EQUATION OF STATE AT DENSITIES GREATER THAN NUCLEAR DENSITY

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Abstract. An equation of state is developed for densities from nuclear density (3×10^{14} g cm^{-3}) to about 10^{16} g cm^{-3}. The repulsive interaction between baryons dominates and empirical arguments for its existence are given. This interaction is attributed to vector meson exchange, and is derived from classical field theory whereupon a Yukawa potential results. The potential actually assumed is a modification of the Reid potential. Arguments are given that the baryons will not form a crystal lattice. The actual calculations were done using Pandharipande's method. The particles present at high density certainly include nucleons, Λ and Σ. The presence of Δ is questionable but that of π is likely. Results are given for the concentration of various species. With the more likely assumption about interactions, the concentration of each permissible species of particle is about equal at $\varrho = 10^{16}$ g cm^{-3}. The relation between energy and density is nearly independent of the assumptions on the species permitted and the energy is about 3 GeV particle^{-1} at $\varrho = 10^{16}$ g cm^{-3}. The relation between pressure and energy density is given, which yields a sound velocity equal to c at a few times 10^{15} g cm^{-3}. Results for the structure of neutron stars are given. The maximum mass is about 2 solar masses and the maximum moment of inertia 10^{45} g cm^2.

1. Density Ranges of Interest

We may distinguish the following ranges of density:

(a) $\varrho < 3 \times 10^{11}$ g cm^{-3}. In this density range we have neutron-rich nuclei and degenerate electrons.

(b) $3 \times 10^{11} < \varrho < 2 \times 10^{14}$ g cm^{-3}, which means less than 0.12 particles fm^{-3}. In this range, we have free neutrons, and some neutron-rich nuclei. Electrons compensate for the charge of the nuclei. With increasing density, the nuclei occupy an increasing fraction of the space.

(c) $2 \times 10^{14} < \varrho < 10^{15}$ g cm^{-3}, i.e. between 0.12 and 0.6 nucleons fm^{-3}. In this range, there are neutrons, a few percent protons, and an equal number of electrons as protons.

(d) $\varrho > 10^{15}$ g cm^{-3}, i.e. more than 0.6 nucleons fm^{-3}. In this density range, we have neutrons, protons, and excited baryons, especially Σ. There are few electrons. There may be some π^- at the higher densities.

Density range (a) has been well treated by Baym *et al.* (1971b). Density range (b) is dominated by the free neutrons whose equation of state has been calculated by Siemens and Pandharipande (1971) and is reported in some detail by Baym *et al.* (1971a). The behavior of the nuclei in this range is reported in this volume by John Negele. The pairing of neutrons gives a very important contribution to the energy; this has been calculated by Yang and Clark (1971), and has been combined with Siemens' equation of state by Bethe (1971).

In density range (c), the equation of state can be obtained directly from nuclear matter theory; this has been done by Siemens and Pandharipande (1971).

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In this paper we are concerned with range (d). We shall restrict this range at the upper end by requiring $\varrho < 10^{16}$ g cm^{-3} because this is the highest density which is likely to occur in neutron stars. By doing this we avoid complicated problems regarding higher excited states of baryons, and perhaps some others.

2. Repulsive Interaction

2.1. Empirical

At high densities, the baryons will come very close to each other; the radius of the sphere containing one baryon is about 0.35 fm at $\varrho = 10^{16}$ g cm^{-3}. It is very difficult to get direct experimental evidence on the interaction of nucleons at such short distances. Experiments at energies $E < 400$ MeV involve de Broglie wavelengths > 0.5 fm, and are therefore not able to give fine details of the interaction at shorter distances. Experiments at higher energy involve inelastic scattering of nucleons in which pions or heavier mesons are produced; it is impossible to deduce a potential from these experiments.

It must be recognized that the high density problem is not similar to the high energy problem. In fact, we know that neutron stars are at very low temperature, perhaps 10 keV, so that the baryons take the lowest energy configuration compatible with the density. There is no excess energy. The momenta of the baryons may be quite high especially when two of them come close to each other, but their energy is not. The situation is therefore entirely analogous to that in normal atomic nuclei, and in low energy scattering experiments, and we should get our guidance from these.

All potentials that have been proposed to explain low energy scattering include a strong repulsive core. The existence of this repulsion is shown directly by the phase shift of the *s*-state scattering at energies above 300 MeV. This core is also essential to explain the saturation of nuclear forces. Calogero and Simonov (1969, 1970a, b, 1972) have shown that it is necessary to have an ordinary repulsive force at short distances which can overcome other forces such as tensor, spin orbit, and exchange. If such a force does not exist, heavy nuclei will collapse.

It has been suggested that perhaps the repulsion derived from scattering data exists only at intermediate distances like 0.3–0.7 fm, and is followed by an attraction at smaller distances. It can easily be shown that such an assumption would lead to collapse of nuclei if the volume integral of the potential for $r < 0.7$ fm is zero or negative. In view of all this, we shall assume that the observed repulsion persists to small r.

2.2. Vector meson exchange

A natural explanation of the repulsion is given by the exchange of vector mesons, chiefly the ω meson which has isospin 0, and is neutral. The interaction due to a neutral vector meson field is very similar to the familiar interaction due to the electromagnetic field, except that the mesons have finite mass while light quanta have zero mass. It was shown in the early days of meson theory that such a field has a static limit (Wentzel, 1949). It is possible to separate the effect of stationary, point nucleons from any effects due to the motion of the nucleons. This is similar to the separation of the Coulomb force from magnetic interactions in electrodynamics. The analog of the Coulomb force is the Yukawa force,

$$Y = g^2 e^{-\mu r}/r, \qquad (1)$$

where g is the coupling constant and μ the reciprocal Compton wavelength of the meson.

Corrections to (1) are proportional to v^2/c^2, where v is the velocity of the nucleons. In our theory it will turn out that v/c is generally less than 1/2 so that these corrections are small. They can be calculated. Renormalization terms are expected to be of the relative order, $\hbar c/g^2$ which in our case is a small number, about 0.1 or less, and they also contain at least a factor of v^2/c^2. We therefore claim that the major interaction at small distances is given by a sum of Yukawa terms (1).

The great advantage of this classical treatment compared to the usual treatments of high energy theory is that the latter require the separate consideration of the exchange of one, two, or more mesons. As more mesons are exchanged, the theory becomes very complicated with cuts, many Riemann sheets, etc. In our case it will turn out that $g^2/\hbar c$ is about 20 so that 20 or more mesons can easily be exchanged. The classical treatment is precisely the formalism adapted to this situation.

In addition to the repulsion, there is a somewhat weaker attraction at intermediate range as is shown clearly by low energy scattering experiments. This is generally attributed to the simultaneous exchange of two pions which together behave much as a scalar particle, $s=0$ and isospin$=0$. This quasi-particle is often called σ.

The most important exchanged particles, ω and σ, are both isoscalar, $t=0$. Therefore they should interact equally with any baryon. This will be assumed in our calculations, and the coupling constant g will be derived from nucleon interaction (see Section 4).

3. Method of Calculation

3.1. Nuclear Matter Theory

In our work, we first tried to use nuclear matter theory. This however is not applicable when the parameter κ of that theory becomes greater than about 1/2. This is the case in density range (d). Our attempts to use nuclear matter theory lead to very peculiar results which are clearly false.

3.2. Crystal Lattice

Next we assumed that the neutrons (or other baryons) form a regular crystal arrangement at high density. In fact, this has been suggested by Anderson and Palmer (1971), in analogy with condensed helium. Some calculations of this type will be reported in this volume by Canuto.

We do not believe that neutron and baryon matter at high density is crystalline. At very high density the interaction (1) goes over into the Coulomb interaction. It is well known to solid state physicists that particles repelling each other with Coulomb

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forces, and having strong quantum effects, do not form a crystalline lattice at high density but only at low density. Therefore there should not be crystal structure for, say, $\mu r_0 < 1$. If μ is taken to correspond to ω mesons this means $r_0 < 0.26$ fm.

There is still the possibility of crystal structure at intermediate densities. However, the interaction (1) is rather soft. Near a given r it will behave as r^{-n} where

$$n = 1 + \mu r. \tag{2}$$

Now clearly r must be chosen small enough that the repulsive potential dominates over the attractive potential. If we assume it to be twice as large as the attractive potential, then for interaction II of Table I, we get $n = 2.6$; for interaction III n is hardly greater than unity. A soft potential is completely different from the $n = 12$ Lennard-Jones potential which is customary for condensed helium. Therefore the analogy between helium and nuclear matter is not valid, and there is no reason to assume crystal structure.

This agrees with our own experience. Coldwell (1972) has tried to calculate dense neutron matter using a crystal model. His energy, derived from a variational calculation, turned out much greater than ours, calculated from a liquid model, indicating that there is no crystallization. Pandharipande in 1970 found no significant difference between the crystal and the liquid energy. Johnson and I, using the Guyer-Zane method for treating quantum crystals, found an energy somewhat lower than Coldwell but still much higher than our liquid calculations. We therefore do not believe that there is a crystal arrangement at any of the densities we have considered.

3.3. Actual method used

We use the method developed by Pandharipande which is a special case of the Jastrow method. He will himself report on his method in this volume. Johnson and I have used the simplest form of Pandharipande's method which he calls "lowest order constrained variation." In his paper, Pandharipande will show that this method agrees with a more sophisticated cluster expansion to within 5 or 10% for dense nuclear matter. He will also show that his method gives excellent results for liquid ^{4}He and ^{3}He. The case of nuclear matter is simpler than that of liquid helium, (i) because the repulsion is softer, and (ii) because both potential and kinetic energy are positive so there is no cancellation.

4. Details of the Potential

4.1. The Reid potential and modifications

It is clear that we need a soft core repulsion to make the calculations succeed. If a hard core was assumed, as in the Hamada-Johnston potential, then the highest possible density would be one in which all the hard cores touch and form a close packed lattice. Beyond this density, nuclear matter could not exist. This is clearly absurd, as is also the physical concept of a potential which goes to infinity at some distance.

The first choice is therefore the Reid (1968) potential. This is the only available potential which has a core of Yukawa shape (1) which we have argued to be the reasonable behavior of the interaction at short distances. The Reid potential also has the proper behavior at large r, namely one-pion exchange. In fact, the potential is of the form

$$V = \sum c_n e^{-nx}/x\,, \qquad x = \mu_\pi r\,, \tag{3}$$

where μ_π is the reciprocal Compton wavelength of the pion, $\mu_\pi = 0.7\ \text{fm}^{-1}$. The c_n are coefficients which are determined by a phenomenological analysis of nucleon scattering up to 400 MeV. There is one c_n, usually $n=7$, which is repulsive and is interpreted as the exchange of ω mesons. A term $n=4$ or 3 has negative c_n and represents the attraction due to exchange of σ mesons. Finally, $n=1$ represents one-pion exchange and is a small contribution for our purposes.

The trouble with the Reid potential for our purpose is that it has very different form for states of different angular momenta L, S and J. This was acceptable for the purpose for which Reid constructed his potential, namely to form a basis for calculations of nuclear matter at normal density. However, in our case we want especially the repulsive term to have the same n for all LSJ. In Reid's potential, the repulsion is $n=7$ for even-l states, $n=6$ for $^3P_2$, but $n=3$ for $^3P_1$ and ^1P. The $n=3$ makes no sense in connection with ω-mesons.

Since we need uniformity in the repulsive force, Pandharipande took all odd-l states to have $n=6$. Mikkel Johnson and I took all states to be $n=7$ because the repulsion in the ^1S state is best determined, and this was chosen to be $n=7$ by Reid. We also took the coefficient c_7 to be the same for all states LSJ. This will be further discussed in sub-Section 4.4. In our most recent calculations, we changed to $n=5.5$ (see sub-Section 4.2) (Interaction III).

The intermediate range attraction for odd-l states was taken by Pandharipande to be the same as $^3P_2$. Johnson and I assumed $n=4$ in our older calculations; in our newest calculation Johnson found $n=3.5$ to be the best fit to scattering data. Table I shows the potential constants c_n used in various calculations; I, II and III are successive models used by Johnson and Bethe.

The table shows that generally odd-l states have less attraction than even-l states. Our odd-l attraction is less than that chosen by Pandharipande. This will be important in the following: The attraction in the ^1D state is less than in $^1S_1$ – this is definitely established by the scattering data. We assumed the ^1D interaction to persist for all higher even-l states, and likewise the Pandharipande interaction to be valid for all odd-l states.

4.2. Modified range of repulsion

The mass of the ω meson is 5.5 times that of π; therefore the chief repulsion should really be $n=5.5$. Johnson is now making new fits using $n=5.5$ for even-l. For odd-l, we still use $n=6$. The corresponding coefficients are listed in Table I. The coefficients are of course smaller if n is smaller. The effect of the repulsion both on nuclear scattering data and on high density nuclear matter is approximately proportional to the

TABLE 1

The coefficients c_n and 'masses' n in the Reid-type interactions of Equation (3). 'Pand' is the interaction used by Pandharipande (1971a, b). The others are 3 interactions used by Johnson and Bethe. The c_n are in MeV

Interaction		n	c_n	n	c_n
Pand.	$l=0$	4	-1650	7	6484
	l even $\neq 0$	4	-1113	7	6484
	l odd	4	-933	6	4152
I	l even	4	-1113	7	6484
	l odd	4	-370	7	6484
II	^1S	4	-1650	7	6484
	^3S	4	-934	7	6484
	l even $\neq 0$	4	-1113	7	6484
	l odd	4	-774	7	6484
III	^1S	3.5	-1240	5.5	3120
	^3S	3.5	-650	5.5	3120
	even $l \neq 0$	3.5	-800	5.5	3120
	odd l, $S=1$	4	-808	6	3750
	odd l, $S=0$	3	-600	6	12140

volume integral of the interaction which is proportional to

$$\int V \, \mathrm{d}\tau \approx c_n/n^2 . \tag{4}$$

This quantity is nearly preserved when going from $n=7$ to the new data with $n=5.5$.

4.3. Comparison with High Energy Data

High energy experiments give results for $g^2/\hbar c$. For ω, the best value is about 8 to 10. With our form (3) of the interaction we have simply.

$$g_n^2/\hbar c = c_n/m_\pi . \tag{5}$$

Taking the mass of the pion $m_\pi = 140$ MeV, Johnson's coefficient $c_{5.5} = 3120$ gives

$$g_\omega^2/\hbar c = 22 . \tag{6}$$

This is more than twice the high energy experimental value. The discrepancy is unexplained. According to a conversation with J. Hamilton there seems to be some extra repulsion of short range between nucleons and some mesons. The cause for this repulsion is unknown. Some such repulsion may also act in our case, and contribute to our repulsive core. We believe that the nucleon scattering experiments are more reliable evidence than the high energy experiments.

The medium range attraction has the correct range for an exchange of two pions, taking into account that these pions will have some relative kinetic energy. Theoretical attempts (Chemtob *et al.*, 1971) to explain the medium range attraction have so far given too small a coefficient for the attraction. G. Brown (private communication) believes that the attraction between the two pions has to be taken into account to increase this coefficient. In the meantime, the phenomenological value is the only reliable one.

4.4. Interaction between various *LSJ* states of the nucleons

Two vector mesons are known to exist, ω and ϱ, both of about the same mass, i.e. 770 MeV. The ω meson has the same kind of coupling as the electromagnetic field, that is the interaction energy is proportional to $\mathbf{j}\cdot\mathbf{A}$. The ϱ meson has negligible $\mathbf{j}\cdot\mathbf{A}$ coupling but appreciable Pauli coupling proportional to $\boldsymbol{\sigma}\cdot\mathbf{B}$. The resulting central force can then be shown to be proportional to

$$g_{\text{rep}}^2 = g_\omega^2 + g_\varrho^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,. \tag{7}$$

The operator multiplying g_ϱ^2 has the values

$$\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 = \begin{cases} -3 \text{ for } L \text{ even} \\ +1 \text{ for } L \text{ odd, } S=1 \\ +9 \text{ for } L \text{ odd, } S=0\,. \end{cases} \tag{8}$$

Qualitatively therefore the repulsion should obey the inequality

$$g^2(^1S) = g^2(^3S) \leqslant g^2(^3P) \leqslant g^2(^1P)\,. \tag{9}$$

We are constructing our potentials so as to obey this inequality. The potential II used by Johnson and me in most of our calculations puts all g^2 equal; this amounts to assuming no coupling to ϱ. If we take (8) seriously, we should have

$$g^2(^3P) = \tfrac{2}{3}g^2(S) + \tfrac{1}{3}g^2(^1P)\,. \tag{10}$$

There is evidence that $g^2(^1P)$ is indeed very large.

The attraction is purely empirical, as was mentioned in sub-Section 4.3. The attraction for ^3P is less than ^1S. There is good evidence from scattering data that the central attraction for ^3S is also smaller than ^1S. For simplicity in calculation, we have chosen the ^3S attraction such that the mean potential for the np interaction is equal to the potential for the ^1D state. This means

$$\tfrac{1}{4}g_\sigma^2(^1S) + \tfrac{3}{4}g_\sigma^2(^3S) = g_\sigma^2(^1D)\,. \tag{11}$$

At low density, the ^3S state is made very attractive by the action of the tensor force. At densities higher than nuclear matter, this stops being true because the tensor forces saturate (Bethe, 1971). We have assumed that for our high densities, the tensor force has no effect at all. Since the central force in the S state is less attractive for unlike particles than for like ones, this means that it is more advantageous to have like particles in high density matter. The effect of this will be apparent in the results, Section 6.

4.5. Other baryons

Using SU_3 symmetry, and remembering that both the ω and the σ mesons have zero isospin, it is reasonable to assume that the interaction of excited baryons, such as

Λ, Σ and Δ, are the same as for nucleons. We have therefore made this assumption. Suitable averages are taken over spin and isospin of these particles when they interact.

Experimental information exists essentially only on the ΛN interaction. This is not enough for any firm answer, especially because it is restricted to low energy. However, the information is compatible with our assumption, taking into account that there is no spin exchange between nucleons and Λ.

5. Particles Present

5.1. Particles possible

The particles that could in principle be present in nuclear matter are listed in Table II, with their masses. We have omitted particles heavier than Δ because their effect on the equation of state is likely to be small. Even the effect of Δ is not very large. At densities $\varrho > 10^{16}$ g cm^{-3}, this situation may change.

TABLE II

Masses of the lowest-energy states of the baryon, and of 3 important mesons, in MeV

Particle	MeV
Neutron, proton	939
Λ	1116
$\Sigma(+, 0, -)$	1193
$\Delta(+, 0, -)$	1236
$\Xi(0, -)$	1317
π	140
ϱ	765
ω	784

The reason for the appearance of heavier baryons is that thereby the Fermi momentum (k_F) may be reduced. Both kinetic and potential energy (Section 7) are decreased by this fact. Once a pure neutron gas would have a very high k_F, it pays to spend the extra energy involved in the higher rest mass of the heavier baryons, in exchange for smaller k_F.

Another important point is the charge. Protons have the same mass as neutrons, but their charge must be compensated by negative charges. If electrons are used for this purpose, their Fermi energy is extremely high for a given momentum because they move essentially with velocity of light. Muons offer no advantage. At some high density, possible π^- may be useful (see sub-Section 5.4). But the cheapest way to provide negative charge is by using a negative baryon, such as Σ^- or Δ^-. Then k_F is reduced at the same time.

The Δ particles have spin 3/2, therefore each of them has twice the statistical weight of either neutron or proton. The relative statistical weights of the particles (NP), ($\Lambda\Sigma$), (Δ^{0+-}) are therefore 2, 4 and 6.

5.2. Problems concerning the Δ

5.2.1. *The Size*

The Δ may be considered as a π bound to a nucleon N. In this view, the Δ will have a fairly large size. One may then expect that at high pressure, the pion is squeezed out, just like an atom suffers pressure ionization. Like the electrons in the atomic case, the π may then move freely through the baryon matter, and the Δ ceases to exist.

5.2.2. *Self-Energy*

Sawyer (1972a) has pointed out that Δ decays rapidly into nucleon plus π which gives the Δ state a width of about 100 MeV. Connected with the decay, by dispersion theory, is also a real correction to the mass (mass renormalization). Now the Pauli principle inhibits decay to $N+\pi$, because many neutron states are occupied. Therefore it changes the mass correction, and it can easily be shown that the mass of Δ in a sea of nucleons will increase, by one to a few hundred MeV.

Sawyer's argument is almost certainly correct in a theory which only enumerates the particles, and will in this case hold *a fortiori* for heavier baryon states with larger width. However, if the *interaction* between baryons is taken into account as in our theory, M. Johnson has questioned Sawyer's argument because the Goldstone formalism of nuclear matter theory states the Pauli principle should not be taken into account in intermediate states. At present, I am inclined to believe self-energy is different from nuclear interactions, and that Sawyer's argument applies even in our theory. This, of course, reduces the concentration of Δ at any given density.

5.2.3. *Pauli Principle*

The Pauli principle might also act as follows: A π may be transmitted from baryon 1 to baryon 2, thus,

$$\Delta_1 + N_2 \rightarrow N_1 + \Delta_2, \tag{12}$$

i.e. the particle 1 transforms from a Δ into a nucleon while particle 2 makes the reverse transformation. In this case we are not talking about an intermediate state but about the fact that we cannot assign to a given baryon the property of being Δ.

When the transformation (12) occurs, the new nucleon N_1 should show anti-symmetry with all the nucleons already existing. Likewise the old nucleon N_2 should be part of the anti-symmetric wave function. This would indicate that Δ's and nucleons together should satisfy antisymmetry. Therefore we would not increase the available momentum space or statistical weight when we introduce Δ.

However, the correct answer seems to be that the one-pion exchange (12) is simply forbidden if the nucleon state N_1 and/or the Δ state Δ_2 is in the Fermi sea. We therefore believe that Δ and nucleon may occupy the *same* momentum states, and that the introduction of Δ *does* increase the available phase space.

The status of Δ must be considered moot, especially because of the size effect (argument I). In our calculations we have included Δ at the observed mass, but its exclusion or a mass increase is not likely to change the relation between pressure and density significantly.

5.3. π^- MESONS

If Δ^- is not included among the particles, the π^- probably should be. At low energy, π^- is repelled by a neutron because π must be in a relative S state. This increases the effective mass of π^- to

$$m_{\text{eff}}(\pi^-) = 140 + 220(\varrho_n - \varrho_p), \tag{13}$$

(see Bethe, 1971, p. 158) where ϱ_n and ϱ_p are the densities of neutrons and protons in particles fm^{-3}. At high density however, π may be in a P state in which case it is attracted to the nucleon.

5.3.1. *Sawyer's Model*

Sawyer (1972b) suggested that there be a close coupling between a neutron of momentum $\mathbf{p}$ with a proton of momentum $\mathbf{p}-\mathbf{q}$ plus a pion of momentum $\mathbf{q}$. He assumes that all π^- are in the same quantum state of momentum $\mathbf{q}$. This coupling is similar to the 'phase locking' of two atomic states by a strong laser beam in resonance. This possibility certainly exists, and was further elaborated upon by Scalapino (1972). The total energy will be decreased by the close coupling, but we have calculated that this decrease is only about 1% of the energy due to the repulsive potential (1) (at high density).

5.3.2. *Scattering*

We thought at one time that an even greater decrease of energy could be obtained by forward scattering of π by the nucleons. However, it appears that the mechanism for forward scattering essentially leads back to Sawyer's theory.

We have not investigated the relation of this model to the assumption of Δ. It is clear that the use of a negative potential energy for π is closely related to the existence of Δ, and that the Δ in this model is only transient.

5.4. CONCENTRATION OF SPECIES

This is obtained by minimizing the total energy E as a function of the concentrations c_k of the various species. Equivalent to this is the requirement that the chemical potential

$$\mu_k = \frac{\partial E}{\partial c_k} \tag{14}$$

be the same for all species of the same charge.

At the same time one has to satisfy the condition of charge neutrality. This requires among other things that

$$\mu_e = \mu_\mu = \mu_0 - \mu_+ = \mu_- - \mu_0, \tag{15}$$

where μ_0 is the chemical potential for neutral baryons, μ_+ and μ_- that of positively and negatively charged ones.

6. Results

6.1. Concentration of species

The calculation was made by Mikkel Johnson, using interaction I of Table I. The result is shown in Figure 1. The concentration of every species approaches

$$\varrho_k \approx \tfrac{1}{2} \text{ baryon fm}^{-3} \tag{16}$$

at high density. The concentration for each species of Δ is twice this amount because of the spin of 3/2. The fact that ϱ_k remains small keeps k_F down to about 2.5 fm^{-1}, corresponding to a velocity $v_F = 1/2\, c$. This is the justification for our use of non-relativistic Schrödinger theory in all our calculations, and also for the neglect of magnetic-like forces, cf. Section 2. In Section 7 we shall discuss the effect of this small k_F on the total energy.

The concentration of Σ^- begins to increase at rather low total density, about 0.3 fm^{-3}. Δ^-, because of its higher mass, follows later. The neutral species, even Λ,

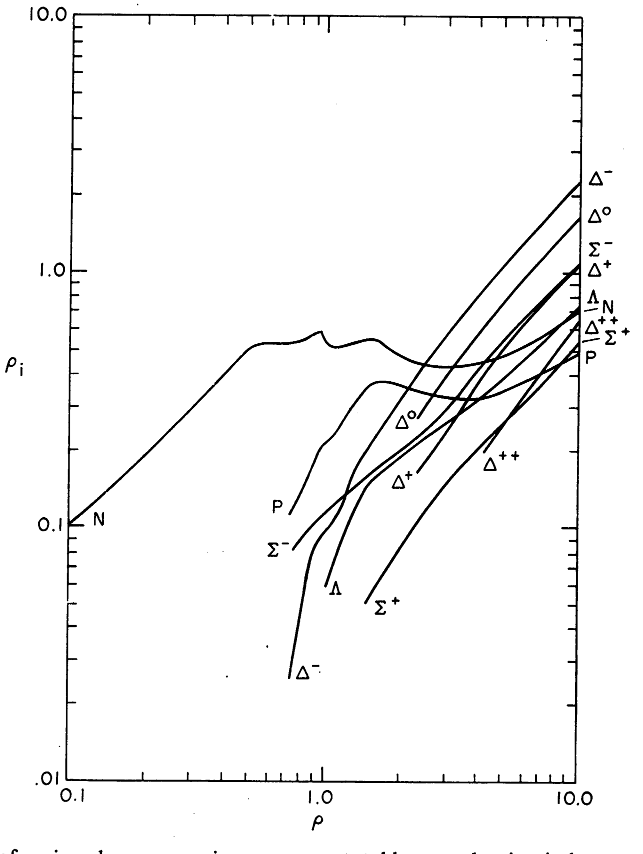

Fig. 1. Density of various baryon species, ϱ_i, versus total baryon density, in baryons fm^{-3} (1 baryon fm$^{-3} = 1.66 \times 10^{15}$ g cm^{-3}). Interaction I of Table I was assumed.

come in substantially later because they do not contribute to neutralizing the protons. Σ^+ and Δ^+ are even less important, and Δ^{++} has minimal concentration even at high density.

Similar results had been obtained earlier by Pandharipande (1971a). He also calculated a pure neutron gas (Pandharipande, 1971b). His interaction is also given in Table I; it is more attractive especially in odd states.

A second calculation was done by Johnson using Interaction II (Figure 2). This interaction has been described in Section 4.4. The ^3S attraction has been reduced, and the ^1P repulsion increased. Both of these facts make the interaction between unlike particles more repulsive. This favors a pure neutron gas. Indeed, the neutron partial

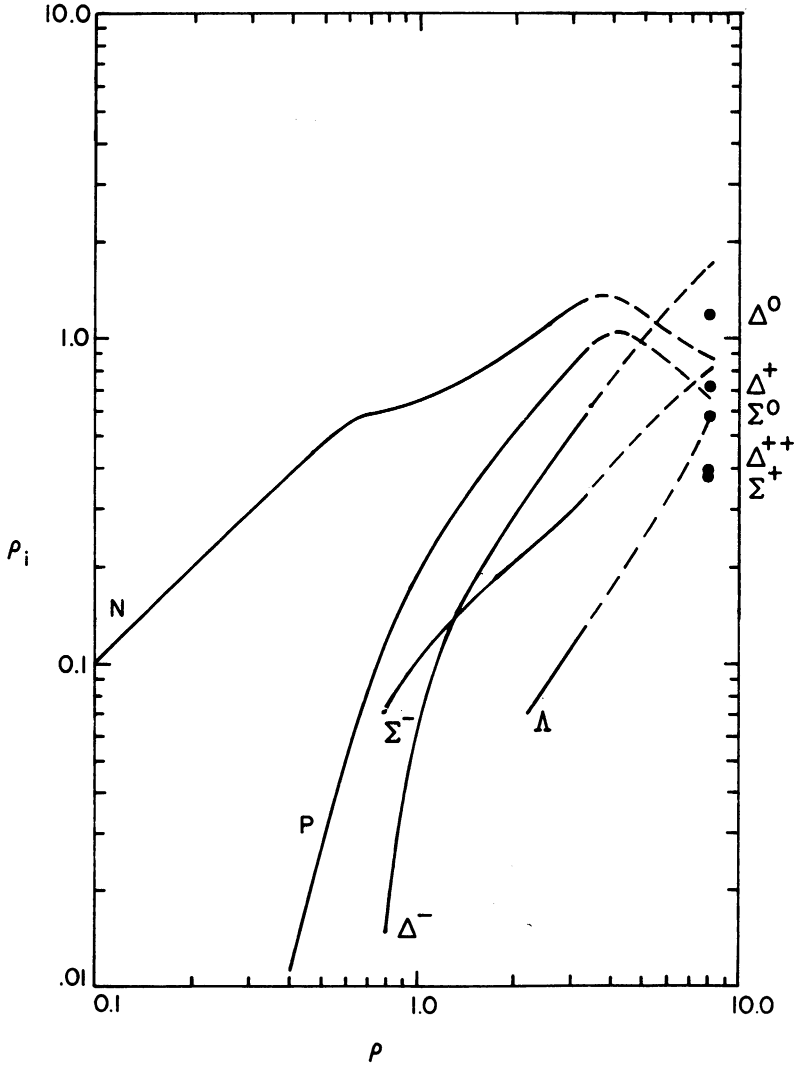

Fig. 2. Same as in Figure 1, but with interaction II of Table I.

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density goes up to about 1 fm^{-3}, and the other particles stay lower. In this calculation, as with Interaction I, the existance of Δ has been assumed. The qualitative features discussed with Figure 1 persist to an even greater extent.

Pandharipande, using somewhat more radical assumptions about the difference in potential between like and unlike nucleons, found that a pure neutron gas persists to the highest density. All this shows that the concentration of species is rather sensitive to the assumed interaction between baryons.

6.2. Energy versus density

Having obtained the concentrations we can now calculate the total energy per baryon in high density matter. This energy excludes the mass energy Mc^2.

Figure 3 compares three calculations. The upper solid curve was obtained by Johnson and Bethe, using interaction I and assuming all excited baryons. It is rather

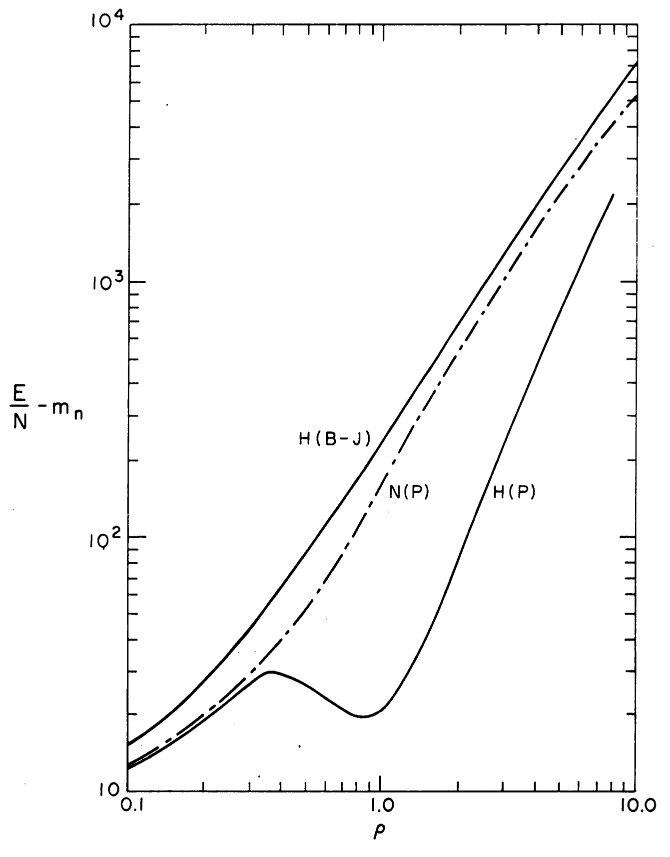

Fig. 3. Energy per baryon (excluding the rest energy of the neutron, $M_n c^2$) in MeV, versus density in baryons fm^{-3}. N(P) = pure neutron matter, according to Pandharipande (interaction Pand. of Table I). H(P) same interaction, but including excited baryons (hyperons and Δ) in the calculation. This is interaction A of Pandharipande (1971b). H(B-J) interaction I of Table I, including excited baryons.

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close to the dashed curve which is Pandharipande's result for pure neutron matter, using his interaction as given in Table I. The lower solid curve is Pandharipande's (1971a) result for a gas including excited baryons; it is substantially lower than that of Johnson and Bethe. The reason for this is of course that Pandharipande used a smaller repulsive, and greater attractive force for odd states than Johnson and Bethe.

The Pandharipande excited curve has the peculiarity that it shows a decrease between densities 0.4 and 1 fm^{-3}. This would mean negative pressure. In reality, one must make the Maxwell construction which then shows a phase transition, going from about 0.2 to 1 fm^{-3} at constant pressure. If this equation of state were correct, such a phase transition would have to occur. I have thought briefly what this might mean to neutron stars, and it does not seem to be useful to explain any of their properties, especially because it would occur at rather low pressure. Actually we do not think that Pandharipande's choice of force constants is justified; therefore his solid curve in Figure 3 should be disregarded.

Figure 4 shows calculations using interaction II (solid curve). The dashed curve would result if only neutrons existed in the medium. It is seen that the effect of excited baryons on the energy is quite small, *viz.*, a reduction of energy by about 20%. Therefore, the resulting equation of state is not sensitive to the assumed baryon interaction. Comparison between Figures 3 and 4 shows also rather small differences between interactions I and II.

The lower solid curve in Figure 4 was obtained by Pandharipande assuming excited baryons, but reducing all attractive potentials by 10% as compared to his lowest curve in Figure 3. It is seen that this small change of interaction has eliminated the peculiar phase transition of Figure 3.

Figure 4 shows that the energy at a density of 10^{16} g cm^{-3} (6 baryons per fm^3) is about 3 GeV per particle or three times the rest energy.

As has already been stated, Pandharipande finds that a consistent cluster expansion will reduce the energy of high density nuclear matter by about 5–10%.

6.3. Pressure

Figure 5 plots the ratio

$$p/\varepsilon \tag{17}$$

against ε. Here ε is the energy density while p is the pressure. These two quantities have the same dimension. We plot p/ε because a plot of p itself would go through so many decades that it would not be very revealing. The energy density ε includes the rest mass energy Mc^2. Figure 5 shows that as ε increases from 10^2 to 10^4 MeV fm^{-3}, the pressure goes from about 1% of ε to 100%. At still higher energy densities, $p > \varepsilon$. An ε of 10^4 corresponds to about 4 baryons fm^{-3}.

The fact that p exceeds ε is slightly disturbing. We should expect that the sound velocity

$$c\, dp/d\varepsilon \tag{18}$$

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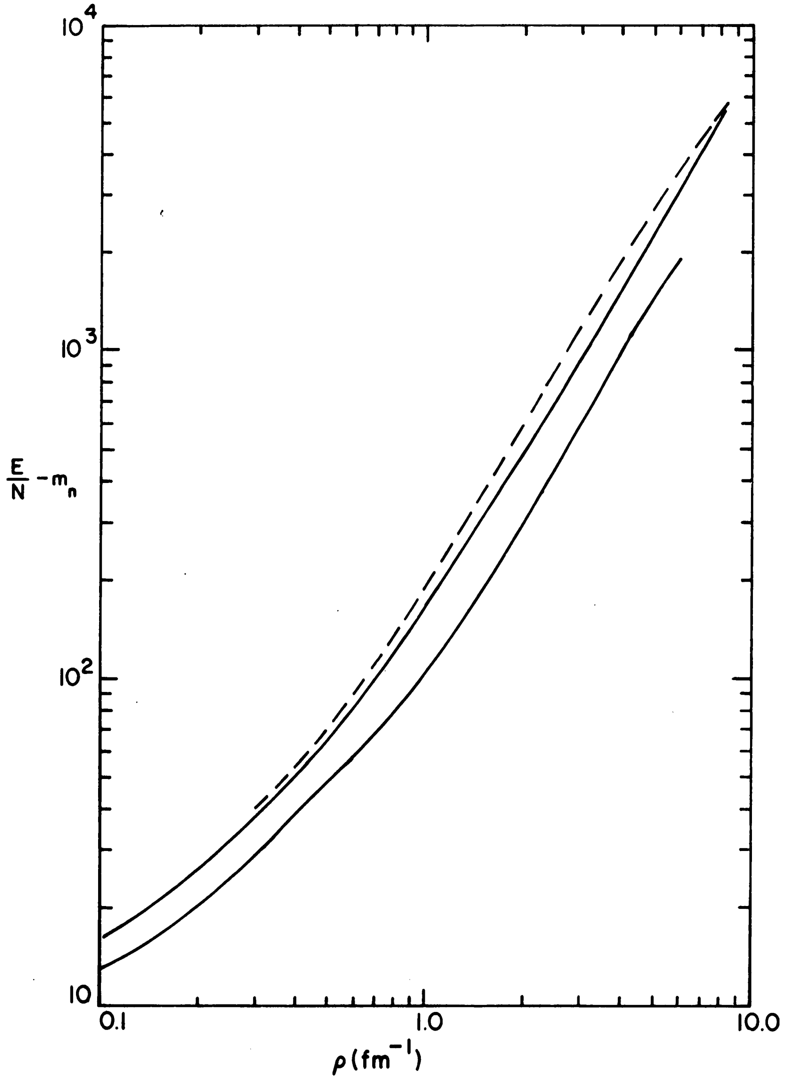

Fig. 4. Same as Figure 3, but upper solid curve is interaction II of Table I, including excited baryons. Dashed curve same with neutrons only. Lowest curve:Pandharipande (1971b), interaction C (with excited baryons).

should be less than the velocity of light, c. The reason that our theory violates this is due to the neglect of special relativity. It has been shown from general principles by Aichelburg *et al.* (1971) that in fact the sound velocity will never exceed the velocity of light. As a practical prescription one might use our theory up to the point where (18) becomes c, and thereafter replace (18) simply by c.

The two curves in Figure 5 refer to interactions I and II.

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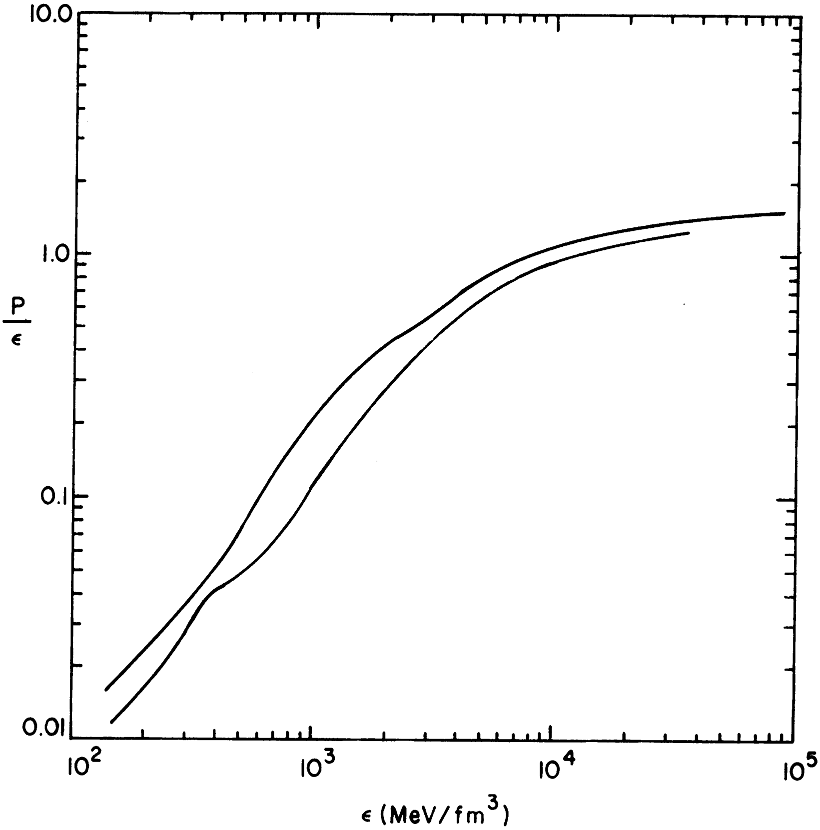

Fig. 5. The ratio of pressure P to energy density ε, a dimensionless quantity, is plotted versus ε itself. The ε includes the rest energy of the neutrons, and is given in MeV fm^{-3}; one of these units equals 1.6×10^{33} erg cm^{-3}.

7. Some Features of the Results

7.1. Contribution of various parts of the interaction

Our potential is more attractive in even than in odd states. Therefore the potential energy will be smaller if the relative momentum of two interacting baryons is small, because in this case $L=0$ states are favored.

It is convenient and possible in our density range to expand the potential energy is powers of the relative momentum, and hence of k_F. Of course only even powers will contribute. Table III lists these contributions for various densities. The contribution proportional to k^2 is always positive (repulsive), corrected by a negative contribution proportional to k^4. The sum of these momentum-dependent contributions to the potential energy is equal to the kinetic energy at $\varrho=0.5$ fm^{-3}; at higher density, the potential energy contribution is much larger. Therefore the advantage of having low k_F comes chiefly from the potential energy and not, as might be suspected, from the kinetic.

TABLE III

Contributions to the energy in MeV per particle, from various parts of the interaction (see Section 7 of text). ϱ is the density in baryons fm^{-3}

ϱ	k^0	k^2	k^4	Kin
10	5340	1950	−520	83
4.5	1480	717	−175	50
1.5	120	308	−102	54
1.0	14	186	−54	54
0.5	−49	101	−25	74
0.25	−33	32	−6	47
0.1	−16	7	−0.5	26

The contribution independent of momentum is negative up to $\varrho = 0.9\ fm^{-3}$ because for these low densities the attraction dominates over the repulsion (disregarding the k^2 terms). At higher density the repulsion dominates. At $\varrho = 4.5\ fm^{-3}$, the momentum-independent part is much greater than the momentum-dependent part of the potential energy; from this point on it does not make very much difference whether we have even or odd states.

7.2. Mass of the Nucleons

It may seem somewhat alarming that the potential energy is as much as 3 Mc^2 at high density. Is it then justified to use the normal rest mass of the nucleon?

The answer is affirmative because the quantity which occurs in the Klein-Gordon equation is

$$(E - V(r))^2 - M^2c^4. \tag{19}$$

Now the total energy arises primarily from the average potential energy, and therefore $E - V(r)$ is very close to Mc^2 for nearly all r, except when two nucleons come very close together – but in that case the wave function is very small. The potential is very uniform at high density, and E simply reflects the existence of a large positive average potential. Since (19) is likely to be small, nonrelativistic theory is justified, and the ordinary nucleon rest mass may be retained.

For some time we thought that possibly the gravitational potential energy should also be included. We have been assured by C. Møller in a private communication that this is not the case but that we should calculate the equation of state in an inertial system regardless of gravitation. The result will then be inserted into the gravitational equations.

8. Results for Neutron Stars

Baym *et al.* (1971b) have used the equation of state of Pandharipande in a calculation of the hydrostatic equilibrium of a neutron star. Malone (unpublished) has done the same for our interaction II. The equations of general relativity have been used. Since our equation of state with excited baryons is similar to Pandharipande's neutron equation the results are similar.

According to Baym *et al.* (1971b) as well as Malone, neutron stars are stable with masses between about 0.1 and 2 solar masses. Table IV gives the results of Malone's calculations. The second column gives the gravitational mass M_g of the star, in units of the mass of the Sun, and the first gives the 'conserved' mass M_c, i.e. the total mass which all the baryons would have if they had no binding energy. The central density is listed, in units of 10^{15} g cm^{-3}; it increases with the mass which is a condition for stability of the star. The radius for most neutron stars is close to 10 km. The binding energy, which is the sum of nuclear and gravitational energies, varies from 4 to over 100 MeV per baryon; probably only neutron stars with binding of more than about 10 MeV can actually be formed.

The most interesting quantity is the moment of inertia which is seen to be of the order of 10^{44} g cm^2 units with the maximum occurring at a mass of 1.8 solar masses and having the value

$$I_{\max} = 11 \times 10^{44} \text{ g cm}^2 . \tag{20}$$

By contrast if a soft equation of state is used, such as that of Leung and Wang (1971), the maximum moment of inertia is 1 or 0.15 units (of 10^{44} g cm^2) according to the nuclear interaction they use at low density.

Useful observations exist only on the Crab Nebula. For this pulsar, the period and its time rate of change have been very accurately observed. If one attributes all the luminosity of the outer part of the nebula to energy transfer from the pulsar, and if one assumes that no pulsar energy escapes from the nebula in an invisible form, then

TABLE IV

Properties of neutron stars of various masses, using the interaction II of Table I, and the equation of state given by the upper solid line in Figure 4. Mass of neutron stars in units of the mass of the Sun, other details are described in Section 8 of text

$M_c/M_\odot$	$M_g/M_\odot$	ϱ_c (10^{15} g cm^{-3})	R km	$B.E.$ MeV	I 10^{44} g cm^2
0.0940	0.0936	0.16	138	4.4	1.13
0.1495	0.148	0.30	19 0	8.4	0.66
0.229	0.226	0.40	13.7	15	1.05
0.374	0.364	0.50	11.8	27	1.9
0.567	0.544	0.70	11.1	38	3.2
0.817	0.770	0.95	10.7	53	4.9
1.13	1.04	1.2	10.4	73	7.3
1.47	1.32	1.6	10.0	97	9.6
1.76	1.54	2.3	9.5	118	11.0
1.95	1.67	4.1	8.5	134	10.7
	LW 0.43		6.8	[a]	1.05
	LW 0.22		3.9	[b]	0.15

[a] Leung and Wang, with Reid potential at low density.
[b] Leung and Wang, with Lomon and Feshbach potential.

the total energy emitted by the pulsar can be deduced from the observations. Together with Ω and $\dot{\Omega}$ this leads to a moment of inertia

$$I_{\text{obs}} = 2.5 \times 10^{44} \text{ g cm}^2 . \tag{21}$$

With our equation of state, this corresponds to a mass of about 0.5 $M_\odot$ which is very satisfactory. With the soft equation of state this moment of inertia could not be obtained at all. We consider this some confirmation of our equation of state, but not a very strong one because the interpretation of the Crab energy is uncertain. The main point is that the order of magnitude of the 'observed' moment of inertia is that which can reasonably be expected from a neutron star.

I am much indebted to Dr M. Johnson for doing the calculations of the equation of state, Dr Pandharipande for devising the method, and Mr R. Malone for calculating the properties of neutron stars. I am also grateful to the many persons mentioned in this report for interesting suggestions and discussions.

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DISCUSSION

Ruderman: The 'disease' of some of the highest density calculations, that $p > \varrho c^2$ or $c_s^2 = \mathrm{d}p/\mathrm{d}\varrho > c^2$, may not be cured by relativity. Some years ago (*Phys. Rev.* (1968) **170**, 1176 and **172**, 1286) Sidney Bludman and I explored the fully relativistic theory of many classical sources repelling each other *via* a neutral vector meson field. The same disease arises there also, as soon as one replaces the positive self energy of the source (infinite for point sources) by a smaller 'renormalized' source mass. In the quantum field theoretic treatment of such models (*Phys. Rev.* (1970) **1D**, 3243) instability against pair production of a source particle-antiparticle pair would apparently occur before the $c_s^2 > c^2$ disease could, so that the latter would never be expected. This suggests a test that should be made in the very dense neutron matter calculations ($\varrho \gtrsim 10^{16}$ g cm^{-3}) whenever $p > \varrho c^2$. Because the forces are described as coming from a vector meson (ω) exchange and a dipion scalar (σ) exchange an antineutron is as strongly attracted in such matter as a neutron is repelled. By appropriately anti-correlating an additional neutron and correlating an anti-neutron wavefunction with respect to the positions of the already present neutrons, the very strong interactions will always lower the total energy to make the pair below 2 Mc^2. If the total $\leqslant 0$ the matter is, of course, unstable.

Bethe: It is comforting to know a mechanism by which the sound velocity will always be kept below the velocity of light. The actual calculations of the correlation function of an additional neutron and antineutron at high density may prove rather difficult.

Itoh: I would like to ask about the shape of the nuclear soft-core potential. Could you exclude experimentally the Gaussian soft-core potential proposed by Tamagaki?

Bethe: It is difficult to exclude it purely from experiment. However, I think one should combine experimental evidence with theoretical arguments. In our problem, this means assuming the exchange of ω mesons which gives a Yukawa potential.

Ruderman: David Pines and I wish to make a joint comment on the question of the nature of the high density limit of a Fermi gas with inverse square law repulsions among the fermions. It is indeed true as Professor Bethe remarked that the high density limit of a degenerate electron gas (in a uniform positive background) is a gas and not a solid. This is expected because as the average separation (b) between electrons decreases, their Fermi energy ($\sim \hbar^2/mb^2$) becomes much larger than their repulsive energy (e^2/b) so in the ground state it is more important to minimize the former than the latter. However this is no longer true when $e^2/\hbar c \sim 1/137$ is replaced by $g^2/\hbar c \sim 10$. For non-relativistic fermions the ratio $(g^2/b)\,(\hbar^2/mb^2)^{-1} > 10$. Its minimum is reached for relativistic baryons when the Fermi energy $\sim \hbar c/b$. Then the ratio of repulsive energy to Fermi energy becomes just $g^2/\hbar c \sim 10$. Thus this ratio is always sufficiently large at all densities that minimizing the repulsive energy by forming a crystal might be expected.

Bethe: The comment by Dr Ruderman is certainly correct. Therefore one cannot, on general principles, prove that neutron matter must remain liquid at high density. Whether it *actually* becomes solid, can only be shown by an explicit calculation with the actual forces between baryons. Our calculations have given negative answers to this question. This was true of Pandharipande's calculation a year ago with a method similar to the one he will describe in his contribution to this conference. About the same time Mikkel Johnson and I used the method of Guyer and Zane, and also found that the energy of the solid was much higher than that of the liquid. Canuto and Chitre, in their contribution to this conference, do find a positive result, viz. that neutron matter does crystallize at high density. It will be necessary to examine very carefully whether their method converges, or whether clusters of more than two bodies will change the result. For the present, the question whether baryon matter actually becomes solid must remain open.