A CYCLIC INEQUALITY AND AN EXTENSION OF IT. II.

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1. Introduction

Throughout this paper, unless otherwise stated, n and L stand for positive integers and α, t, x, x_1, x_2, ... for positive real numbers. Let

$$S_n(x_1, \ldots, x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}}, \quad \ldots\ldots(1)$$

where

$$x_{n+r} = x_r \text{ (all } r), \quad \ldots\ldots(2)$$

and

$$\lambda(n) = \frac{1}{n} \inf_{x_1,\ldots,x_n} S_n(x_1, \ldots, x_n). \quad \ldots\ldots(3)$$

Then, it is known (see (2)) that

$$\begin{aligned} \lambda(n) &= \tfrac{1}{2} \quad (n \leqq 8), \\ &< \tfrac{1}{2} \quad (\text{even } n \geqq 14, \text{ odd } n \geqq 27). \end{aligned}$$

Also, as Rankin (4) has proved, $\lambda(n)$ has a finite limit as $n \to \infty$ and

$$\lambda = \lim_{n\to\infty} \lambda(n) = \inf_n \lambda(n). \quad \ldots\ldots(4)$$

Further (6),

$$\lambda \leqq \lambda(24) < 0{\cdot}49950317. \quad \ldots\ldots(5)$$

In a paper (1) to appear shortly, we have shown that

$$\lambda(n) \geqq \lambda \geqq \tfrac{1}{2}(\sqrt{2}-\tfrac{1}{2}) = 0{\cdot}457107, \quad \ldots\ldots(6)$$

thus improving Rankin's result (5)

$$\lambda(n) \geqq \lambda \geqq 0{\cdot}330232,$$

which he obtained by a method involving the use of properties of convex functions. Our result (6) was first obtained by a development of Rankin's method, although later a simpler proof was found (see (1)). In this paper we shall develop Rankin's method further and prove that

$$\lambda(n) \geqq \lambda \geqq 0{\cdot}461238. \quad \ldots\ldots(7)$$

We shall also prove that

$$\lambda \leqq \lambda(24) < 0{\cdot}499197. \quad \ldots\ldots(8)$$

The upper and lower bounds for λ, appearing in (7) and (8), have a gap which is less than 90 per cent. of that between the best previously known bounds which are given in (5) and (6).

2. Some Lemmas

Lemma 1. *Let α, x_1, x_2, ... be any real numbers satisfying* (2). *Then there are integers $a_1, a_2, \ldots, a_{s+1} (s>0)$, with*

$$a_{s+1} \equiv a_1 \pmod{n}, \quad \ldots\ldots(9)$$

such that, for $k = 1, 2, \ldots, s$,

(i) $$a_{k+1} \geqq a_k + 2 \quad \ldots\ldots(10)$$

and

(ii) *either* $a_{k+1} - a_k$ *is even,* $x_{a_{k+1}} \geqq \alpha x_{a_{k+1}+1}$

and $x_{a_k+2} < \alpha x_{a_k+3} < x_{a_k+4} < \alpha x_{a_k+5} < \ldots < x_{a_{k+1}}$, $\quad \ldots\ldots(11)$

or $a_{k+1} - a_k$ *is odd,* $\alpha x_{a_{k+1}} \geqq x_{a_{k+1}+1}$

and $x_{a_k+2} < \alpha x_{a_k+3} < x_{a_k+4} < \alpha x_{a_k+5} < \ldots < \alpha x_{a_{k+1}}$. $\quad \ldots\ldots(12)$

Proof. First let a_k be an arbitrary integer. Consider the infinite chain C of inequalities

$$x_{a_k+2} < \alpha x_{a_k+3} < x_{a_k+4} < \alpha x_{a_k+5} < x_{a_k+6} < \ldots.$$

If all these inequalities are true then

$$x_{a_k+2} < x_{a_k+4} < x_{a_k+6} < \ldots$$

and so

$$x_{a_k+2} < x_{a_k+2n+2}.$$

This contradicts (2). Hence the inequalities in C are not all true. Suppose that the first $b_k = a_{k+1} - (a_k+2)$ inequalities in C are true and the next one false. Then we have (10) since $b_k \geqq 0$. Also we have (11) if b_k is even and (12) if b_k is odd. Thus there is an integer a_{k+1} satisfying both (i) and (ii).

Hence, starting with an arbitrary integer a_1, we can find successively integers a_2, a_3, a_4, ... satisfying (i) and (ii) for $k = 1, 2, 3, \ldots$ respectively. Consider now the infinite sequence of integers $a_1, a_2, a_3, \ldots$. Since there are only n residue classes (mod n), it follows that there are positive integers s and t such that $a_{s+t} \equiv a_t \pmod{n}$. Also (i) and (ii) are satisfied for $k = t$, $t+1, \ldots, s+t-1$. Since (for any fixed s and t) $a_t, a_{t+1}, \ldots, a_{s+t}$ can be renamed $a_1, a_2, \ldots, a_{s+1}$ respectively, the lemma follows. (I am indebted to the referee for commenting that my original proof needed further clarification.)

Following Rankin (5), we write

$$(\phi_L x_0, x_1, \ldots, x_{L+1}) = \sum_{r=0}^{L-1} \frac{x_r}{x_{r+1}+x_{r+2}}. \quad \ldots\ldots(13)$$

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We write also

$$\psi_L(x_0, x_1, \ldots, x_L) = \frac{x_0}{x_1+x_2} + \frac{x_1+x_2}{x_3+x_4} + \frac{x_3+x_4}{x_5+x_6} + \ldots + \frac{x_{L-3}+x_{L-2}}{x_{L-1}+x_L} + \frac{\alpha x_{L-1}}{(1+\alpha)x_L} \text{ (even } L \geqq 2), \qquad \ldots\ldots(14)$$

$$= \frac{x_0}{x_1+x_2} + \frac{x_1+x_2}{x_3+x_4} + \frac{x_3+x_4}{x_5+x_6} + \ldots + \frac{x_{L-4}+x_{L-3}}{x_{L-2}+x_{L-1}} + \frac{x_{L-2}+x_{L-1}}{(1+\alpha)x_L} \text{ (odd } L \geqq 3). \qquad \ldots\ldots(15)$$

Lemma 2. *Let L be even and* $\geqq 2$. *Suppose that*

$$x_2 < \alpha x_3 < x_4 < \alpha x_5 < \ldots < x_L \text{ and } x_L \geqq \alpha x_{L+1}. \qquad \ldots\ldots(16)$$

Then

$$\phi_L(x_0, x_1, \ldots, x_{L+1}) \geqq \psi_L(x_0, x_1, \ldots, x_L). \qquad \ldots\ldots(17)$$

Proof. This follows from (13), (14) and (16).

Lemma 3. *Let L be odd and* $\geqq 3$. *Suppose that*

$$x_2 < \alpha x_3 < x_4 < \alpha x_5 < \ldots < \alpha x_L \text{ and } \alpha x_L \geqq x_{L+1}. \qquad \ldots\ldots(18)$$

Then (17) *is true.*

Proof. This follows from (13), (15) and (18).

For each t, we define functions $f_t(x)$, $g_t(x)$, $F_t(x)$ and $G_t(x)$ for $x \geqq 0$ as follows.

$$\left.\begin{aligned} f_t(x) &= \tfrac{1}{2} t x^{2/t} && \text{for } x \leqq \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}L}, \\ &= \tfrac{1}{2}(t+2)\left(\frac{\alpha x}{1+\alpha}\right)^{2/(t+2)} - \frac{\alpha}{1+\alpha} && \text{for } x \geqq \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}L}. \end{aligned}\right\} \qquad \ldots\ldots(19)$$

$$g_t(x) = \tfrac{1}{2}(t+1)\left(\frac{x}{1+\alpha}\right)^{2/(t+1)} \qquad \ldots\ldots(20)$$

$$\left.\begin{aligned} F_t(x) = \frac{2}{t} f_t(x^{\frac{1}{2}t}) &= x && \text{for } x \leqq \frac{\alpha}{1+\alpha}, \\ &= \frac{t+2}{t}\left(\frac{\alpha}{1+\alpha}\right)^{2/(t+2)} x^{t/(t+2)} - \frac{2}{t}\,\frac{\alpha}{1+\alpha} && \text{for } x \geqq \frac{\alpha}{1+\alpha}. \end{aligned}\right\} \ldots(21)$$

$$G_t(x) = \frac{2}{t} g_t(x^{\frac{1}{2}t}) = \frac{t+1}{t}\left(\frac{1}{1+\alpha}\right)^{2/(t+1)} x^{t/(t+1)}. \qquad \ldots\ldots(22)$$

The functions defined above are all convex functions of $\log x$ for $x > 0$, but we shall use this convexity property only of $F_2(x)$ and $G_3(x)$.

Lemma 4.

$$\left.\begin{aligned} F_2(x) &= x \text{ for } x \leqq \frac{\alpha}{1+\alpha}, \\ &= 2\sqrt{\frac{\alpha x}{1+\alpha}} - \frac{\alpha}{1+\alpha} \text{ for } x \geqq \frac{\alpha}{1+\alpha}; \end{aligned}\right\} \qquad (23)$$

and

$$G_3(x) = \frac{4}{3}\frac{x^{\frac{3}{4}}}{(1+\alpha)^{\frac{1}{2}}} \qquad (24)$$

are convex functions of $\log x$ *for* $x>0$. *Also*

$$F_2(x) \geqq 2\sqrt{\frac{\alpha x}{1+\alpha}} - \frac{\alpha}{1+\alpha}. \qquad (25)$$

Proof. The convexity properties follow from the following facts: For $x>0$ (i) $F_2(x)$ and $G_3(x)$ are continuous and have continuous derivatives, (ii) except at $x = \frac{\alpha}{1+\alpha}$, $F_2''(x)$ exists and $xF_2''(x)+F_2'(x) \geqq 0$ and (iii) $G_3''(x)$ exists and $xG_3''(x)+G_3'(x) \geqq 0$.

(25) follows from (23) since $x + \frac{\alpha}{1+\alpha} \geqq 2\sqrt{\frac{\alpha x}{1+\alpha}}$, by the inequality of the (arithmetic and geometric) means.

Lemma 5. $F_t(x) \geqq F_{t'}(x)$ *for* $t \geqq t' > 0$.

Proof. From (21), the result is true for $x \leqq \frac{\alpha}{1+\alpha}$. When $x \geqq \frac{\alpha}{1+\alpha}$,

$$F_t'(x) - F_{t'}'(x) = \left(\frac{\alpha}{(1+\alpha)x}\right)^{2/(t+2)} - \left(\frac{\alpha}{(1+\alpha)x}\right)^{2/(t'+2)} \geqq 0.$$

Hence

$$F_t(x) - F_{t'}(x) \geqq F_t\left(\frac{\alpha}{1+\alpha}\right) - F_{t'}\left(\frac{\alpha}{1+\alpha}\right) = 0.$$

Lemma 6. $G_t(x) \geqq F_2(x)$

if

$$t > 1 \text{ and } \alpha(1+\alpha) \leqq \left(\frac{t}{t-1}\right)^{t-1}. \qquad (26)$$

Proof. Let

$$F(x) = 2\sqrt{\frac{\alpha x}{1+\alpha}} - \frac{\alpha}{1+\alpha}. \qquad (27)$$

Then, from (22) and (27),

$$\begin{aligned} G_t'(x) - F'(x) &= (1+\alpha)^{-2/(t+1)} x^{-1/(t+1)} - \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \\ &= 0 \text{ at } x = \frac{\alpha}{1+\alpha}\{\alpha(1+\alpha)\}^{2/(t-1)}, \end{aligned}$$

where $G_t(x)-F(x)$ has the minimum value

$$\frac{\alpha}{1+\alpha}-\frac{t-1}{t}\alpha^{t/(t-1)}(1+\alpha)^{(2-t)/(t-1)}\geqq 0,$$

from (26). Hence $G_t(x)\geqq F(x)=F_2(x)$ for $x\geqq\dfrac{\alpha}{1+\alpha}$, from (23) and (27).

For $x\leqq\dfrac{\alpha}{1+\alpha}$, $G_t(x)\geqq F_2(x)$ is equivalent, from (22) and (27), to

$$x\leqq\left(\frac{t+1}{t}\right)^{t+1}(1+\alpha)^{-2}$$

which is satisfied if $\alpha(1+\alpha)\leqq\left(\dfrac{t+1}{t}\right)^{t+1}$. But (26) is true, and

$$\left(\frac{t}{t-1}\right)^{t-1}\leqq\left(\frac{t+1}{t}\right)^{t+1} \text{ since } \left(\frac{t}{t-1}\right)^{t-1}\left(\frac{t}{t+1}\right)^{t+1}\leqq 1$$

by the inequality of the means. Hence $G_t(x)\geqq F_2(x)$ for $x\leqq\dfrac{\alpha}{1+\alpha}$ also.

Lemma 7. $\psi_L(x_0, x_1, \ldots, x_L)\geqq f_L(x_0/x_L)$ (*even* $L\geqq 2$), ...(28)

$\geqq g_L(x_0/x_L)$ (*odd* $L\geqq 3$). ....(29)

Proof. For odd $L\geqq 3$, (29) follows from (15), (20) and the inequality of the means. Let $x=x_0/x_L$.

For even $L\geqq 2$ and $x\geqq\left(\dfrac{\alpha}{1+\alpha}\right)^{\frac{1}{2}L}$,

$$\psi_L(x_0, x_1, \ldots, x_L)+\frac{\alpha}{1+\alpha}=\frac{x_0}{x_1+x_2}+\frac{x_1+x_2}{x_3+x_4}+\frac{x_3+x_4}{x_5+x_6}+\ldots$$
$$+\frac{x_{L-3}+x_{L-2}}{x_{L-1}+x_L}+\frac{\alpha(x_{L-1}+x_L)}{(1+\alpha)x_L},$$

from (14). Hence (28) follows from the inequality of the means and (19).

For even $L\geqq 2$ and $x\leqq\left(\dfrac{\alpha}{1+\alpha}\right)^{\frac{1}{2}L}$,

$$\psi_L(x_0, x_1, \ldots, x_L)\geqq\tfrac{1}{2}L\left(\frac{x_0}{x_{L-1}+x_L}\right)^{2/L}+\frac{\alpha}{1+\alpha}\frac{x_{L-1}}{x_L},$$

from (14) and the inequality of the means. Hence

$$\psi_L(x_0, x_1, \ldots, x_L)\geqq h\left(\frac{x_0}{x_L}, \frac{x_{L-1}}{x_L}\right)$$

where

$$h(x, u)=\tfrac{1}{2}L\left(\frac{x}{1+u}\right)^{2/L}+\frac{\alpha u}{1+\alpha}.$$

Now

$$\frac{\partial}{\partial u} h(x, u) = -\frac{1}{1+u}\left(\frac{x}{1+u}\right)^{2/L} + \frac{\alpha}{1+\alpha}$$

$$\geqq 0 \text{ for } u \geqq 0 \text{ and } x \leqq \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}L}.$$

Thus $\psi_L(x_0, x_1, \ldots, x_L) \geqq h(x_0/x_L, 0) = f_L(x_0/x_L)$, from (19). This completes the proof of the lemma.

Lemma 8. $\left(\frac{t}{t-1}\right)^{t-1}$ *increases for* $t > 1$.

Proof. This follows from Theorem 140 of (3).

Lemma 9. $G_t(x) \geqq F_2(x)$ *for* $t \geqq 3$ *if* $\alpha(1+\alpha) \leqq \frac{9}{4}$.

Lemma 10. $G_t(x) \geqq F_2(x)$ *for* $t \geqq 5$ *if* $\alpha(1+\alpha) \leqq \frac{625}{256}$.

Proofs. Lemmas 9 and 10 follow from Lemmas 6 and 8.

Lemma 11. $\sqrt{x} - \frac{1}{2}x$ *increases for* $0 \leqq x \leqq 1$.

Lemma 12. $(1+x)e^{-x}$ *decreases for* $x \geqq 0$.

Proofs. Lemmas 11 and 12 have obvious proofs.

3. First Improvement of (6)

We can improve (6) to

$$\lambda(n) \geqq \lambda \geqq \frac{8\sqrt{10}-17}{18} = 0{\cdot}461012, \quad \ldots\ldots(30)$$

without much computational work, as follows.

Let $\alpha, x_1, x_2, \ldots$ be positive and (2) be satisfied. Then we can find an increasing sequence of integers $a_1, a_2, \ldots, a_{s+1}$ in accordance with Lemma 1. From (1), (2) and (13),

$$\frac{a_{s+1}-a_1}{n} S_n(x_1, \ldots, x_n) = \sum_{k=1}^{s} \phi_{d_k}(x_{a_k}, x_{a_k+1}, \ldots, x_{a_{k+1}+1}), \quad \ldots\ldots(31)$$

where

$$d_k = a_{k+1} - a_k \geqq 2, \quad \ldots\ldots(32)$$

from (10). From (31), (32) and Lemmas 2, 3 and 7,

$$\frac{a_{s+1}-a_1}{n} S_n(x_1, \ldots, x_n) \geqq \sum_{\substack{k=1 \\ d_k \text{ even}}}^{s} f_{d_k}(x_{a_k}/x_{a_{k+1}}) + \sum_{\substack{k=1 \\ d_k \text{ odd}}}^{s} g_{d_k}(x_{a_k}/x_{a_{k+1}}),$$

since (11) or (12) is satisfied. Hence, by (21) and (22),

$$\frac{a_{s+1}-a_1}{n} S_n(x_1, \ldots, x_n) \geqq \sum_{\substack{k=1 \\ d_k \text{ even}}}^{s} \tfrac{1}{2} d_k F_{d_k}(x_{a_k}^{2/d_k}/x_{a_{k+1}}^{2/d_k})$$

$$+ \sum_{\substack{k=1 \\ d_k \text{ odd}}}^{s} \tfrac{1}{2} d_k G_{d_k}(x_{a_k}^{2/d_k}/x_{a_{k+1}}^{2/d_k}). \quad \ldots\ldots(33)$$

Suppose that $\alpha(1+\alpha)\leqq\frac{9}{4}$. Then, by (32), (33) and Lemmas 5, 6 and 9,

$$\frac{a_{s+1}-a_1}{n}S_n(x_1, \ldots, x_n)\geqq\sum_{k=1}^{s}\tfrac{1}{2}d_kF_2(x_{a_k}^{2/d_k}/x_{a_{k+1}}^{2/d_k})\geqq\tfrac{1}{2}(a_{s+1}-a_1)F_2(1),$$

by the convexity property of $F_2(x)$, given in Lemma 4, and the fact that, as a consequence of (2) and (9),

$$\prod_{k=1}^{s}(x_{a_k}/x_{a_{k+1}})=1. \qquad (34)$$

Hence, by (25),

$$\frac{1}{n}S_n(x_1, \ldots, x_n)\geqq\sqrt{\frac{\alpha}{1+\alpha}}-\frac{1}{2}\,\frac{\alpha}{1+\alpha};$$

and so, from (3) and (4),

$$\lambda(n)\geqq\lambda\geqq\sqrt{\frac{\alpha}{1+\alpha}}-\frac{1}{2}\,\frac{\alpha}{1+\alpha} \qquad (35)$$

when $\alpha(1+\alpha)\leqq\frac{9}{4}$. If $\alpha=1$ we get the inequality (6); and if $\alpha(1+\alpha)=\frac{9}{4}$, so that $\alpha=\frac{1}{2}(\sqrt{10}-1)$, we get the inequality (30). That this is the best inequality, obtainable from (35), for $\alpha(1+\alpha)\leqq\frac{9}{4}$ follows from Lemma 11.

4. Further Improvements of (6)

We next consider $\alpha(\geqq 1)$ satisfying

$$\alpha(1+\alpha)\leqq\frac{625}{256}. \qquad (36)$$

As in § 3, we can obtain (33) in this case also. It is convenient to write

$$\sum_{\substack{k=1\\ d_k\neq 3}}^{s}d_k=p,\quad \sum_{\substack{k=1\\ d_k=3}}^{s}d_k=q \text{ and } p+q=N. \qquad (37)$$

Then,

$$a_{s+1}-a_1=N\geqq p\geqq 0, \qquad (38)$$

from (32) and (37). It is also convenient to write, in conformity with (34),

$$\prod_{\substack{k=1\\ d_k\neq 3}}^{s}\frac{x_{a_k}}{x_{a_{k+1}}}=x \quad\text{and}\quad \prod_{\substack{k=1\\ d_k=3}}^{s}\frac{x_{a_k}}{x_{a_{k+1}}}=\frac{1}{x}. \qquad (39)$$

Then, using (32), (37) to (39) and Lemmas 4 to 6 and 10, we get, from (33),

$$\begin{aligned}\frac{N}{n}S_n(x_1, \ldots, x_n)&\geqq\tfrac{1}{2}pF_2(x^{2/p})+\tfrac{1}{2}qG_3(x^{-2/q})\\ &\geqq-\tfrac{1}{2}p\,\frac{\alpha}{1+\alpha}+p\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}x^{1/p}+\tfrac{2}{3}q\left(\frac{1}{1+\alpha}\right)^{\frac{1}{2}}x^{-3/2q}\\ &\geqq-\tfrac{1}{2}p\,\frac{\alpha}{1+\alpha}+(p+\tfrac{2}{3}q)\left(\frac{1}{1+\alpha}\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}p/(p+\frac{2}{3}q)},\end{aligned}$$

by the inequality of the means. Thus we get, using (37),

$$\frac{1}{n}S_n(x_1, \ldots, x_n) \geqq H(\alpha, p) = -\frac{p\alpha}{2N(1+\alpha)} + \frac{p+2N}{3N}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} \alpha^{1-3N/(p+2N)}. \quad (40)$$

Thus, from (3), (4), (38) and (40),

$$\lambda(n) \geqq \lambda \geqq \max_{\alpha \leqq \alpha_0} \min_{0 \leqq p \leqq N} H(\alpha, p), \quad \ldots\ldots(41)$$

where (36) is equivalent to $\alpha \leqq \alpha_0 = \dfrac{\sqrt{689}-8}{16} = 1{\cdot}14055$. Now

$$\frac{\partial H}{\partial p} = -\frac{\alpha}{2N(1+\alpha)} + \frac{\alpha}{3N}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}\left(1+\frac{3N\log\alpha}{p+2N}\right)\exp\frac{-3N\log\alpha}{p+2N}, \quad \ldots(42)$$

and, in virtue of (38),

$$\tfrac{3}{2}\log\alpha \geqq \frac{3N\log\alpha}{p+2N} \geqq \log\alpha.$$

From Lemma 12, it follows that

$$\frac{1}{3N}\left(\frac{1}{1+\alpha}\right)^{\frac{1}{2}}(1+\tfrac{3}{2}\log\alpha) \leqq \frac{\partial H}{\partial p} + \frac{\alpha}{2N(1+\alpha)} \leqq \frac{1}{3N}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}(1+\log\alpha).$$

Hence, for all p satisfying (38),

$$\frac{\partial H}{\partial p} \leqq 0 \text{ if } \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} \geqq \tfrac{3}{2}(1+\log\alpha), \quad \text{i.e. } \alpha \leqq \alpha_1 = 1{\cdot}08571$$

and

$$\frac{\partial H}{\partial p} \geqq 0 \text{ if } \alpha\left(\frac{1}{1+\alpha}\right)^{\frac{1}{2}} \leqq \tfrac{2}{3} + \log\alpha, \quad \text{i.e. } \alpha \geqq \alpha_2 = 1{\cdot}09277$$

(since $1 \leqq \alpha \leqq \alpha_0$). It is seen that $1 < \alpha_1 < \alpha_2 < \alpha_0$.

If $\alpha_1 \leqq \alpha \leqq \alpha_2$ then $\dfrac{\partial H}{\partial p} = 0$ for some p satisfying (38). For this p, from (40) and (42),

$$\{\alpha(1+\alpha)\}^{\frac{1}{2}} = \tfrac{3}{2}\left(1+\frac{3N\log\alpha}{p+2N}\right)^{-1}\exp\frac{3N\log\alpha}{p+2N} \quad \ldots\ldots(43)$$

and

$$H(\alpha, p) = \tfrac{1}{2}\frac{\alpha}{1+\alpha}\left\{2 - 3\log\alpha\left(1+\frac{3N\log\alpha}{p+2N}\right)^{-1}\right\}. \quad \ldots\ldots(44)$$

If $\alpha \leqq \alpha_1$, $\dfrac{\partial H}{\partial p} \leqq 0$ for all p. Thus, from (40),

$$\min_{0 \leqq p \leqq N} H(\alpha, p) = H(\alpha, N) = \sqrt{\frac{\alpha}{1+\alpha}} - \tfrac{1}{2}\frac{\alpha}{1+\alpha}.$$

Hence we have (35) and thus, from Lemma 11, the best inequality obtainable from (35) is when $\alpha = \alpha_1$. This is

$$\lambda(n) \geqq \lambda \geqq 0{\cdot}461216. \quad \ldots\ldots(45)$$

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If $\alpha \geqq \alpha_2$, $\dfrac{\partial H}{\partial p} \geqq 0$ for all p. Thus, from (40),

$$\min_{0 \leqq p \leqq N} H(\alpha, p) = H(\alpha, 0) = \tfrac{2}{3}\sqrt{\frac{1}{1+\alpha}};$$

and we have

$$\lambda(n) \geqq \lambda \geqq \tfrac{2}{3}\sqrt{\frac{1}{1+\alpha}}$$

which is best when $\alpha = \alpha_2$. The inequality then is

$$\lambda(n) \geqq \lambda \geqq 0{\cdot}460838,$$

which is not so good as (45).

If $\alpha_1 \leqq \alpha \leqq \alpha_2$, from (43) and (44) we find (by computation) that, for α and p satisfying (43), $H(\alpha, p)$ has its maximum value when $\alpha = 1{\cdot}0868$ and $p = 0{\cdot}7214N$. This maximum value is 0·461238. Thus (41) is equivalent to (7) which, we note, is only a slight improvement of (45), which itself is better than (6).

5. Proof of (8)

From (1) and (3), we easily get (8) if we let n be 24 and $x_1, \ldots, x_{24}$ be 0, 15, 0, 17, 0, 19, 0, 21, 2, 22, 5, 21, 7, 18, 7, 16, 6, 14, 5, 13, 3, 13, 1, 14 respectively, in (1), and use considerations of continuity.

6. Addendum to (1)

Near the end of (1) we proved an inequality equivalent to

$$\Sigma_n(x_1, \ldots, x_n) = \frac{4}{n}\sum_{r=1}^{n} \frac{x_r}{3x_{r+1}+x_{r+2}+|x_{r+1}-x_{r+2}|} \geqq 2^{[\frac{1}{2}n]/n} - \frac{[\frac{1}{2}n]}{n} \quad \ldots(46)$$

if (2) is satisfied. We can now prove more, namely, that

$$\inf_{x_1, \ldots x_n} \Sigma_n(x_1, \ldots, x_n) = 2^{[\frac{1}{2}n]/n} - \frac{[\frac{1}{2}n]}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(47)$$

if (2) *is satisfied.*

Proof. For even n, (47) follows from (46) since we have equality in (46) if

$$x_1 = x_3 = \ldots = x_{n-1} \text{ and } x_2 = x_4 = \ldots = x_n = (\sqrt{2}-1)x_1,$$

when
$$\Sigma_n = \frac{x_1}{x_2+x_3} + \frac{x_2}{2x_3} = \sqrt{2} - \tfrac{1}{2}.$$

For $n = 1$, (47) is trivially true. For odd $n > 1$, (47) follows from (46) since we have equality in (46) if

$$x_1 : x_2 : x_3 = x_3 : x_4 : x_5 = \ldots = x_{n-2} : x_{n-1} : x_n = 2^{-1/n} : 2^{(n-1)/2n} - 1 : 1,$$

when
$$\Sigma_n = \frac{n-1}{n}\left(\frac{x_1}{x_2+x_3} + \frac{x_2}{2x_3}\right) + \frac{x_n}{nx_1} = 2^{(n-1)/2n} - \frac{n-1}{2n}.$$

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