\beta}_2(\tilde{z}-t)}F_3(\tilde{\Phi},\tilde{\Psi})(t)dt\right\}=:P_{1,c}(F_3)(\tilde{z}),\\ \tilde{\Psi}(\tilde{z})&=\frac{c^2}{\tilde{\beta}_2-\tilde{\beta}_1}\left\{\int_{-\infty}^{\tilde{z}}e^{\tilde{\beta}_1(\tilde{z}-t)}F_4(\tilde{\Phi},\tilde{\Psi})(t)dt+\int_{\tilde{z}}^{\infty}e^{\tilde{\beta}_2(\tilde{z}-t)}F_4(\tilde{\Phi},\tilde{\Psi})(t)dt\right\}=:P_{2,c}(F_4)(\tilde{z}). \end{aligned} \tag{3.19}$$

When $c\to\infty$, we have

$$\begin{aligned} P_{1,\infty}(F_3)(\tilde{z})&=\int_{\tilde{z}}^{\infty}e^{\tilde{\alpha}_1(\tilde{z}-t)}F_3(\tilde{\Phi},\tilde{\Psi})(t)dt,\\ P_{2,\infty}(F_4)(\tilde{z})&=\int_{\tilde{z}}^{\infty}e^{\tilde{\alpha}_1(\tilde{z}-t)}F_4(\tilde{\Phi},\tilde{\Psi})(t)dt. \end{aligned}$$

Assume that $(\tilde{\Phi}^*(\tilde{z}),\tilde{\Psi}^*(\tilde{z})):=(\tilde{\Phi}_1(\tilde{z})+\tilde{W}_1(\tilde{z}),\tilde{\Psi}_1(\tilde{z})+\tilde{W}_2(\tilde{z}))$, where $\tilde{W}_1(\tilde{z})$ and $\tilde{W}_2(\tilde{z})$ are functions in B_0 that need to be determined. Here, B_0 is defined as

$$B_0=\{u\in C(-\infty,\infty):u(\pm\infty)=0\}.$$

Therefore, we need to prove the existence of $\tilde{W}_1(\tilde{z})$ and $\tilde{W}_2(\tilde{z})$ in B_0 such that $(\tilde{\Phi}^*(\tilde{z}),\tilde{\Psi}^*(\tilde{z}))$ satisfies (3.10) with $c=\frac{1}{\varepsilon}$. Plugging $(\tilde{\Phi}^*(\tilde{z}),\tilde{\Psi}^*(\tilde{z}))$ into (3.10) and utilising (3.11) lead to

$$\begin{aligned} &\varepsilon^2\tilde{W}_1{}''+\tilde{W}_1{}'-\tilde{\alpha}_1\tilde{W}_1=-(\tilde{F}_0+\tilde{F}_{1,\varepsilon}+\tilde{F}_{2,\varepsilon}+\tilde{F}_h),\\ &\varepsilon^2\tilde{W}_2{}''+\tilde{W}_2{}'-\tilde{\alpha}_1\tilde{W}_2=-(\tilde{G}_0+\tilde{G}_\varepsilon+\tilde{G}_h), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \tilde{F}_0&=\left[\tilde{\alpha}_1-2\tilde{\Phi}_1+\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{\Psi}_1(\tilde{z}+s)dyds\right]\tilde{W}_1+\tilde{\Phi}_1\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{W}_2(\tilde{z}+s)dyds,\\ \tilde{F}_{1,\varepsilon}&=\varepsilon^2\tilde{\Phi}_1''+\tilde{\Phi}_1\int_0^\infty\int_{\mathbb{R}}K(s,y)\big[\tilde{\Psi}_1(\tilde{z}+s-\varepsilon y)-\tilde{\Psi}_1(\tilde{z}+s)\big]dyds\\ \tilde{F}_{2,\varepsilon}&=\tilde{\Phi}_1\int_0^\infty\int_{\mathbb{R}}K(s,y)\big[\tilde{W}_2(\tilde{z}+s-\varepsilon y)-\tilde{W}_2(\tilde{z}+s)\big]dyds\\ &\quad+\tilde{W}_1\int_0^\infty\int_{\mathbb{R}}K(s,y)\big[\tilde{\Psi}_1(\tilde{z}+s-\varepsilon y)-\tilde{\Psi}_1(\tilde{z}+s)\big]dyds\\ \tilde{F}_h&=\tilde{W}_1\int_0^\infty\int_{\mathbb{R}}K(s,y)\tilde{W}_2(\tilde{z}+s-\varepsilon y)dyds-\tilde{W}_1{}^2,\\ \tilde{G}_0&=(\tilde{\alpha}_1-b\tilde{\Phi}_1)\tilde{W}_2+b(1-\tilde{\Psi}_1)\tilde{W}_1,\\ \tilde{G}_\varepsilon&=\varepsilon^2\tilde{\Psi}_1'',\\ \tilde{G}_h&=-b\tilde{W}_1\tilde{W}_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \tilde{W}_1&=P_{1,\infty}(\tilde{F}_0)+P_{1,\frac{1}{\varepsilon}}(\tilde{F}_0)-P_{1,\infty}(\tilde{F}_0)+P_{1,\frac{1}{\varepsilon}}(\tilde{F}_{1,\varepsilon})+P_{1,\frac{1}{\varepsilon}}(\tilde{F}_{2,\varepsilon})+P_{1,\frac{1}{\varepsilon}}(\tilde{F}_h),\\ \tilde{W}_2&=P_{2,\infty}(G_0)+P_{2,\frac{1}{\varepsilon}}(G_0)-P_{2,\infty}(G_0)+P_{2,\frac{1}{\varepsilon}}(\tilde{G}_\varepsilon)+P_{2,\frac{1}{\varepsilon}}(\tilde{G}_h). \end{aligned} \tag{3.21}$$

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Define

$$\tilde{P}(\tilde{W}) = \begin{pmatrix} P_{1,\infty}(\tilde{F}_0(\tilde{W})) \\ P_{2,\infty}(\tilde{G}_0(\tilde{W})) \end{pmatrix}, \quad \tilde{W} = (\tilde{W}_1, \tilde{W}_2).$$

By a simple estimate, we have

$$P_{1,\frac{1}{\varepsilon}}(\tilde{F}_0) - P_{1,\infty}(\tilde{F}_0) = O(\varepsilon^2 \tilde{W}), P_{2,\frac{1}{\varepsilon}}(G_0) - P_{2,\infty}(G_0) = O(\varepsilon^2 \tilde{W}),$$
$$P_{1,\frac{1}{\varepsilon}}(\tilde{F}_{1,\varepsilon}) = O(\varepsilon), P_{1,\frac{1}{\varepsilon}}(\tilde{F}_{2,\varepsilon}) = O(\varepsilon \tilde{W}), P_{2,\frac{1}{\varepsilon}}(\tilde{G}_\varepsilon) = O(\varepsilon^2),$$

and $P_{1,\frac{1}{\varepsilon}}(\tilde{F}_{1,\varepsilon}) = o(\tilde{z}^{-3})$ as $\tilde{z} \to \infty$. The linear operator $\tilde{P}$ is both compact and strongly positive. It possesses a simple principal eigenvalue $\lambda = 1$ with the corresponding positive eigenfunction $(-\tilde{\Phi}_1', -\tilde{\Psi}_1')$. To eliminate this eigenfunction from B_0, we introduce the weighted functional space as follows

$$\mathcal{H} = \{h(\tilde{z}) \in B_0 : h\tilde{z}^3 = o(1) \quad \text{as} \quad \tilde{z} \to \infty\}.$$

Since the eigenfunction $(-\tilde{\Phi}_1', -\tilde{\Psi}_1')$ does not belong to the space $\mathcal{H} \times B_0$, it implies that $\tilde{P}$ does not possess the eigenvalue $\lambda = 1$ in the functional space $\mathcal{H} \times B_0$. Consequently, $I - \tilde{P}$ has a bounded inverse in $\mathcal{H} \times B_0$, where I is the identity operator. By applying the abstract implicit function theorem in $\mathcal{H} \times B_0$, we conclude that there exists ε_0 such that a solution $(\tilde{W}_1, \tilde{W}_2)$ of the system (3.20) exists for any $\varepsilon \in [0, \varepsilon_0)$. Consequently, there exists a solution $(\tilde{\Phi}^*(\tilde{z}), \tilde{\Psi}^*(\tilde{z}))$ of (3.10) for all $c \geq \tilde{M} > \frac{1}{\varepsilon_0}$, with decay behaviour (3.16).

Step 3. By continuation and the abstract implicit function theorem, we further prove the existence of monostable monotone travelling waves of system (3.10) for $c = \tilde{M} - \delta$, with δ is a sufficiently small and to-be-determined value. Let $(\tilde{\Phi}_2(\tilde{z}), \tilde{\Psi}_2(\tilde{z}))$ be the solution of (3.10) for $c = c_2 = \tilde{M}$. Thus, we have

$$\begin{cases} \dfrac{1}{c_2^2}\tilde{\Phi}_2''(\tilde{z}) + \tilde{\Phi}_2'(\tilde{z}) + \tilde{\Phi}_2(\tilde{z})\left[-\tilde{\Phi}_2(\tilde{z}) + \displaystyle\int_0^\infty \int_{\mathbb{R}} K(s,y)\tilde{\Psi}_2(\tilde{z} + s - \frac{1}{c_2}y)dyds\right] = 0, \\ \dfrac{1}{c_2^2}\tilde{\Psi}_2''(\tilde{z}) + \tilde{\Psi}_2'(\tilde{z}) + b\tilde{\Phi}_2(\tilde{z})(1 - \tilde{\Psi}_2(\tilde{z})) = 0, \\ (\tilde{\Phi}_2, \tilde{\Psi}_2)(-\infty) = e_\beta, \quad (\tilde{\Phi}_2, \tilde{\Psi}_2)(+\infty) = e_0, \end{cases} \tag{3.22}$$

Assuming $c = c_\delta = c_2 - \delta$. To find the solution of (3.17), say $(\bar{\Phi}_\delta(\tilde{z}), \bar{\Psi}_\delta(\tilde{z}))$ at $c = c_\delta$, we let $\bar{\Phi}_\delta(\tilde{z}) = \tilde{\Phi}_2(\tilde{z}) + \bar{W}_1(\tilde{z})$ and $\bar{\Psi}_\delta(\tilde{z}) = \tilde{\Psi}_2(\tilde{z}) + \bar{W}_2(\tilde{z})$, where $\bar{W}_1(\tilde{z})$ and $\bar{W}_2(\tilde{z})$ are functions in B_0 that need to be determined. Thus, we aim to prove the existence of $\bar{W}_1(\tilde{z})$ and $\bar{W}_2(\tilde{z})$ in B_0 such that $(\bar{\Phi}_\delta(\tilde{z}), \bar{\Psi}_\delta(\tilde{z}))$ satisfies (3.10) or (3.17) with speed c_δ. Substituting $(\bar{\Phi}_\delta(\tilde{z}), \bar{\Psi}_\delta(\tilde{z}))$ into (3.10) and using (3.22) give

$$\begin{aligned} \bar{W}_1 &= P_{1,c_2}(\bar{F}_0) + P_{1,c_\delta}(\bar{F}_0) - P_{1,c_2}(\bar{F}_0) + P_{1,c_\delta}(\bar{F}_{1,\delta}) + P_{1,c_\delta}(\bar{F}_{2,\delta}) + P_{1,c_\delta}(\bar{F}_h), \\ \bar{W}_2 &= P_{2,c_2}(\bar{G}_0) + P_{2,c_\delta}(\bar{G}_0) - P_{2,c_2}(\bar{G}_0) + P_{2,c_\delta}(\bar{G}_\delta) + P_{2,c_\delta}(\bar{G}_h), \end{aligned} \tag{3.23}$$

where

$$\bar{F}_0 = \left[\tilde{\alpha}_1 - 2\tilde{\Phi}_2 + \int_0^\infty \int_{\mathbb{R}} K(s,y)\tilde{\Psi}_2(\tilde{z} + s - \frac{1}{c_2}y)dyds\right]\bar{W}_1 + \tilde{\Phi}_2 \int_0^\infty \int_{\mathbb{R}} K(s,y)\bar{W}_2(\tilde{z} + s - \frac{1}{c_2}y)dyds,$$
$$\bar{F}_{1,\delta} = [\frac{1}{(c_2 - \delta)^2} - \frac{1}{c_2^2}]\tilde{\Phi}_2'' + \tilde{\Phi}_2 \int_0^\infty \int_{\mathbb{R}} K(s,y)[\tilde{\Psi}_2(\tilde{z} + s - \frac{1}{c_2 - \delta}y) - \tilde{\Psi}_2(\tilde{z} + s - \frac{1}{c_2}y)]dyds$$

$$\bar{F}_{2,\delta} = \tilde{\Phi}_2 \int_0^\infty \int_{\mathbb{R}} K(s,y)\big[\bar{W}_2(\tilde{z}+s-\frac{1}{c_2-\delta}y) - \bar{W}_2(\tilde{z}+s-\frac{1}{c_2}y)\big]dyds$$
$$+ \bar{W}_1 \int_0^\infty \int_{\mathbb{R}} K(s,y)\big[\tilde{\Psi}_2(\tilde{z}+s-\frac{1}{c_2-\delta}y) - \tilde{\Psi}_2(\tilde{z}+s-\frac{1}{c_2}y)\big]dyds$$
$$\bar{F}_h = \bar{W}_1 \int_0^\infty \int_{\mathbb{R}} K(s,y)\bar{W}_2(\tilde{z}+s-\frac{1}{c_2-\delta}y)dyds - \bar{W}_1^2,$$
$$\bar{G}_0 = (\tilde{\alpha}_1 - b\tilde{\Phi}_2)\bar{W}_2 + b(1-\tilde{\Psi}_2)\bar{W}_1,$$
$$\bar{G}_\delta = [\frac{1}{(c_2-\delta)^2} - \frac{1}{c_2^2}]\tilde{\Psi}_2'',$$
$$\bar{G}_h = -b\bar{W}_1\bar{W}_2.$$

Define

$$P(\bar{W}) = \begin{pmatrix} P_{1,c_2}(\bar{F}_0(\bar{W})) \\ P_{2,c_2}(\bar{G}_0(\bar{W})) \end{pmatrix}, \ \bar{W} = (\bar{W}_1, \bar{W}_2).$$

Clearly, we have

$$P_{1,c_\delta}(\bar{F}_0) - P_{1,c_2}(\bar{F}_0) = O(\delta^2\bar{W}), P_{2,c_\delta}(\bar{G}_0) - P_{2,c_2}(\bar{G}_0) = O(\delta^2\bar{W}),$$
$$P_{1,c_\delta}(\bar{F}_{1,\delta}) = O(\delta), P_{1,c_\delta}(\bar{F}_{2,\delta}) = O(\delta\bar{W}), P_{2,c_\delta}(\bar{G}_\delta) = O(\delta^2),$$

and $P_{1,c_\delta}(\bar{F}_{1,\delta}) = o(\tilde{z}^{-3})$ as $\tilde{z} \to \infty$. The linear operator P is both compact and strongly positive. It possesses a simple principal eigenvalue $\lambda = 1$ with the corresponding positive eigenfunction $(-\tilde{\Phi}_2', -\tilde{\Psi}_2')$. By a similar argument as in Step 2, the eigenfunction $(-\tilde{\Phi}_2', -\tilde{\Psi}_2')$ does not belong to the space $\mathcal{H} \times B_0$. It follows from the abstract implicit function theorem in $\mathcal{H} \times B_0$ that there exists δ_0 such that a solution $(\bar{W}_1, \bar{W}_2)$ of the system (3.23) exists for any $\delta \in [0, \delta_0)$. Consequently, there exists a solution $(\bar{\Phi}_\delta(\tilde{z}), \bar{\Psi}_\delta(\tilde{z}))$ of (3.10) with $c = c_\delta$.

Step 4. Note that $\mu_1 = 0$ and $\mu_2 = c$ from (3.2) for $r = 1$. When $c > c_{\min}^{\bar{r}}$, Theorem 3.2 implies that if travelling wave solutions of (1.5)-(1.6) with $r = 1$ exist, they do not decay exponentially to 0 as $z \to \infty$. This indicates that travelling wave solutions of (3.10), if they exist, exhibit algebraic decay behaviour (3.16) for all $c > c_{\min}^{\bar{r}}$. By repeating the continuation process from Step 3, we establish the existence of monostable monotone travelling waves of system (3.10) for all $c \in (c_{\min}^{\bar{r}}, \tilde{M})$. Therefore, we prove that when $r = 1$, there is a finite $c_{\min}^{\bar{r}} > 0$ such that monotone and positive travelling wave solutions of (1.5)-(1.6) exist if and only if $c \geq c_{\min}^{\bar{r}}$. When $c > c_{\min}^{\bar{r}}$, the travelling wave solution $(\Phi, \Psi)(z)$ of (1.5)-(1.6) has the following algebraic decay behaviour

$$\Phi(z) \sim \frac{2c^2}{b} z^{-2}, \quad \Psi(z) \sim 2cz^{-1}, \quad \text{as } z \to \infty.$$

□

3.2. Speed selection

In this subsection, we shall derive some conditions for speed selection. The subsequent theorem provides a necessary and sufficient condition for nonlinear selection, which is a development of the abstract results of speed selection in [23] to the system (1.1).

Theorem 3.4. *Assume that H2 and (K1) hold. The minimal wave speed $c_{\min}$ of (1.5)-(1.6) is nonlinearly selected if and only if there exists a wavefront (c_2, Φ, Ψ), $c_2 > c_0$ such that*

$$\Phi(z) \sim A_2 e^{-\mu_2(c_2)z}, \ \text{as } z \to \infty, \ z = \nu \cdot x - c_2 t, \ ||\nu|| = 1,$$

with $A_2 > 0$. Furthermore, $c_2 = c_{\min}$.

Proof. To establish the necessity, we assume that the minimal wave speed is nonlinearly selected, denoted as $c_{\min} > c_0$. Our goal is to demonstrate that the travelling wave $(\Phi_2(z), \Psi_2(z))$ of system

(1.5)-(1.6), with a speed $c = c_2 = c_{\min}$, exhibits the following behaviour

$$\Phi_2(z) \sim A_2 e^{-\mu_2(c_2)z}, \text{ as } z \to \infty, \; z = \nu \cdot x - c_2 t, \; ||\nu|| = 1,$$

with $A_2 > 0$. To the contrary, we assume that

$$\Phi_2(z) \sim A_2 e^{-\mu_1(c_2)z}, \text{ as } z \to \infty, \quad \text{for } r \in (0, 1) \tag{3.24}$$

and

$$\Phi_2(z) \sim \frac{2c_2^2}{b} z^{-2}, \text{ as } z \to \infty, \quad \text{for } r = 1. \tag{3.25}$$

By a similar argument in Step 3 of Theorem 3.3, we can prove that there is a monotone travelling wave solution of (1.5)-(1.6) when $c = c_\delta = c_2 - \delta$, where δ is a sufficiently small and to-be-determined value. It implies that c_2 is not the minimum wave speed. This contradiction establishes the necessity. Clearly, for $r = 1$, Step 3 of Theorem 3.3 shows the existence of monotone travelling wave solution of (1.5)-(1.6) when $c = c_2 - \delta$ with a small δ. Thus, we only focus on $r \in (0, 1)$. Let α be large enough so that

$$F(\Phi, \Psi) := \alpha\Phi + \Phi(1 - r - \Phi + r(K \star_c \Psi))$$

and

$$G(\Phi, \Psi) := \alpha\Psi + b\Phi(1 - \Psi)$$

are non-decreasing functions of Φ and Ψ. Then we can express system (1.5) as

$$\begin{aligned} \Phi'' + c\Phi' - \alpha\Phi &= -F(\Phi, \Psi), \\ \Psi'' + c\Psi' - \alpha\Psi &= -G(\Phi, \Psi). \end{aligned} \tag{3.26}$$

Define β_1 and β_2 as

$$\beta_1 = \frac{-c - \sqrt{c^2 + 4\alpha}}{2} < 0, \quad \beta_2 = \frac{-c + \sqrt{c^2 + 4\alpha}}{2} > 0. \tag{3.27}$$

In view of the variation of parameters, the integral form of (3.26) is given by

$$\begin{aligned} \Phi(z) &= \frac{1}{\beta_2 - \beta_1}\left\{ \int_{-\infty}^{z} e^{\beta_1(z-t)} F(\Phi, \Psi)(t)dt + \int_{z}^{\infty} e^{\beta_2(z-t)} F(\Phi, \Psi)(t)dt \right\} =: T_{1,c}(F)(z), \\ \Psi(z) &= \frac{1}{\beta_2 - \beta_1}\left\{ \int_{-\infty}^{z} e^{\beta_1(z-t)} G(\Phi, \Psi)(t)dt + \int_{z}^{\infty} e^{\beta_2(z-t)} G(\Phi, \Psi)(t)dt \right\} =: T_{2,c}(G)(z). \end{aligned} \tag{3.28}$$

Note that $(\Phi_2(z), \Psi_2(z))$ satisfies

$$\begin{cases} \Phi_2'' + c_2\Phi_2' + \Phi_2(1 - r - \Phi_2 + r(K \star_{c_2} \Psi_2)) = 0, \\ \Psi_2'' + c_2\Psi_2' + b\Phi_2(1 - \Psi_2) = 0, \\ (\Phi_2, \Psi_2)(-\infty) = e_\beta, \quad (\Phi_2, \Psi_2)(\infty) = e_0. \end{cases} \tag{3.29}$$

Define $\bar{\Phi}(z) = \Phi_2(z)\omega(z)$, where

$$\omega = \frac{1}{1 + \delta \exp\{(\mu_1(c_\delta) - \mu_1(c_2))z\}},$$

Using asymptotic analysis, we can deduce that

$$\bar{\Phi}(z) \sim \frac{A_2}{\delta} e^{-\mu_1(c_\delta)z} \text{ as } z \to \infty.$$

To find the solution of (3.26), say $(\Phi_\delta(z), \Psi_\delta(z))$, at $c = c_\delta$, we introduce $\Phi_\delta(z)$ as $\Phi_\delta(z) = \bar{\Phi}(z) + W_1(z)$ and $\Psi_\delta(z)$ as $\Psi_\delta(z) = \Psi_2(z) + W_2(z)$, where $W_1(z)$ and $W_2(z)$ are functions in B_0 that need to be determined. Here, B_0 is defined inTheorem 3.3. By substituting $(\Phi_\delta(z), \Psi_\delta(z))$ into (1.5) and utilising (3.29), we observe that W_1 and W_2 fulfill the following equations

$$\begin{aligned} W_1 &= T_{1,c_2}(F_0) + T_{1,c_\delta}(F_0) - T_{1,c_2}(F_0) + T_{1,c_\delta}(F_{1,\delta}) + T_{1,c_\delta}(F_{2,\delta}) + T_{1,c_\delta}(F_h), \\ W_2 &= T_{2,c_2}(G_0) + T_{2,c_\delta}(G_0) - T_{2,c_2}(G_0) + T_{2,c_\delta}(G_{1,\delta}) + T_{2,c_\delta}(G_{2,\delta}) + T_{2,c_\delta}(G_h) \end{aligned} \tag{3.30}$$

where

$$F_0 = \{\alpha + 1 - r - 2\Phi_2 + r(K \star_{c_2} \Psi_2)\} W_1 + r\Phi_2(K \star_{c_2} W_2),$$
$$F_{1,\delta} = (\omega\Phi_2)'' + (c_2 - \delta)(\omega\Phi_2)' - \omega\Phi_2'' - c_2\omega\Phi_2' + (1-\omega)\omega\Phi_2^2 + r\Phi_2\omega[K \star_{c_\delta} \Psi_2 - K \star_{c_2} \Psi_2],$$
$$F_{2,\delta} = (1-\omega)\Phi_2[2W_1 - r(K \star_{c_2} W_2)] + r\Phi_2\omega[K \star_{c_\delta} W_2 - K \star_{c_2} W_2] + rW_1[K \star_{c_\delta} \Psi_2 - K \star_{c_2} \Psi_2],$$
$$F_h = rW_1(K \star_{c_\delta} W_2) - W_1^2,$$
$$G_0 = (\alpha - b\Phi_2)W_2 + b(1 - \Psi_2)W_1,$$
$$G_{1,\delta} = -\delta\Psi_2' - (1-\omega)b\Phi_2(1-\Psi_2),$$
$$G_{2,\delta} = (1-\omega)b\Phi_2 W_2,$$
$$G_h = -bW_1W_2.$$

Define

$$T(W) = \begin{pmatrix} T_{1,c_2}(F_0(W)) \\ T_{2,c_2}(G_0(W)) \end{pmatrix}, \ W = (W_1, W_2).$$

It is worth noting that $\omega' = -(\mu_1(c_\delta) - \mu_1(c_2))\omega(1-\omega)$ and $\omega'' = (\mu_1(c_\delta) - \mu_1(c_2))^2\omega(1-\omega)(1-2\omega)$. Clearly, we have

$$T_{1,c_\delta}(F_0) - T_{1,c_2}(F_0) = O(\delta W), T_{2,c_\delta}(G_0) - T_{2,c_2}(G_0) = O(\delta W),$$
$$T_{1,c_\delta}(F_{1,\delta}) = O(\delta), T_{1,c_\delta}(F_{2,\delta}) = O(\delta W), T_{2,c_\delta}(G_{1,\delta}) = O(\delta), T_{2,c_\delta}(G_{2,\delta}) = O(\delta W),$$

and $T_{1,c_\delta}(F_{1,\delta}) = o(e^{-\mu_1(c_\delta)z})$ as $z \to \infty$. The linear operator T is both compact and strongly positive. It possesses a simple principal eigenvalue $\lambda = 1$ with the corresponding positive eigenfunction $(-\Phi_2', -\Psi_2')$. Here, we introduce another weighted functional space

$$\mathcal{H}_1 = \{h(\xi) \in B_0 : he^{\mu_1(c_\delta)\xi} = o(1) \quad \text{as} \quad \xi \to \infty\}$$

to remove eigenfunction $(-\Phi_2', -\Psi_2')$ from B_0. Similar to Step 3 of Theorem 3.3, by applying the abstract implicit function theorem in $\mathcal{H}_1 \times B_0$, we conclude that there exists δ_1 such that a solution (W_1, W_2) of the system (3.30) exists for any $\delta \in [0, \delta_1)$ when $r \in (0, 1)$. Consequently, there exists a solution $(\Phi_\delta(z), \Psi_\delta(z))$ of (1.5)-(1.6) with $c = c_\delta$ when $r \in (0, 1)$.

The sufficiency result follows from Theorem 3.2. □

The result above relies on an assumption about the wave solution, which is generally unknown. To overcome this difficulty, we will provide a theorem that offers a convenient approach for the nonlinear or linear selection of the minimal wave speed in system (1.5)-(1.6).

Theorem 3.5. *(Linear, nonlinear selection and estimate of $c_{\min}$) Assume that H2 and (K1) hold.*

(i) For $c_1 > c_0$, if there exists a nonnegative and monotonic lower solution $(\underline{\Phi}, \underline{\Psi})(z)$ of system (1.5)-(1.6) such that $\underline{\Phi}(z) \sim \underline{A}e^{-\mu_2 z}$ as $z \to \infty$ with $\underline{A} > 0$, and $\limsup_{z\to-\infty} \underline{\Phi}(z) < 1$, then no travelling wave solution exists for $c \in [c_0, c_1)$ and there exists $c_{\min} > c_1$.

(ii) For $r \in (0, 1)$ and $c = c_0 + \varepsilon$ where ε is any small positive number, if there exists a nonnegative and monotonic upper solution $(\overline{\Phi}, \overline{\Psi})(z)$ of system (1.5)-(1.6) such that $\overline{\Phi}(z) \sim \overline{A}e^{-\mu_1(c)z}$ as $z \to \infty$ with $\overline{A} > 0$, and $\limsup_{z\to-\infty} \overline{\Phi}(z) \geq 1$, then we have $c_{\min} = c_0$.

Proof. (i) On the contrary, we assume that for $c \in (c_0, c_1)$, there exists a monotone travelling wave solution $(\Phi, \Psi)(\nu \cdot x - ct)$ of the system (1.2) with the initial conditions

$$\phi(0, x) = \Phi(\nu \cdot x), \quad \psi(0, x) = \Psi(\nu \cdot x).$$

Note that $\underline{\Phi}(\nu \cdot x) \leq \Phi(\nu \cdot x)$ for $x \in \mathbb{R}^n$ by shifting if necessary. According to Ψ-equation of (1.5), we have $(\underline{\Phi}, \underline{\Psi})(\nu \cdot x) \leq (\Phi, \Psi)(\nu \cdot x)$ for $x \in \mathbb{R}^n$. By the comparison theorem, we obtain

$$\underline{\Phi}(\nu \cdot x - c_1 t) \leq \Phi(\nu \cdot x - ct),$$
$$\underline{\Psi}(\nu \cdot x - c_1 t) \leq \Psi(\nu \cdot x - ct).$$

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Fix $z^* = \nu \cdot x - c_1 t$ such that $\underline{\Phi}(z^*) > 0$. Since

$$\Phi(\nu \cdot x - ct) = \Phi(z^* + (c_1 - c)t) \to 0, \text{ as } t \to \infty,$$

it follows that $\underline{\Phi}(z^*) \leq 0$, which leads to a contradiction.

For part (ii), Lemma 3.1 provides a lower solution. Our result follows by the standard upper-lower solution technique. Hence, the proof is complete. □

The following theorem shows that the minimal speed $c_{\min}$ has a monotonic relationship with the parameter b. It also presents a necessary and sufficient criterion to distinguish between linear and nonlinear selection based on the value of parameter b. Specifically, we can show that there exists a critical value b^* that determines the transition between linear and nonlinear selection.

Theorem 3.6. *Assume that H2 and (K1) hold. For fixed r, the minimal speed $c_{\min}$ exhibits the following properties with respect to the parameter b:*

(1) $c_{\min}$ is non-decreasing with respect to b for fixed $r \in (0, 1)$;

(2) If $r \in (0, 1)$, then there exists a finite value $b^* > 0$ such that $c_{\min} = c_0$ for $b \in (0, b^*]$, and $c_{\min} \in (c_0, 2]$ for $b > b^*$; If $r = 1$, then $c_{\min} > c_0$ for all positive b.

Proof. Let $r \in (0, 1)$ be fixed and $b_2 > b_1 > 0$. According to Theorem 3.3, there exists a monotone travelling wave solution (Φ_{b_2}, Ψ_{b_2}) of system (1.5)-(1.6) with speed $c = c_{\min}(b_2)$ for $b = b_2$. By applying the comparison theorem, it can be shown that (Φ_{b_2}, Ψ_{b_2}) serves as an upper solution of the system for $b = b_1$. A lower solution is defined as Lemma 3.1. Utilising the upper-lower solution method, it follows that monotone travelling wave solutions of the system exist for $b = b_1$ when $c = c_{\min}(b_2)$. Consequently, we obtain $c_{\min}(b_1) \leq c_{\min}(b_2)$. Therefore, the minimal speed $c_{\min}$ demonstrates a non-decreasing behaviour with respect to b.

Now we prove that there is a finite value of b so that transition of the speed selection is realised for the fixed $r \in (0, 1)$. Lemma 2.1 b) implies $\Psi(z) \to 0$ as $b \to 0$. Then Φ-equation of (1.5) becomes the Fisher–KPP equation as $b \to 0$. Therefore, $c_{\min} = 2\sqrt{1-r} = c_0$ (see [17]). Similarly, Lemma 2.1 b) leads us to derive $c_{\min} = 2 > c_0$ as $b \to \infty$. Consequently, due to the non-decreasing property of $c_{\min}$ with respect to b, there exists a finite b^* such that $c_{\min} = c_0$ for $b \in (0, b^*]$, and $c_{\min} \in (c_0, 2]$ for $b > b^*$. If $r = 1$, then $c_0 = 0$. We know the wave speed $c_{\min}$ of system (1.5)-(1.6) is positive. It follows that $c_{\min}$ is always bigger than c_0. Therefore, the proof is complete. □

Next, we shall provide some estimates about b^* based on the upper-lower solution method.

Theorem 3.7. *Assume that $r \in (0, 1)$ and (K1) hold. If parameters b and r satisfy*

$$\begin{cases} 0 < b \leq \dfrac{2(1-r)^2}{r}, \\ 0 < r \leq \dfrac{1}{2}, \\ \bar{\Theta} \leq 1, \end{cases} \tag{3.31}$$

where

$$\bar{\Theta} = \int_0^\infty \int_{-\infty}^{c_0 s} K(s, y) dy ds + \int_0^\infty \int_{c_0 s}^\infty K(s, y) e^{-\bar{\mu}(c_0 s - y)} dy ds,$$

then $c_{\min} = c_0$.

Proof. We can use part (ii) of Theorem 3.5. For illustration, We will alternatively use the upper-lower solution method to prove it. Define a pair of continuous function $(\bar{\Phi}_1, \bar{\Psi}_1)(z)$ as

$$\bar{\Phi}_1(z) = \frac{1}{1 + e^{\mu_1 z}}, \quad \bar{\Psi}_1(z) = \begin{cases} 1, & z \leq z_4, \\ \dfrac{2(1-r)}{r} \bar{\Phi}_1, & z > z_4, \end{cases} \tag{3.32}$$

where $z_4 = \frac{1}{\mu_1}\ln\frac{2(1-r)}{r}$. Let $c = c_0 + \varepsilon_3$, where ε_3 is sufficiently small. It follows that $\mu_1 \sim \bar{\mu} = \sqrt{1-r}$. We shall prove $(\bar{\Phi}_1, \bar{\Psi}_1)(z)$ is an upper solution of (1.5)-(1.6) under condition (3.32). Substituting $(\bar{\Phi}_1, \bar{\Psi}_1)(z)$ into (1.5) gives

$$\bar{\Psi}_1''(z) + c\bar{\Psi}_1'(z) + b\bar{\Phi}_1(z)(1-\bar{\Psi}_1(z)) = 0, \ z \le z_4,$$

$$\begin{aligned}
&\bar{\Psi}_1''(z) + c\bar{\Psi}_1'(z) + b\bar{\Phi}_1(z)(1-\bar{\Psi}_1(z)) \\
&= \frac{2(1-r)}{r}\bar{\Phi}_1(1-\bar{\Phi}_1)\left\{\mu_1^2 - c\mu_1 - 2\mu_1^2\bar{\Phi}_1 + \frac{b(1-\frac{2(1-r)}{r}\bar{\Phi}_1)}{\frac{2(1-r)}{r}(1-\bar{\Phi}_1)}\right\} \\
&\le \frac{2(1-r)}{r}\bar{\Phi}_1(1-\bar{\Phi}_1)[-2\mu_1^2\bar{\Phi}_1 - (1-r) + \frac{br}{2(1-r)}] \\
&\le \frac{2(1-r)}{r}\bar{\Phi}_1(1-\bar{\Phi}_1)[\frac{br}{2(1-r)} - (1-r)] \le 0, \ z > z_4,
\end{aligned}$$

by (3.31), and

$$\begin{aligned}
&\bar{\Phi}_1''(z) + c\bar{\Phi}_1'(z) + \bar{\Phi}_1(z)(1-r-\bar{\Phi}_1(z) + r(K\star_c\bar{\Psi}_1)(z)) \\
&= \bar{\Phi}_1(1-\bar{\Phi}_1)\left\{\mu_1^2 - c\mu_1 - 2\mu_1^2\bar{\Phi}_1 + \frac{1-r-\bar{\Phi}_1 + r(K\star_c\bar{\Psi}_1)}{1-\bar{\Phi}_1}\right\} \\
&= \bar{\Phi}_1^2(1-\bar{\Phi}_1)\left\{-2\mu_1^2 + \frac{r(K\star_c\bar{\Psi}_1) - r\bar{\Phi}_1}{(1-\bar{\Phi}_1)\bar{\Phi}_1}\right\} \\
&:= \bar{\Phi}_1^2(1-\bar{\Phi}_1)\left\{-2\mu_1^2 + J\right\}.
\end{aligned}$$

Note that

$$J = \begin{cases} \dfrac{r}{\bar{\Phi}_1} \le 2(1-r), \ z \le z_4, \\ \dfrac{2(1-r)\Theta_c - r}{1-\bar{\Phi}_1} \le \dfrac{2(1-r)[2(1-r)\Theta_c - r]}{2-3r}, z > z_4, \end{cases} \tag{3.33}$$

where

$$\Theta_c = \int_0^\infty\int_{-\infty}^{cs} K(s,y)dyds + \int_0^\infty\int_{cs}^\infty K(s,y)e^{-\mu_1(cs-y)}dyds.$$

Then $-2\mu_1^2 + J \le 0$ by (3.31). Therefore, $(\bar{\Phi}_1, \bar{\Psi}_1)(z)$ is an upper solution of (1.5)-(1.6). By Theorem 3.5, we know $c_{\min} = c_0$ when (3.31) holds. The proof is complete. □

Theorem 3.8. *Assume that H2 and (K1) hold. If b and r satisfy*

$$b > \frac{3(1-r)}{\min\{r\chi - (1-r), 1\}} \ge 0, \tag{3.34}$$

where

$$\chi = \int_0^\infty\int_{-\infty}^{c_0 s} K(s,y)e^{-\bar{\mu}(c_0 s - y)}dyds + \int_0^\infty\int_{c_0 s}^\infty K(s,y)dyds,$$

then $c_{\min} > c_0$.

Proof. We shall use Theorem 3.5 to prove this theorem. The key point is to find a suitable lower solution satisfying all conditions in Theorem 3.5. Define a pair of continuous function $(\underline{\Phi}_1, \underline{\Psi}_1)(z)$ as

$$\underline{\Phi}_1(z) = \frac{k_1}{1+e^{\mu_2 z}}, \quad \underline{\Psi}_1(z) = \frac{\underline{\Phi}_1}{k_1},$$

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where $0 < k_1 < 1$. We need to prove $(\underline{\Phi}_1, \underline{\Psi}_1)(z)$ is the lower solution of system (1.5)-(1.6). According to (3.34), k_1 can be chosen such that

$$\frac{3(1-r)}{b} < k_1 < \min\{r\chi - (1-r), 1\}. \tag{3.35}$$

We have

$$\begin{aligned}
&\underline{\Phi}_1''(z) + c\underline{\Phi}_1'(z) + \underline{\Phi}_1(z)(1 - r - \underline{\Phi}_1(z) + r(K \star_c \underline{\Psi}_1)(z)) \\
&= \underline{\Phi}_1(1 - \frac{\underline{\Phi}_1}{k_1})\Big\{\mu_2^2 - c\mu_2 + 1 - r + \frac{\underline{\Phi}_1}{k_1}[-2\mu_2^2 \\
&+ \frac{(1-r)\frac{\underline{\Phi}_1}{k_1} - \underline{\Phi}_1 + r\frac{\underline{\Phi}_1}{k_1}\int_0^\infty \int_{\mathbb{R}} K(s,y)\frac{1+e^{\mu_2 z}}{1+e^{\mu_2 z}e^{\mu_2(cs-y)}}dyds}{(1-\frac{\underline{\Phi}_1}{k_1})\frac{\underline{\Phi}_1}{k_1}}]\Big\} \\
&= \frac{\underline{\Phi}_1^2}{k_1}(1 - \frac{\underline{\Phi}_1}{k_1})\Big\{-2\mu_2^2 + \frac{(1-r) - k_1 + r\int_0^\infty \int_{\mathbb{R}} K(s,y)\frac{1+e^{\mu_2 z}}{1+e^{\mu_2 z}e^{\mu_2(cs-y)}}dyds}{1 - \frac{\underline{\Phi}_1}{k_1}}\Big\} \\
&\geq \frac{\underline{\Phi}_1^2}{k_1}(1 - \frac{\underline{\Phi}_1}{k_1})\Big\{-2\mu_2^2 + (1-r) - k_1 + r[\int_0^\infty \int_{-\infty}^{cs} K(s,y)e^{-\mu_2(cs-y)}dyds \\
&+ \int_0^\infty \int_{cs}^\infty K(s,y)dyds]\Big\} \\
&\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\underline{\Psi}_1''(z) + c\underline{\Psi}_1'(z) + b\underline{\Phi}_1(z)(1 - \underline{\Psi}_1(z)) \\
&= \frac{\underline{\Phi}_1}{k_1}(1 - \frac{\underline{\Phi}_1}{k_1})[\mu_2^2 - c\mu_2 - 2\mu_2^2\frac{\underline{\Phi}_1}{k_1} + bk_1] \\
&\geq \frac{\underline{\Phi}_1}{k_1}(1 - \frac{\underline{\Phi}_1}{k_1})[-2\mu_2^2 + bk_1 - (1-r)] \geq 0,
\end{aligned}$$

by using (3.35) and $\mu_2 \sim \sqrt{1-r}$ for $c = c_0 + \varepsilon_3$, where $\varepsilon_3 > 0$ is small. Thus, $(\underline{\Phi}_1, \underline{\Psi}_1)(z)$ is a lower solution of (1.5)-(1.6). It follows that $c_{\min} > c_0$ by Theorem 3.5. Thus, the proof is complete. □

Hasík et al. [12] also derived a linear speed selection condition

$$0 < b \leq \frac{1-r}{r\Theta}, \tag{3.36}$$

where

$$\Theta = \int_0^\infty \int_{\mathbb{R}} K(s,y)e^{-\bar{\mu}(c_0 s - y)}dyds.$$

Here, the condition (3.36) can be regards as an estimate for b^*.

3.3. The decay behaviour of the travelling wave with the minimal speed

In this subsection, we shall analyse how the travelling wave with the minimal speed decays in different parameter b and r, respectively. The subsequent theorem provides information on the rate at which the travelling wave with the minimal speed decays, with respect to the parameter b. Our aim is to demonstrate that as b approaches b^* from below, the decay rate of the travelling wave with the minimal speed transitions from $ze^{-\mu_1(c_0)z}$ to $e^{-\mu_1(c_0)z}$. In previous work by Wu et al. [41, Theorems 1.2 and 1.5], a similar outcome was established for scalar reaction-diffusion equations and two-species Lotka–Volterra competition systems. However, their approach involved constructing an intricate upper solution, which poses

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challenges to its applicability to our non-local model. Here, we have developed a novel technique to prove this decay behaviour for this Belousov–Zhabotinsky system with spatiotemporal interaction.

Theorem 3.9. *Assume that H2 and (K1) hold. Let b^* be the turning point for the minimal speed selection for fixed r, specifically $c_{\min} = c_0$ for $b \in (0, b^*]$, and $c_{\min} \in (c_0, 2]$ for $b > b^*$. The behaviour of the minimal-speed travelling wave solution $(\Phi(z), \Psi(z))$ of system (1.5)-(1.6) can be described as follows:*

(1) if $b \in (0, b^*)$, then $\Phi(z) \sim Aze^{-\mu_1(c_0)z}$ as $z \to \infty$, where $A > 0$;

(2) if $b = b^*$, then $\Phi(z) \sim Be^{-\mu_1(c_0)z}$ as $z \to \infty$, where $B > 0$;

(3) if $b > b^*$, then $\Phi(z) \sim Be^{-\mu_2(c_{\min})z}$ as $z \to \infty$, where $B > 0$.

Proof. Note that $c_{\min} > c_0 = 0$ for all $b > 0$ when $r = 1$. It follows from Theorem 3.4 that when $r = 1$, the result (3) holds for $b > 0$. Thus, we only consider $r \in (0, 1)$. For $b \in (0, b^*]$, Theorem 3.6 implies $c_{\min} = c_0$. Then we have

$$\Phi(z) \sim Aze^{-\mu_1(c_0)z} + Be^{-\mu_1(c_0)z}, \quad \text{as } z \to \infty, \tag{3.37}$$

where $A > 0$, or $B > 0$ if $A = 0$, see [4]. To establish result (2), we utilise a method similar to that employed in Theorem 3.4. Denote the minimal-speed travelling wave solution $(\Phi(z), \Psi(z))$ by $(\Phi^*(z), \Psi^*(z))$ for $b = b^*$. Then $(\Phi^*(z), \Psi^*(z))$ satisfies

$$\begin{cases} \Phi^{*''} + c_0\Phi^{*'} + \Phi^*(1 - r - \Phi^* + r(K \star_{c_0} \Psi^*)) = 0, \\ \Psi^{*''} + c_0\Psi^{*'} + b^*\Phi^*(1 - \Psi^*) = 0. \end{cases} \tag{3.38}$$

To the contrary, we assume that $\Phi^*(z) \sim Aze^{-\mu_1(c_0)z}$ as $z \to \infty$ with $A > 0$. Consider $b_\varepsilon = b^* + \varepsilon$, where ε is sufficiently small to be determined. Let

$$\Phi_\varepsilon(z) = \Phi^*(z) + W_3(z) \text{ and } \Psi_\varepsilon(z) = \Psi^*(z) + W_4(z),$$

where $W_3, W_4 \in B_0 = \{u \in C(-\infty, \infty), u(\pm\infty) = 0\}$. Our goal is to show the existence of W_3 and W_4 such that $(\Phi_\varepsilon(z), \Psi_\varepsilon(z))$ satisfies (1.5) with $b = b_\varepsilon$ and $c = c_0$. If we can demonstrate this, it would contradict the definition of b^* and therefore establish result (2). Suppose that $(\Phi_\varepsilon(z), \Psi_\varepsilon(z))$ is a solution of (1.5) with $b = b_\varepsilon$ and $c = c_0$. It implies

$$\begin{cases} \Phi_\varepsilon'' + c_0\Phi_\varepsilon' + \Phi_\varepsilon(1 - r - \Phi_\varepsilon + r(K \star_{c_0} \Psi_\varepsilon)) = 0, \\ \Psi_\varepsilon'' + c_0\Psi_\varepsilon' + b_\varepsilon\Phi_\varepsilon(1 - \Psi_\varepsilon) = 0. \end{cases} \tag{3.39}$$

By (3.38) and (3.39), W_3 and W_4 satisfy

$$\begin{aligned} &W_3'' + c_0W_3' - \alpha W_3 = -(P_0 + P_h), \\ &W_4'' + c_0W_4' - \alpha W_4 = -(R_0 + R_{1,\varepsilon} + R_{2,\varepsilon} + R_h), \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} &P_0 = \{\alpha + 1 - r - 2\Phi^* + r(K \star_{c_0} \Psi^*)\}W_3 + r\Phi^*(K \star_{c_0} W_4), \\ &P_h = -W_3^2 + rW_3(K \star_{c_0} W_4), \\ &R_0 = (\alpha - b^*\Phi^*)W_4 + b^*(1 - \Psi^*)W_3, \\ &R_{1,\varepsilon} = \varepsilon\Phi^*(1 - \Psi^*), \\ &R_{2,\varepsilon} = \varepsilon(1 - \Psi^*)W_3 - \varepsilon\Phi^*W_4, \\ &R_h = -(b^* + \varepsilon)W_3W_4. \end{aligned}$$

Therefore, for $c = c_0$, we have

$$\begin{aligned} &W_3 = T_{1,c_0}(P_0) + T_{1,c_0}(P_h), \\ &W_4 = T_{2,c_0}(R_0) + T_{2,c_0}(R_{1,\varepsilon}) + T_{2,c_0}(R_{2,\varepsilon}) + T_{2,c_0}(R_h), \end{aligned} \tag{3.41}$$

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where T_{1,c_0} and T_{2,c_0} are defined in (3.28). Define

$$\bar{T}(\bar{W}) = \begin{pmatrix} T_{1,c_0}(P_0(\bar{W})) \\ T_{2,c_0}(R_0(\bar{W})) \end{pmatrix}, \quad \bar{W} = (W_3, W_4).$$

Clearly, $T_{2,c_0}(R_{1,\varepsilon}) = O(\varepsilon)$ and $T_{2,c_0}(R_{2,\varepsilon}) = O(\varepsilon\bar{W})$. The linear operator $\bar{T}$ is both compact and strongly positive, and it possesses a simple principal eigenvalue $\lambda = 1$ associated with the positive eigenfunction $(-\Phi^{*'}, -\Psi^{*'})$. To eliminate this eigenfunction from B_0, we introduce a weighted functional space defined as

$$\mathcal{H}_2 = \{h(\xi) \in B_0 : he^{\mu_1(c_0)\xi} = o(1) \quad \text{as} \quad \xi \to \infty\}.$$

Since $(-\Phi^{*'}, -\Psi^{*'})$ does not belong to the space $\mathcal{H}_2 \times B_0$, it follows that $\bar{T}$ does not possess the eigenvalue $\lambda = 1$ for (W_3, W_4) in $\mathcal{H}_2 \times B_0$. In other words, $I - \bar{T}$ (where I denotes the identity operator) has a bounded inverse in $\mathcal{H}_2 \times B_0$. Consequently, there exists a solution (W_3, W_4) to (3.41) for any $\varepsilon \in [0, \varepsilon_0)$, where ε_0 is a small positive number. This implies that there exists a ε_0 such that the solution $(\Phi_\varepsilon, \Psi_\varepsilon)$ to (1.5)-(1.6) exists with $c = c_0$ for $b = b_\varepsilon = b^* + \varepsilon < b^* + \varepsilon_0$. However, this contradicts the conclusion in Theorem 3.6. Thus, result (2) holds true.

We can establish result (1) using a similar approach as presented in [41, Theorem 1.2]. Suppose there exists $b \in (0, b^*)$ such that the minimal-speed travelling wave solution satisfies

$$\Phi_b(z) \sim B_1 e^{-\mu_1(c_0)z}, \text{ as } z \to \infty, \ B_1 > 0. \tag{3.42}$$

Then by asymptotic analysis, we have

$$\Psi_b(z) \sim B_2 e^{-\mu_1(c_0)z}, \text{ as } z \to \infty, \ B_2 > 0.$$

By result (2), for $b = b^*$, the minimal-speed travelling wave (Φ_{b^*}, Ψ_{b^*}) satisfies

$$\begin{cases} \Phi_{b^*}(z) \sim Be^{-\mu_1(c_0)z}, \text{ as } z \to \infty, \ B > 0, \\ \Psi_{b^*}(z) \sim B_3 e^{-\mu_1(c_0)z}, \text{ as } z \to \infty, \ B_3 > 0. \end{cases} \tag{3.43}$$

As $z \to -\infty$, we obtain

$$\begin{cases} \Phi_b(z) \sim \begin{cases} 1 - \bar{B}_1 e^{\min\{\frac{-c_0+\sqrt{c_0^2+4b}}{2}, \frac{-c_0+\sqrt{c_0^2+4}}{2}\}z}, \text{ as } z \to -\infty, b \neq 1, \bar{B}_1 > 0, \\ 1 - \bar{B}_1 |z| e^{\frac{-c_0+\sqrt{c_0^2+4}}{2}z}, \text{ as } z \to -\infty, b = 1, \bar{B}_1 > 0, \end{cases} \\ \Psi_b(z) \sim 1 - \bar{B}_2 e^{\frac{-c_0+\sqrt{c_0^2+4b}}{2}z}, \text{ as } z \to -\infty, \bar{B}_2 > 0, \end{cases}$$

$$\begin{cases} \Phi_{b^*}(z) \sim \begin{cases} 1 - \bar{B} e^{\min\{\frac{-c_0+\sqrt{c_0^2+4b^*}}{2}, \frac{-c_0+\sqrt{c_0^2+4}}{2}\}z}, \text{ as } z \to -\infty, b^* \neq 1, \bar{B} > 0, \\ 1 - \bar{B} |z| e^{\frac{-c_0+\sqrt{c_0^2+4}}{2}z}, \text{ as } z \to -\infty, b^* = 1, \bar{B} > 0, \end{cases} \\ \Psi_{b^*}(z) \sim 1 - \bar{B}_3 e^{\frac{-c_0+\sqrt{c_0^2+4b^*}}{2}z}, \text{ as } z \to -\infty, \bar{B}_3 > 0. \end{cases}$$

Consequently, there exists an $L > 0$ such that $\Phi_{b^*}(z - L) > \Phi_b(z)$ and $\Psi_{b^*}(z - L) > \Psi_b(z)$ for all $z \in \mathbb{R}$. Define

$$L^* := \inf\{L \in \mathbb{R} | \Phi_{b^*}(z - L) \geq \Phi_b(z), \ \Psi_{b^*}(z - L) \geq \Psi_b(z), \ \forall z \in \mathbb{R}\}.$$

Suppose there exists $z_1 \in \mathbb{R}$ such that $\Phi_{b^*}(z_1 - L^*) = \Phi_b(z_1)$. By the strong maximum principle, it follows that $\Phi_{b^*}(z - L^*) = \Phi_b(z)$ for all $z \in \mathbb{R}$. However, this contradicts the fact that $(\Phi_{b^*}(z - L^*), \Psi_{b^*}(z - L^*))$ and $(\Phi_b(z), \Psi_b(z))$ satisfy different equations. Thus, $\Phi_{b^*}(z - L^*) > \Phi_b(z)$ for all $z \in \mathbb{R}$. Similarly, we can prove $\Psi_{b^*}(z - L^*) > \Psi_b(z)$ for all $z \in \mathbb{R}$.

Based on the above argument, we claim that

$$\lim_{z\to\infty} \frac{\Phi_{b^*}(z - L^*)}{\Phi_b(z)} = 1. \tag{3.44}$$

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Assuming $\lim_{z\to\infty} \frac{\Phi_{b^*}(z-L^*)}{\Phi_b(z)} > 1$, then by the second equation in (1.5), we can deduce that $\lim_{z\to\infty} \frac{\Psi_{b^*}(z-L^*)}{\Psi_b(z)} > 1$. Therefore, there exists a sufficiently small $\varepsilon_1 > 0$ such that $\Phi_{b^*}(z-(L^*-\varepsilon_1)) \geq \Phi_b(z)$ and $\Psi_{b^*}(z-(L^*-\varepsilon_1)) \geq \Psi_b(z)$ for $z \in \mathbb{R}$. However, this contradicts the definition of L^*. Hence, the claim (3.44) holds.

With the established claim, we have $Be^{\mu_1(c_0)L^*} = B_1$. Let $\hat{W}(z) = \Phi_{b^*}(z-L^*) - \Phi_b(z)$. Then $\hat{W}(z)$ satisfies

$$\hat{W}'' + c_0\hat{W}' + G_1(b^*, c_0, \Phi_{b^*}(z-L^*), \Psi_{b^*}(z-L^*)) - G_1(b, c_0, \Phi_b(z), \Psi_b(z)) = 0, \tag{3.45}$$

where

$$G_1(b, c, \Phi, \Psi) = \Phi(1 - r - \Phi + r(K \star_c \Psi)).$$

Note that

$$\hat{W}(z) \sim (Dz + E)e^{-\mu_1(c_0)z}, \text{ as } z \to \infty,$$

where $D \geq 0$, or $E > 0$ if $D = 0$. By (3.42), (3.43), and $Be^{\mu_1(c_0)L^*} = B_1$, we know $E = 0$ and $D = 0$. This leads to a contradiction. Thus, result (1) holds.

Result (3) can be obtained by combining Theorems 3.4 and 3.6. Therefore, the proof is complete. □

Similarly, we can consider the decay details of the minimal-speed wave solution based on different r. To this end, we first offer a necessary and sufficient condition based on r when b is fixed.

Theorem 3.10. *Assume that H2 and (K1) hold. For fixed b, there exists a unique turning point $r^* \in (0, 1)$ for the minimal speed selection, that is, $c_{\min} = c_0$ for $r \in (0, r^*]$, and $c_{\min} \in (c_0, 2]$ for $r \in (r^*, 1]$.*

Proof. Since Φ-equation in system (1.5) becomes the Fisher–Kpp equation when $r = 0$. That means $c_{\min} = c_0 = 2$, which can be seen in [17]. Since $c_{\min}$ is non-increasing in r by Theorem 3.3, we have $c_{\min} \in [c_0, 2]$ when $r \in (0, 1]$. When $r = 1$, we know $c_{\min} > c_0$ by Theorem 3.8. Therefore, there exists at least one turning point r^* at which speed selection changes from linearly selected to nonlinearly selected. We shall prove the uniqueness of the turning point r^* by contradiction. Let $r_1^*, r_2^* \in (0, 1)$ be two turning points for speed selection.

For $r \in [0, 1)$, we shall rewrite system (1.2) by the scaling

$$x\sqrt{1-r} \to x, \quad (1-r)t \to t, \quad \phi(t, x) \to \phi(t, x), \quad \psi(t, x) \to \psi(t, x).$$

Let $(\phi, \psi) = (\Phi, \Psi)(\nu \cdot x - ct)$, where $||\nu|| = 1$. Then we have the following wavefront system

$$\begin{cases} \Phi''(z) + c\Phi'(z) + \Phi(z)(1 - \dfrac{1}{1-r}\Phi(z) + \dfrac{r}{1-r}(K \star \Psi)(z)) = 0, \\ \Psi''(z) + c\Psi'(z) + \dfrac{b}{1-r}\Phi(z)(1 - \Psi(z)) = 0, \end{cases} \tag{3.46}$$

with the boundary condition (1.6), where

$$(K \star \Psi)(z) = \int_0^{+\infty}\int_{\mathbb{R}} K(s, y)\Psi(z + (1-r)cs - \sqrt{1-r}y)dyds.$$

Linearising (3.46) at e_0, we can get the characteristic equation $\mu^2 - c\mu + 1 = 0$, which yields two roots

$$\mu_3 = \mu_3(c) = \frac{c - \sqrt{c^2-4}}{2}, \quad \mu_4 = \mu_4(c) = \frac{c + \sqrt{c^2-4}}{2}. \tag{3.47}$$

Here, the linear speed $\bar{c}_0 = 2$.

Let (r_1^*, Φ_1, Ψ_1) and (r_2^*, Φ_2, Ψ_2) be two solutions of (3.46)-(1.6). According to a similar argument in the proof of Theorem 3.9 (2), we know

$$\Phi_1(z) \sim A_3e^{-\mu_3(\bar{c}_0)z}, \text{ as } z \to \infty, \ A_3 > 0,$$
$$\Phi_2(z) \sim A_4e^{-\mu_3(\bar{c}_0)z}, \text{ as } z \to \infty, \ A_4 > 0.$$

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By asymptotic analysis, we have

$$\Psi_1(z) \sim A_5 e^{-\mu_3(\bar{c}_0)z}, \text{ as } z \to \infty, \ A_5 > 0,$$
$$\Psi_2(z) \sim A_6 e^{-\mu_3(\bar{c}_0)z}, \text{ as } z \to \infty, \ A_6 > 0.$$

As $z \to -\infty$, we obtain

$$\begin{cases} \Phi_1(z) \sim \begin{cases} 1 - \bar{A}_3 e^{\min\{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4b}{1-r_1^*}}}{2}, \frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4}{1-r_1^*}}}{2}\}z}, \text{ as } z \to -\infty, b \neq 1, \bar{A}_3 > 0, \\ 1 - \bar{A}_3 |z| e^{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4}{1-r_1^*}}}{2}z}, \text{ as } z \to -\infty, b = 1, \bar{A}_3 > 0, \end{cases} \\ \Psi_1(z) \sim 1 - \bar{A}_5 e^{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4b}{1-r_1^*}}}{2}z}, \text{ as } z \to -\infty, \bar{A}_5 > 0, \end{cases}$$

$$\begin{cases} \Phi_2(z) \sim \begin{cases} 1 - \bar{A}_4 e^{\min\{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4b}{1-r_2^*}}}{2}, \frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4}{1-r_2^*}}}{2}\}z}, \text{ as } z \to -\infty, b \neq 1, \bar{A}_4 > 0, \\ 1 - \bar{A}_4 |z| e^{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4}{1-r_2^*}}}{2}z}, \text{ as } z \to -\infty, b = 1, \bar{A}_4 > 0, \end{cases} \\ \Psi_2(z) \sim 1 - \bar{A}_6 e^{\frac{-\bar{c}_0+\sqrt{\bar{c}_0^2+\frac{4b}{1-r_2^*}}}{2}z}, \text{ as } z \to -\infty, \bar{A}_6 > 0, \end{cases}$$

Consequently, there exists an $L > 0$ such that $\Phi_{b^*}(z - L) > \Phi_b(z)$ and $\Psi_{b^*}(z - L) > \Psi_b(z)$ for all $z \in \mathbb{R}$. Define

$$L_1^* := \inf\{L \in \mathbb{R} | \Phi_2(z - L) \geq \Phi_1(z), \ \Psi_2(z - L) \geq \Psi_1(z), \ \forall z \in \mathbb{R}\}.$$

By a similar analysis in the proof of Theorem 3.9 (1), we will derive a contradiction. Therefore, there exists a unique turning point r^* for speed selection. Hence, the proof is complete. □

Based on the above Theorem 3.10, we give the decay details of the minimal-speed wave solution of (3.46)-(1.6) based on different r. Since the proof of this assertion is identical to the proof of Theorem 3.9, we omit it here.

Theorem 3.11. *Assume that H2 and (K1) hold. For fixed b, we let $r^* < 1$ be the turning point for speed selection of (3.46)-(1.6), that is, $c_{\min} = \bar{c}_0$ for $r \in (0, r^*]$, and $c_{\min} > \bar{c}_0$ for $r \in (r^*, 1)$. The minimal-speed travelling wave solution $(\Phi(z), \Psi(z))$ of system (3.46)-(1.6) satisfies:*

(1) if $r \in (0, r^*)$, then $\Phi(z) \sim Aze^{-\mu_3(\bar{c}_0)z}$ as $z \to \infty$, where $A > 0$;
(2) if $r = r^*$, then $\Phi(z) \sim Be^{-\mu_3(\bar{c}_0)z}$ as $z \to \infty$, where $B > 0$;
(3) if $r \in (r^*, 1)$, then $\Phi(z) \sim Be^{-\mu_4(c_{\min})z}$ as $z \to \infty$, where $B > 0$.

4. Numerical analysis

In this section, we will use MATLAB software to simulate our model (1.2) under cases H1-H2 and provide several examples to demonstrate our numerical findings. We consider system (1.2) in $x \in \mathbb{R}$ and choose the kernel function as

$$K(s, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{-s}. \tag{4.1}$$

Clearly, conditions (K1) and (K2) hold. We construct the initial conditions of (1.2) as

$$\phi(\theta, x) = \frac{0.5}{1 + e^x}, \quad \psi(\theta, x) = \frac{0.5}{1 + e^x}, \quad x \in \mathbb{R}, \quad \theta \in [-10, 0]. \tag{4.2}$$

to calculate the numerical speeds. Although they are not semi-compact supported, the exponentiate decay rate is not smaller than $\sqrt{1 - r}$ when $z \to \infty$ so that the solution still evolves to the wave with minimal speed (see [11, 38]). Here we take $\theta \in [-10, 0]$ in the initial data since the integral

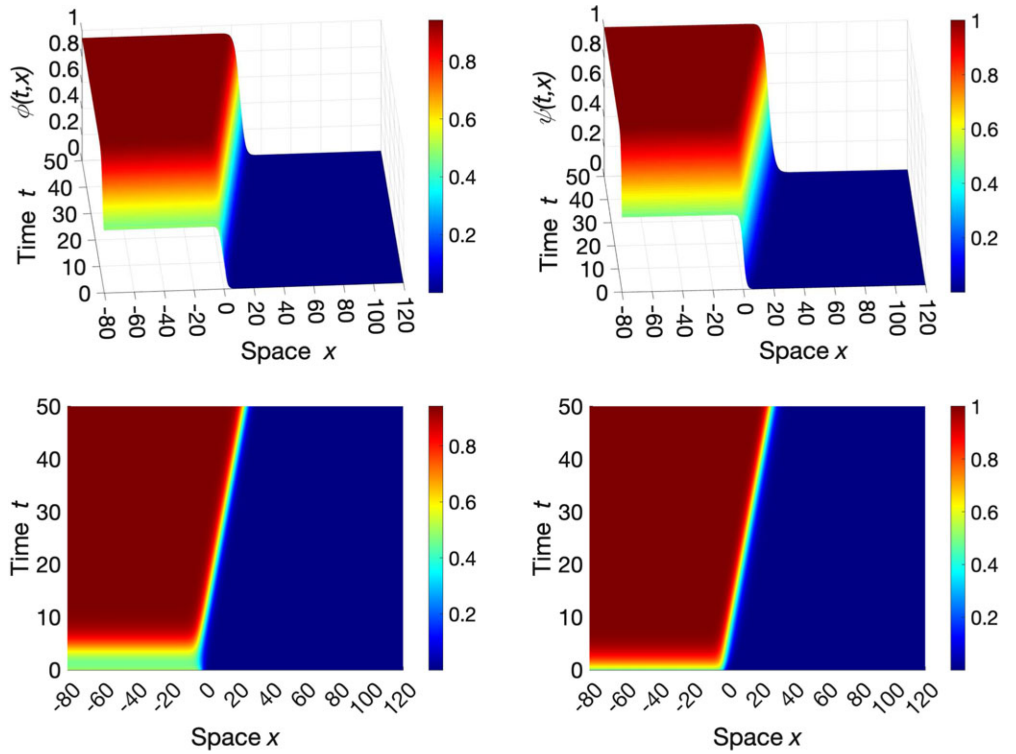

Figure 1. *Simulation of $(\phi, \psi)(t, x)$ in accordance with case H1, where the parameters are defined as $b = 1$ and $r = 1.2$. The first row depicts the spatiotemporal movements of ϕ and ψ, respectively. The bottom row displays their respective 2-D plots observed from a top view.*

$\int_{10}^{\infty}\int_{\mathbb{R}} K(s, y)\psi(t-s, x-y)dyds < e^{-10} \simeq 4.5 \times 10^{-5}$ is very small in the reaction term of system (1.2) when the delay $s > 10$.

Example 4.1. *Let $b = 1$ and $r = 1.2$. The kernel function and the initial data are defined in* (4.1) *and* (4.2)*, respectively.*

Since $r = 1.2 > 1$, this example corresponds to the bistable case H1. The evolution of ϕ and ψ is depicted in Figure 1. In the case H1, both e_β and e_0 are stable. As illustrated in Figure 1, the solution progresses towards the right (i.e. $c > 0$), indicating that bromide ion within the system persists as time t is large.

Example 4.2. *Let $b = 5$ and $r = 0.5$. The kernel function and the initial data are defined in* (4.1) *and* (4.2)*, respectively.*

In this case, we have $r = 0.5 < 1$, and then H2 holds. The detailed simulations are presented in Figure 2. These figures clearly demonstrate that the solution propagates to the right, stablising to a constant speed, consistent with the findings when e_β is stable and e_0 is unstable under H2. To calculate the numerical speeds, we utilise the level sets depicted in Figure 3. For the computation of the numerical moving speed c_ϕ^* of $\phi(t, x)$, we locate the level set $x(t)$ such that $\phi(t, x(t)) = 0.5$ for different values of t. Similarly, we determine $x(t)$ such that $\psi(t, x(t)) = 0.5$ to compute the numerical moving speed c_ψ^* of $\psi(t, x)$. By calculating the difference in x coordinates in the level set and dividing it by the corresponding time difference at each snapshot time, we can obtain the numerical speed. Taking snapshots of ϕ and ψ at $t = 40, 41, 42, \ldots 50$ (shown in Figure 3), we can derive the moving speeds at different times. Once

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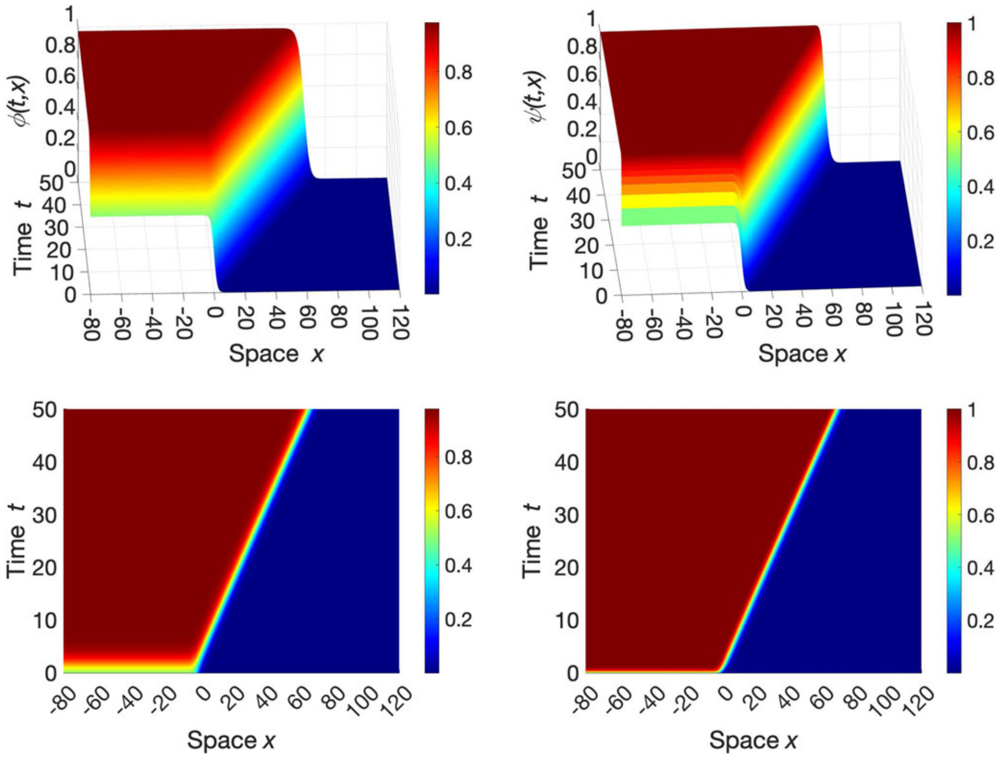

Figure 2. The simulation of $(\phi, \psi)(t, x)$ *is carried out based on case H2, with parameter values set as* $b = 5$ *and* $r = 0.5$*. The spatiotemporal dynamics of* ϕ *and* ψ *are presented in the first row, while the corresponding 2-D plots viewed from the top are shown in the bottom row.*

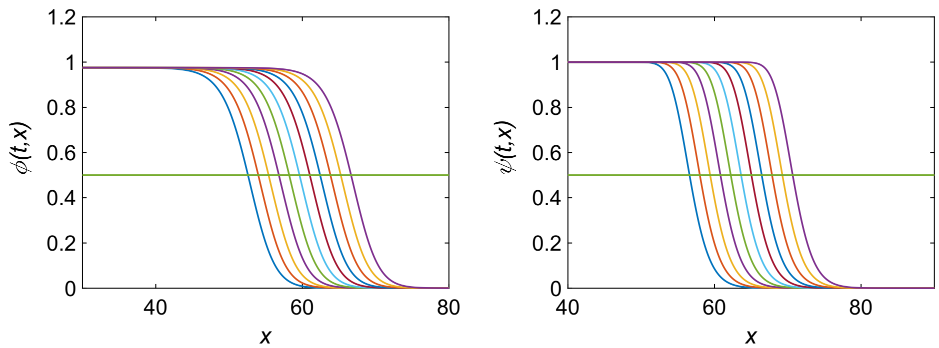

Figure 3. Snapshots of $\phi(t, x)$ *and* $\psi(t, x)$*'s movements. The graph on the left represents* $\phi(t, x)$*, while the graph on the right represents* $\psi(t, x)$*. These snapshots correspond to different time values* $t = 40, 41, 42, \ldots 50$.

the speed difference among them becomes less than 5×10^{-3}, we stop the computation and consider their values as the final result. Through this computation, we find that $c^*_\phi = 1.4140$ and $c^*_\psi = 1.4143$. Note that the linear speed $c_0 = 2\sqrt{1-r} = 1.4142$. Therefore, we conclude that $c^*_\phi \simeq c^*_\psi \simeq c_0$.

Theorem 3.6 shows that the minimal speed $c_{\min}$ has a monotonic relationship with the parameter b. Additionally, this theorem offers a necessary and sufficient criterion for distinguishing between linear

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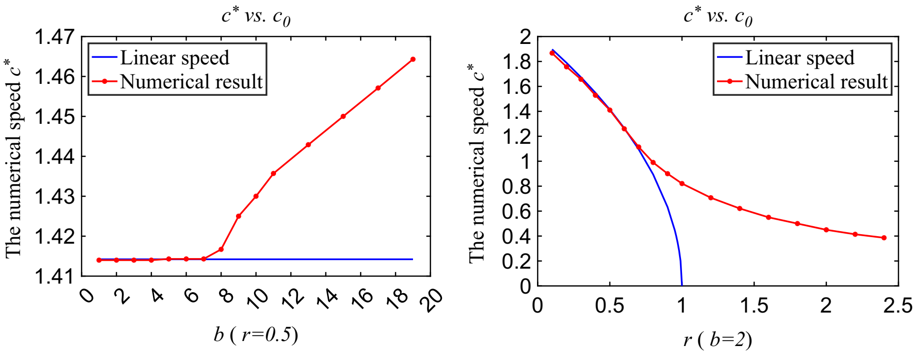

Figure 4. *In the left figure, we have $r = 0.5$. It shows the relationship between the numerical moving speed c^* and b for fixed r. The relationship between c^* and r, when $b = 2$, is in the right figure.*

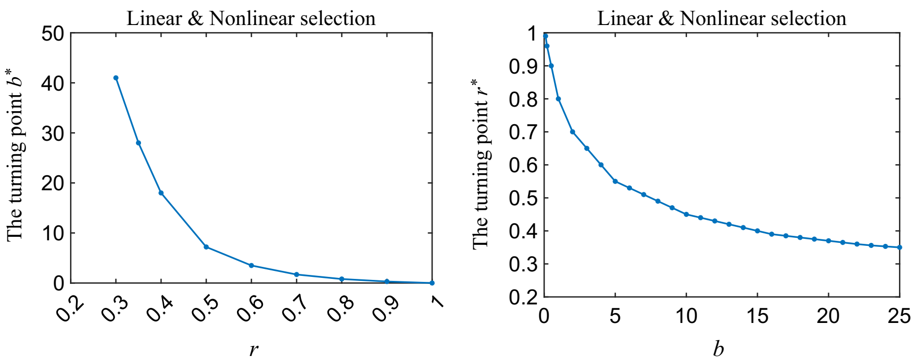

Figure 5. *The left diagram illustrates the relationship between b^* and r, while the right figure depicts the relation between r^* and b.*

and nonlinear selection based on the value of b. Theorem 3.10 illustrates that there exists a unique turning point r^* for speed selection when b is fixed. The numerical simulation presented in Figure 4 aligns with the findings outlined in Theorems 3.6 and 3.10.

For the left picture in Figure 4, we set $r = 0.5$. It is observed that the numerical speed c^* increases monotonically with respect to b for a fixed value of r, as shown by the dot-sign (red) curve. The straight (blue) line represents the linear speed, where $c_0 = 2\sqrt{1-r} \approx 1.4142$. There exists a turning point b^* at which the two curves coincide ($c^* = c_0$) for $b \leq b^*$, and $c^* > c_0$ for $b > b^*$. In our numerical analysis with $r = 0.5$, we find that $b^* \approx 7$. In the right figure, we let $b = 2$. The solid (blue) curve is the linear speed $c_0 = 2\sqrt{1-r}$, and the dot-sign (red) curve is the numerical speed c^*. we can see that there is also a turning point r^* such that $c^* = c_0$ for $r \leq r^*$, and $c^* > c_0$ for $r > r^*$. From this figure, we know $r^* \approx 0.7$.

For the two pictures in Figure 5, if the parameter value is above the curve, the minimal wave speed is nonlinearly selected; otherwise, if the parameter value is below the curve, the minimal wave speed is linearly selected.

5. Conclusion and discussion

In this work, we investigated the existence, uniqueness, speed sign of bistable travelling waves, and the minimal wave speed selection for monostable travelling waves to the Belousov–Zhabotinsky system with spatiotemporal interaction (1.2). As (1.2) exhibits transition dynamics from monostable to

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bistable for different domains of the system parameter r, we analysed it in two cases, H1(bistable) and H2(monostable). The presence of a non-isolated equilibrium e_0 and a non-local delayed reaction term in the system proposes a challenge for directly applying most existing theories to establish the existence of bistable travelling waves. By developing new ideas such as construction of auxiliary parabolic non-local model, we successfully established the existence and uniqueness of travelling waves under bistable cases H1. It is important to note that the ideas and techniques utilised in this work can be extended and applied to various new systems, particularly those with non-isolated equilibrium points and non-local reaction terms, for which most existing theories do not apply.

In the monostable case H2, we focused on the speed selection. We established necessary and sufficient conditions for linear and nonlinear selections, as presented in Theorems 3.4, 3.6 and 3.10. The existence of transition points b^* (when r is fixed) and r^* (when b is fixed) for the selection of the minimal speed was also illustrated in Figure 4. Additionally, we found the decay rate of the travelling wave with the minimal speed as $z \to \infty$ in terms of the parameters b and r, respectively, as shown in Theorems 3.9 and 3.11. Moreover, we provided a few explicit estimates for b^* in Theorems 3.7 and 3.8.

Finally, we would like to emphasise and compare the novelty of our findings to previous works. We want to point out that our results can be directly generalised to the localised case ($(\bar{K} * v)(t,x) = v(t,x)$) and the local delayed case ($(\bar{K} * v)(t,x) = v(t-h,x)$) by letting $\bar{K}(s,y) = \delta(s)\delta(y)$ and $\bar{K}(s,y) = \delta(s-h)\delta(y)$, respectively. The existence, uniqueness and speed sign of the bistable travelling wave for the localised case and the local delayed case in the system (1.1) can also be derived by Theorems 1.1 and 1.2. For the minimal speed of the system (1.1) with $(\bar{K} * v)(t,x) = v(t,x)$, Trofimchuk et al. [31] gave information of the minimal speed in the monostable case as follows:

$$\begin{cases} c_{\min} = 2\sqrt{1-r}, \text{ if } rb + r \le 1; \\ c_{\min} \le 2\sqrt{b}, \text{ if either } b + r > 1, b < 1, r \in (0,1] \text{ or } b = 1, r < 1; \\ c_{\min} \le 2, \text{ if } b > 1, r \in (0,1]. \end{cases} \tag{5.1}$$

We are more concentrated on the speed selection by providing general methods, and our results (Theorems 3.4-3.10) can be directly applied to this case when $\bar{K}(s,y) = \delta(s)\delta(y)$ by construction of various upper or lower solutions. In particular, by letting $\bar{K}(s,y) = \delta(s)\delta(y)$, the linear selection condition

$$\begin{cases} 0 < b \le \dfrac{2(1-r)^2}{r}, \\ 0 < r \le \dfrac{1}{2}. \end{cases} \tag{5.2}$$

in Theorem 3.7 is better than $rb + r \le 1$ in (5.1) when $r \le \frac{1}{2}$, and we also have the nonlinear selection condition $b > \frac{3(1-r)}{2r-1} \ge 0$ which are completely new. Similarly, when $\bar{K}(s,y) = \delta(s-h)\delta(y)$, we can calculate that the condition (3.31) in Theorem 3.7 is better than the linear selection condition

$$rbe^{-2h(1-r)} + r \le 1 \tag{5.3}$$

in [31] of (1.1) with $(\bar{K} * v)(t,x) = v(t-h,x)$ for some domain of r and h. We should mention that we can provide a series of explicit conditions on speed selections by constructing various upper or lower solutions in Theorem 3.5. For instance, readers can also apply nonlinear selection condition (Theorem 3.8) and necessary and/or sufficient speed selection conditions (Theorems 3.4, 3.6 and 3.10) to the model with $\bar{K}(s,y) = \delta(s-h)\delta(y)$.

As we mentioned, the exact or explicit value of the minimal wave speed $c_{\min}$ for the system (1.1) with $(\bar{K} * v)(t,x) = v(t,x)$ or $(\bar{K} * v)(t,x) = v(t-h,x)$ cannot be worked out [31, 32] in the case of nonlinear selection. Our results can provide some new estimates for $c_{\min}$, whether the system (1.1) is in the localised case or in the local delayed case or in the non-local delayed case.

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