European Journal of Applied Mathematics (2024), 1–24
doi:10.1017/S0956792524000846

CAMBRIDGE UNIVERSITY PRESS

PAPER

Propagation dynamics for an epidemic patch model with variable incubation period

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Received: 13 January 2024; **Revised:** 15 November 2024; **Accepted:** 24 November 2024

Keywords: epidemic patch model; travelling wave solution; distributed latency

2020 Mathematics Subject Classification: 39A36 (Primary); 37N25, 39A60, 92D30 (Secondary)

Abstract
We study an epidemic patch model that describes the disease spread in population with variable latency due to the differences in immunologic tolerance between individuals. We focus on whether the disease can spread in space that leads to the emergence of epidemic wave, that is the travelling wave solution with constant speed. We first establish some properties of the linearized wave profile equations, which are helpful in obtaining the priori estimates of travelling waves and wave speeds. Then, applying the truncation method and limiting arguments, we can obtain threshold propagation dynamics of the epidemic model. Our result gives a complete characterization of the existence, nonexistence and minimal wave speed of travelling waves. To the best of our knowledge, this is the first time to characterize the propagation dynamics of epidemic patch model with variable latency, which contributes to the understanding of the transmission phenomenon of disease.

1. Introduction

With the emergence of various infectious diseases in the development of human society, mathematical models have become an important tool for understanding the transmission dynamics of infectious diseases. In past decades, great attention has been paid to investigate the evolution of diseases through different mathematical models. For example, many different reaction-diffusion and integro-differential continuous models were established under the assumption that the populations disperse among continuous spaces, see [2, 8, 9, 13–16, 19, 21, 22, 25, 27, 29–31, 33, 37]. However, due to the development of modern transportation, the mobility of population in real life usually has the characteristics of large scale and span. When the infected persons travel by buses, trains or airplanes, many epidemics (such as plague, SARS, COVID-19, H1N1 flu, etc.) can be easily spread between different discrete spaces, such as cities, countries or regions. Therefore, it is more realistic and important to consider the impact of population diffusion in patchy environments on the spread of disease.

In order to investigate the dynamics of disease transmission under the influence of a population dispersal among patches, Wang and Zhao [20] proposed the following epidemic dynamical systems with population dispersal among n patches:

$$\begin{cases} \dfrac{dS_j}{dt} = \displaystyle\sum_{i=1}^{n} a_{ji}S_i - \mu_j S_j - \beta_j S_j I_j + \gamma_j I_j + B_j(N_j)N_j, \\ \dfrac{dI_j}{dt} = \displaystyle\sum_{i=1}^{n} b_{ji}I_i - (\mu_j + \gamma_j)I_j + \beta_j S_j I_j, \end{cases} \quad \text{for } j = 1, \cdots, n, \tag{1}$$

https://doi.org/10.1017/S0956792524000846 Published online by Cambridge University Press

where $S_j(t)$ and $I_j(t)$ denote the densities of susceptible individuals and infected individuals respectively in patch j at time $t \geq 0$;

$a_{ii} \leq 0$ represents the emigration rate of susceptible individuals in $i-$th patch;
$a_{ji} \geq 0$ $(j \neq i)$ represents the immigration rate of susceptible individuals from $i-$th patch to $j-$th patch;
$b_{ii} \leq 0$ represents the emigration rate of infective individuals in $i-$th patch;
$N_j = S_j + I_j$ represents the total number of population in patch j;
$B_j(\cdot)$ represents the birth rate of population in the $j-$th patch.

One can see from [20] that system (1) admits a threshold dynamics for the uniform persistence and extinction of disease, provided the birth rate satisfies certain assumptions.

Later, to clarify the effects of habitat connectivity and movement rates on the disease transmission dynamics, Allen et al. [1] proposed the following frequency-dependent *SI* metapopulation model which consists of n patches:

$$\begin{cases} \dfrac{dS_j}{dt} = d_S \displaystyle\sum_{i \in \Omega} \left(L_{ji} S_i - L_{ij} S_j \right) - \beta_j \dfrac{S_j I_j}{S_j + I_j} + \gamma_j I_j, \\ \dfrac{dI_j}{dt} = d_I \displaystyle\sum_{i \in \Omega} \left(L_{ji} I_i - L_{ij} I_j \right) + \beta_j \dfrac{S_j I_j}{S_j + I_j} - \gamma_j I_j, \end{cases} \quad \text{for} \quad j = 1, \cdots, n, \tag{2}$$

where $\Omega = \{1, 2, \cdots, n\}$;

d_S and d_I are positive diffusion rates for the susceptible and infected subpopulations (resp.);
L_{ji} describes the degree of movement from patch i to patch j;
β_j and γ_j are positive rates of disease transmission and recovery (resp.) in patch j.

They proved the existence and stability of disease-free and endemic equilibria and established some threshold type results which can predict whether the disease will persist or die out. Their results link the spatial heterogeneity, habitat connectivity and rates of movement to disease persistence and extinction.

Note that both results of [1, 20] characterized the global dynamics of the disease equilibria in terms of the values of basic reproduction numbers for the considered models. It has been realized that the persistence and extinction of disease are related to whether the infectious source can spread between patches as a wave. This fact prompts some researchers to investigate the propagation phenomena of travelling waves for different epidemic patch models. Guo et al. [6, 11, 23] considered comprehensively the travelling wave solutions for a class of epidemic patch model of the form (1), under the assumption that the population is distributed on infinite patches and spreads only in adjacent patches. The recent works [28, 32, 34, 36] made further generalization and development by introducing different types of nonlinear incidence rates. In these works, some threshold type results were established for the existence and nonexistence of travelling waves connecting two different equilibrium states. Let's point out that all the aforementioned works considered the models that the population disperses between its *adjacent patches*, i.e., the population in the j patch only interacts with those in the $j+1$ and $j-1$ patches. Such a characteristic in mathematics makes the wave equations of the models second-order difference equations. This brings some conveniences for the mathematical analysis to the models that considered in the aforementioned works. However, as mentioned above, the population in real world may spread over a large span due to the development of modern transportation.

On the other hand, it is known that for some diseases, such as the recent epidemic outbreak of COVID-19, the incubation period usually fluctuates in certain range due to the differences in immunologic tolerance between individuals. Therefore, the latent period from infection to onset of symptoms is often a variable. Inspired by the works [3, 7], it is more realistic to describe the incubation period via a weight function, which specifies the probability that an individual from uninfected to infection in a certain time interval.

To explore the spatial dynamics of disease spreads under the effects of large span diffusion and variable latency, we consider the following formulation of epidemic patch model:

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$$\begin{cases} \dfrac{d}{dt}S_j = d_S \displaystyle\sum_{i\in\mathbb{Z}} J_1(i)\big(S_{j-i} - S_j\big) - \beta S_j \int_0^{\tau} f(s) I_j(t-s)ds, \\ \dfrac{d}{dt}I_j = d_I \displaystyle\sum_{i\in\mathbb{Z}} J_2(i)\big(I_{j-i} - I_j\big) + \beta S_j \int_0^{\tau} f(s) I_j(t-s)ds - \gamma I_j, \end{cases} \quad \text{for } j \in \mathbb{Z}, \tag{3}$$

where $\tau > 0$ represents the maximum latent period from infection to onset of symptoms. Different to system (2), we assume that the disease transmission rate and recovery rate are isotropic (i.e., $\beta_i \equiv \beta$, $\gamma_i \equiv \gamma$), and the population in patch j can dispersal to $j-i$ patch with probability $J_k(i)$ $(k = 1, 2)$ for $i \in \mathbb{Z}$. Similar to [3, 7], we assume that the variation of incubation period is described by a probability function $f(\cdot)$ satisfying

$$\int_0^{\tau} f(s)ds = 1.$$

Note that the term $S_j \displaystyle\int_0^{\tau} f(s)I_j(t-s)ds$ measures the infection force of disease in patch j and at time t. Actually, system (3) is a *SIR* (Susceptible-Infectious-Removed) type epidemic model that the recovered individuals are not involved in the transmission of the disease as they will not be re-infected due to the protection of antibodies. This model describes a closed system without births and deaths. Our goal is to explore the propagation phenomena in system (3), especially travelling wave solutions. Although there have been many researches on the travelling waves of different discrete systems modelling epidemic dynamics, the analysis of systems with distributed delay and global interactions should be relatively more difficult. As we know, there seems to have no results on the wave propagation for this type of epidemic dynamical systems.

A travelling wave solution of system (3) means a solution propagating with a constant speed c and a fixed profile. Mathematically, one can consider the ansatz

$$S_j(t) = S(j + ct) = S(\xi) \quad \text{and} \quad I_j(t) = I(j + ct) = I(\xi) \tag{4}$$

for some wave functions $S(\cdot)$ and $I(\cdot)$ defined on $\mathbb{R}$, where $\xi = j + ct$ means the moving coordinate. Substituting the transformations of (4) into (3), we can obtain the profile equation

$$\begin{cases} cS'(\xi) = d_S \displaystyle\sum_{i\in\mathbb{Z}} J_1(i)\,(S(\xi - i) - S(\xi)) - \beta S(\xi) \int_0^{\tau} f(s) I(\xi - cs)ds, \\ cI'(\xi) = d_I \displaystyle\sum_{i\in\mathbb{Z}} J_2(i)\,(I(\xi - i) - I(\xi)) + \beta S(\xi) \int_0^{\tau} f(s) I(\xi - cs)ds - \gamma I(\xi). \end{cases} \tag{5}$$

In order to explain the evolution process of disease from outbreak to extinction, we are interested in finding the positive solutions of system (5) that connect from the initial disease-free state to the final disease-free state, i.e., $(S(\xi), I(\xi))$ satisfies the following conditions:

$$\begin{gathered} 0 < S(\xi) \le S_0, \quad I(\xi) > 0, \quad \text{for all } \xi \in \mathbb{R}, \\ \lim_{\xi\to-\infty} (S(\xi), I(\xi)) = (S_0, 0) \ \text{ and } \ \lim_{\xi\to+\infty} (S(\xi), I(\xi)) = (S_\infty, 0), \end{gathered} \tag{6}$$

The constant $S_0 > 0$ represents the density of susceptible individuals before the onset of epidemics, while $S_\infty \in [0, S_0)$ represents the density of susceptible individuals after the onset of epidemics. Condition (6) means that the travelling wave solutions are of mixed type, i.e., $S-$component is front type and $I-$component is pulse type. Biologically, the mixed type travelling wave indicates that the number of infected individuals increases first, and then decreases gradually until extinction.

In past years, there have been many literature working on the travelling waves of different discrete *SI* type epidemic dynamical systems, see e.g., [6, 11, 18, 23, 24, 28, 36, 38] and so on. However, little is known so far for the models like (3) with non-adjacent diffusion and distributed delay. The main difficulty arises from the fact that the solutions of (3) have no priori upper bound when the basic reproduction number

$$R_0 := \beta S_0/\gamma$$

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is greater than one, which leads to the construction of bounded travelling wave solution becomes very difficult. To overcome the difficulty and for mathematical convenience, throughout this paper, we assume that $J_k(\cdot)$ $(k = 1, 2)$ are compactly supported so that

1. (J) $J_k(i) = 0$ for $|i| > 2$ and $J_k(i) = J_k(-i) > 0$ for $|i| \leq 2$.

Focusing on the dispersal operator, one can see that system (2) is a particular case of system (1). In addition, focusing on the susceptible population equations, the dispersal operator in system (3) is also a particular case of that in system (2), e.g., we may take $L_{j,i} = L_{i,j} = J_1(|j - i|)$. This fact also provides the reason why we impose the symmetry property on $J_k(i)$ in assumption (J). Moreover, the symmetry of the kernel functions can ensure that the travelling wave solutions propagate forward(i.e., wave speed $c > 0$, cf. Lemma 3.1), which is of particular interest in applications.

Based on the above assumption, we first establish some properties of solutions for the linearized equation of the profile equation (5) around the disease-free equilibrium $(S_0, 0)$. With the help of these properties, when the basic reproduction number R_0 is greater than one, we may apply the truncation method and develop some novel analytical techniques to establish the travelling wave solutions of (3) that satisfy the condition (6). Moreover, we consider the minimal speed problem of travelling waves, which is important in epidemiology since it is usually the speed at which the disease spreads. Our main results can be summarized as the following theorem.

Theorem 1.1. *There exists a $c^* > 0$ such that the following statements are valid.*

(1) *If $R_0 > 1$, then system (3) admits a travelling wave $(S(\xi), I(\xi))$ satisfying (6) if and only if $c \geq c^*$.*

(2) *If $R_0 \leq 1$, system (3) has no travelling wave satisfying (6) for any $c \in \mathbb{R}$.*

From Theorem 1.1, one can see that R_0 is a threshold value in determining the occurrence of wave propagation of system (3). The critical speed c^* is the minimal wave speed of travelling waves when $R_0 > 1$. Moreover, it is interesting to see that the population moves at different speed the disease will go extinct. Indeed, if we fix an initial time, then the I component goes to zero as time goes to infinity.

To deal with the problem of minimal wave speed, we make use of some priori estimates and suitable limiting arguments. Let's point out that the limiting arguments were used in many works to study the existence of *front type* minimal travelling waves (i.e., the travelling waves with minimal speed connecting a zero equilibrium and a certain positive equilibrium) for various evolution equations, see [4, 5, 10, 14, 19, 26, 35] and so on. However, for *SI* epidemic systems, little works have been done for the existence of *mixed type* minimal travelling waves connecting *two disease-free equilibria*. The difficulty comes from proving the *non-triviality* ($S-$component is non-constant, and $I-$component is non-zero) of the limiting function and showing its asymptotic behaviour that connects two disease-free equilibria, cf. [23]. To overcome the difficulty, we use some limiting arguments and establish a crucial lemma (see Lemma 3.3) to prove the existence of minimal travelling wave of system (3) that connects two disease-free equilibria. Further, the nonexistence of travelling waves of system (3) is derived by using some priori estimates and the properties of solutions of the linearized profile equation.

Let's remark that Theorem 1.1 provides a complete characterization of the existence, nonexistence and minimal speed of travelling waves. To the best of our knowledge, this is the first result on the propagation dynamics of epidemic patch model with large span diffusion and variable incubation period.

The remainder of this paper is organized as follows. In Section 2, we establish some properties of the solutions for the linearized profile equation around the disease-free equilibrium. Some crucial priori estimates on wave profiles and wave speeds are given in section 3. In section 4, we first establish the existence of solutions for the profile system (5) over large finite domains. Then, we apply the truncation method via some different limiting arguments to prove the results of Theorem 1.1.

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2. Some properties of the linearized profile equation

Linearizing the second equation of (5) around the disease-free equilibrium $(S_0, 0)$ yields to the linear equation

$$cI'(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\,(I(\xi-i) - I(\xi)) + \beta S_0 \int_0^\tau f(s)I(\xi - cs)ds - \gamma I(\xi), \text{ for } \xi \in \mathbb{R}, \tag{7}$$

By the assumption (J), it is clear that $\tilde{\Omega} = \{-2, -1, 1, 2\}$. To establish a more general theoretical framework, we embed (7) into the following general form:

$$c\varphi'(\xi) = d \sum_{i\in\tilde{\Omega}} J(i)\varphi(\xi - i) + b_0 \int_0^\tau f(s)\varphi(\xi - cs)ds + b\varphi(\xi), \text{ for } \xi \in \mathbb{R}, \tag{8}$$

where $d, c > 0$, $b_0 \geq 0$, $b \in \mathbb{R}$ and J satisfies the assumption (J). Clearly, the equation (7) is a special form of (8) with $d := d_I$, $b_0 := \beta S_0$, $b := -(d_I \sum_{i\in\tilde{\Omega}} J_2(i) + \gamma)$ and $J(i) := J_2(i)$.

The characteristic equation of (8) is defined by

$$\Psi(d, b_0, b, c, \lambda) = d \sum_{i\in\tilde{\Omega}} J(i)e^{-\lambda i} + b_0 \int_0^\tau f(s)e^{-c\lambda s}ds - c\lambda + b = 0. \tag{9}$$

It is easy to verify that $\Psi(d, b_0, b, c, \cdot) = 0$ has at most two real roots since it is convex with respect to λ. Especially, when $R_0 = \beta S_0/\gamma > 1$ and $J(i) = J_2(i)$, it is obvious that $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda)$ is decreasing in c with $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, 0) > 0$ and $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, 0, \lambda) > 0$ for all $\lambda > 0$. Thus, the constant c^*, defined by

$$c^* := \inf\left\{c > 0 |\ \Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0 \text{ has a positive real root}\right\},$$

is well-defined. In addition, we have the following properties:

- if $c > c^*$, $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$ has two distinct positive roots $\lambda_1 = \lambda_1(c) < \lambda_2 = \lambda_2(c)$;
- if $c = c^*$, $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$ has a unique positive real root λ^*;
- if $c < c^*$, $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$ has no positive real root.

Let $\varphi(\xi)$ be a positive solution of (8) with $c > 0$, it is clear that $\phi(\xi) := \varphi'(\xi)/\varphi(\xi)$ satisfies the equation

$$c\phi(\xi) = d \sum_{i\in\tilde{\Omega}} J(i)e^{\int_\xi^{\xi-i}\phi(y)dy} + b_0 \int_0^\tau f(s)e^{\int_\xi^{\xi-cs}\phi(y)dy}ds + b, \text{ for } \xi \in \mathbb{R}. \tag{10}$$

Boundedness and smoothness for solutions of (10) are established in the following lemma.

Lemma 2.1. *If $\phi(\cdot)$ is a solution of equation (10), then $\phi(\cdot) \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$.*

Proof. Let 's denote

$$u(\xi) := e^{\nu_1\xi + \int_0^\xi \phi(y)dy} \text{ with } \nu_1 := -b/c.$$

Then, $u(\xi)$ satisfies the equation

$$cu'(\xi) = d \sum_{i\in\tilde{\Omega}} J(i)e^{i\nu_1}u(\xi - i) + b_0 \int_0^\tau f(s)e^{cs\nu_1}u(\xi - cs)ds > 0, \tag{11}$$

which implies that $u(\xi)$ is strictly increasing on $\mathbb{R}$. Thus, for any $p \in \tilde{\Omega}$ with $p > 0$, we have

$$cu'(\xi) > dJ(p)e^{-p\nu_1}u(\xi + p). \tag{12}$$

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Integrating the inequality (12) from $\xi - \frac{p}{2}$ to ξ gives

$$cu(\xi) - cu\Big(\xi - \frac{p}{2}\Big) > \frac{dp}{2} J(p) e^{-pv_1} u\Big(\xi + \frac{p}{2}\Big),$$

which implies

$$u(\xi) > \frac{dpJ(p)u\big(\xi + \frac{p}{2}\big)}{2ce^{pv_1}} > \frac{d^2p^2J^2(p)u(\xi + p)}{4c^2e^{2pv_1}},$$

and hence

$$\frac{u(\xi + p)}{u(\xi)} < \frac{4c^2e^{2pv_1}}{d^2p^2J^2(p)}. \tag{13}$$

Since $u(\xi)$ is strictly increasing on $\mathbb{R}$, it follows from (11) and (13) that $u'(\xi)/u(\xi) \le M_0$, for some $M_0 > 0$. Note that $\phi(\xi) = u'(\xi)/u(\xi) - v_1$. Thus, $\phi(\xi)$ is uniformly bounded. Moreover, by (10), it is easy to see $\phi(\cdot) \in C^\infty(\mathbb{R})$. The proof is complete. □

In addition, we prove that any non-constant solution of (10) has no global extrema.

Lemma 2.2. *Let $\phi(\xi)$ be a solution of (10) that attains its global maxima or minima, then it must be a constant function.*

Proof. The proof of the case $b_0 = 0$ is similar to that of [12, Lemma 2.8], so we only consider the case $b_0 > 0$. Differentiating equation (10) gives

$$c\phi'(\xi) = d \sum_{i \in \tilde{\Omega}} J(i) e^{\int_\xi^{\xi - i} \phi(y)dy} (\phi(\xi - i) - \phi(\xi)) + b_0 \int_0^\tau f(s) e^{\int_\xi^{\xi - cs} \phi(y)dy} (\phi(\xi - cs) - \phi(\xi)) ds. \tag{14}$$

Suppose that $\phi(\xi)$ admits a global maxima at ξ_*, then (14) gives

$$0 = c\phi'(\xi_*) = d \sum_{i \in \tilde{\Omega}} J(i) e^{\int_{\xi_*}^{\xi_* - i} \phi(y)dy} (\phi(\xi_* - i) - \phi(\xi_*)) + b_0 \int_0^\tau f(s) e^{\int_{\xi_*}^{\xi_* - cs} \phi(y)dy} (\phi(\xi_* - cs) - \phi(\xi_*)) ds \le 0,$$

which implies

$$\phi(\xi_* - i) = \phi(\xi_*), \ \text{ for } i \in \{-2, -1, 1, 2\}; \text{ and } \phi(\xi_* - cs) = \phi(\xi_*), \quad \text{for } s \in [0, \tau].$$

By induction arguments, we can conclude that $\phi(\xi) = \phi(\xi_*)$ for all $\xi \in \mathbb{R}$. Similarly, $\phi(\xi)$ is a constant function provided that it admits a global minima. The proof is complete. □

Next, we investigate the asymptotic behaviour of solutions for the equation (10).

Lemma 2.3. *Assume that $\phi(\xi)$ is a solution of (10). Then we have the following statements.*

(1) $\phi(\pm\infty) := \lim_{\xi \to \pm\infty} \phi(\xi)$ exist and $\Psi(d, b_0, b, c, \phi(\pm\infty)) = 0$.

(2) If $\phi(\xi)$ is a non-constant solution, then

$$\phi(\xi) - \phi(-\infty) \in C^1(\mathbb{R}, (0, \infty)) \cap L^1(\mathbb{R}_-) \ \textit{ and } \phi(+\infty) - \phi(\xi) \in C^1(\mathbb{R}, (0, \infty)) \cap L^1(\mathbb{R}_+).$$

Proof. (1) Let $\{y_j\}_{j=1}^\infty$ be a sequence such that

$$\lim_{j \to +\infty} y_j = +\infty \ \text{ and } \ \lim_{j \to +\infty} \phi(y_j) = w^* := \limsup_{\xi \to +\infty} \phi(\xi).$$

Denote $\eta_j(\xi) := \phi(y_j + \xi)$ for $j \in \mathbb{N}$. Clearly, $\{\eta_j(\xi)\}_{j=1}^\infty$ is uniformly bounded and equicontinuous. Then it follows from the Arzela-Ascoli theorem that $\{\eta_j(\xi)\}_{j=1}^\infty$ has a subsequence, still written as $\{\eta_j(\xi)\}_{j=1}^\infty$, such that $\eta_j(\xi) \to \eta(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $j \to +\infty$. Hence, $\eta(\xi)$ is a solution of (10) with $\eta(0) = w^* = \max_{\xi \in \mathbb{R}} \eta(\xi)$.

By Lemma 2.2, we have $\eta(\xi) = w^*$ for all $\xi \in \mathbb{R}$, which implies

$$\lim_{j \to +\infty} \max_{[y_j - 2 - c\tau, y_j + 2 + c\tau]} |\phi(\xi) - w^*| = 0. \tag{15}$$

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We first show that $\phi(+\infty)$ exists. Suppose $w^* > w_* := \liminf_{\xi\to+\infty} \phi(\xi)$, by (15), there exists a sufficiently large j such that

$$\min_{\xi\in[y_j,y_{j+1}]} \phi(\xi) < \frac{w_*+w^*}{2} \quad \text{and} \quad \phi(\xi) > \frac{w_*+w^*}{2}, \text{ for } \xi \in \big[y_j, y_j+2+c\tau\big] \cup \big[y_{j+1}-2-c\tau, y_{j+1}\big]. \tag{16}$$

Let $\hat{y}$ be the point such that $\phi(\hat{y}) = \min_{\xi\in[y_j,y_{j+1}]} \phi(\xi)$. According to (16), we have $\hat{y} \in (y_j+2+c\tau, y_{j+1}-2-c\tau)$. Then, for $s \in [0, \tau]$, it holds that $\phi'(\hat{y}) = 0$,

$$\phi(\hat{y}) \le \min\big\{\phi(\hat{y}+1), \phi(\hat{y}+2)\big\}, \ \phi(\hat{y}) < \min\big\{\phi(\hat{y}-1), \phi(\hat{y}-2)\big\} \ \text{ and } \ \phi(\hat{y}) \le \phi(\hat{y}-cs).$$

Taking $\xi = \hat{y}$ in (14) yields

$$0 = d\sum_{i\in\tilde{\Omega}} J(i)e^{\int_{\hat{y}}^{\hat{y}-i}\phi(y)dy}\big(\phi(\hat{y}-i)-\phi(\hat{y})\big) + b_0\int_0^\tau f(s)e^{\int_{\hat{y}}^{\hat{y}-cs}\phi(y)dy}\big(\phi(\hat{y}-cs)-\phi(\hat{y})\big)\,ds > 0,$$

which leads to a contradiction. Hence $w^* = w_*$, i.e., the limit $\phi(+\infty)$ exists.

In the same way, we can obtain that $\phi(-\infty)$ exists. Letting $\xi \to \pm\infty$ in (10), it follows that $\phi(\pm\infty)$ satisfy the equation $\Psi(d, b_0, b, c, \phi(\pm\infty)) = 0$.

(2) Since $\phi(\xi)$ is a non-constant solution, according to Lemma 2.2, $\phi(\xi)$ cannot attain its global extrema. Thus, $\phi(-\infty) \ne \phi(+\infty)$ and

$$\lambda_1 := \min\{\phi(-\infty), \phi(+\infty)\} < \phi(\xi) < \max\{\phi(-\infty), \phi(+\infty)\} := \lambda_2.$$

We first claim that $\phi(-\infty) < \phi(+\infty)$. If the claim is false, i.e., $\phi(-\infty) > \phi(+\infty)$, one has $\phi(+\infty) = \lambda_1$. Thus, given any small $\epsilon > 0$, by translation if necessary, we may assume that

$$\phi(\xi) < \lambda_1 + \epsilon, \quad \text{for } \xi > 0 \ \text{ and } \ \lambda_1 + \epsilon \le \phi(\xi), \ \text{for } \xi \le 0. \tag{17}$$

According to (10) and (17), we have

$$\begin{aligned} c(\lambda_1+\epsilon) = c\phi(0) &= d\sum_{i=-2}^{-1} J(i)e^{\int_0^{-i}\phi(y)dy} + d\sum_{i=1}^{2} J(i)e^{\int_0^{-i}\phi(y)dy} + b_0\int_0^\tau f(s)e^{\int_0^{-cs}\phi(y)dy}ds + b \\ &\le d\sum_{i=-2}^{-1} J(i)e^{-(\lambda_1+\epsilon)i} + d\sum_{i=1}^{2} J(i)e^{-(\lambda_1+\epsilon)i} + b_0\int_0^\tau f(s)e^{-(\lambda_1+\epsilon)cs}ds + b \\ &= d\sum_{i\in\tilde{\Omega}} J(i)e^{-(\lambda_1+\epsilon)i} + b_0\int_0^\tau f(s)e^{-(\lambda_1+\epsilon)cs}ds + b, \end{aligned}$$

which implies

$$\Psi(d, b_0, b, c, \lambda_1+\epsilon) = d\sum_{i\in\tilde{\Omega}} J(i)e^{-(\lambda_1+\epsilon)i} + b_0\int_0^\tau f(s)e^{-(\lambda_1+\epsilon)cs}ds + b - c(\lambda_1+\epsilon) \ge 0. \tag{18}$$

However, the inequality (18) contradicts to the fact that $\Psi(d, b_0, b, c, \lambda_1+\epsilon) < 0$ for every small $\epsilon > 0$ with $\lambda_1 + \epsilon < \lambda_2$. Therefore, $\lambda_1 = \phi(-\infty) < \phi(+\infty) = \lambda_2$.

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Next we show that $\int_{-\infty}^{0}(\phi(\xi)-\lambda_1)d\xi<+\infty$. Combining the equations $\Psi(d,b_0,b,c,\lambda_1)=0$ and (10), we have

$$\begin{aligned}
c(\phi(\xi)-\lambda_1) =& d\sum_{i\in\tilde{\Omega}}J(i)(e^{\int_{\xi}^{\xi-i}\phi(y)dy}-e^{-i\lambda_1})+b_0\int_0^{\tau}f(s)(e^{\int_{\xi}^{\xi-cs}\phi(y)dy}-e^{-cs\lambda_1})ds\\
=& d\sum_{i=-2}^{-1}J(i)e^{B_1(\xi,i)}\int_{\xi}^{\xi-i}(\phi(y)-\lambda_1)dy+d\sum_{i=1}^{2}J(i)e^{B_2(\xi,i)}\int_{\xi}^{\xi-i}(\phi(y)-\lambda_1)dy+\\
& b_0\int_0^{\tau}f(s)e^{B_3(\xi,s)}\int_{\xi}^{\xi-cs}(\phi(y)-\lambda_1)dyds, \qquad (19)
\end{aligned}$$

where $B_1(\xi,i)\in[-i\lambda_1,-i\max_{y\in[\xi,\xi-i]}\phi(y)]$ for $i\in\{-2,-1\}$;

$$B_2(\xi,i)\in[-i\max_{y\in[\xi-i,\xi]}\phi(y),-i\lambda_1] \text{ for } i\in\{1,2\}; \text{ and } B_3(\xi,s)\in[-cs\max_{y\in[\xi-cs,\xi]}\phi(y),-cs\lambda_1].$$

Let's denote $R(\xi):=\phi(\xi)-\lambda_1$. Integrating (19) from M to 0 with $M<-2-c\tau$ gives

$$\begin{aligned}
c\int_M^0 R(\xi)d\xi =& d\sum_{i=-2}^{-1}J(i)\int_M^0\int_{\xi}^{\xi-i}e^{B_1(\xi,i)}R(y)dyd\xi+d\sum_{i=1}^{2}J(i)\int_M^0\int_{\xi}^{\xi-i}e^{B_2(\xi,i)}R(y)dyd\xi+\\
& b_0\int_0^{\tau}f(s)\int_M^0\int_{\xi}^{\xi-cs}e^{B_3(\xi,s)}R(y)dyd\xi ds. \qquad (20)
\end{aligned}$$

Changing the integration order in (20) yields

$$\begin{aligned}
0=&-c\int_M^0 R(\xi)d\xi+d\sum_{i=-2}^{-1}J(i)\int_M^{M-i}\int_M^{y}e^{B_1(\xi,i)}R(y)d\xi dy+d\sum_{i=-2}^{-1}J(i)\int_M^0\int_{y+i}^{y}e^{B_1(\xi,i)}R(y)d\xi dy\\
&-d\sum_{i=-2}^{-1}J(i)\int_M^{M-i}\int_{y+i}^{y}e^{B_1(\xi,i)}R(y)d\xi dy+d\sum_{i=-2}^{-1}J(i)\int_0^{-i}\int_{y+i}^{0}e^{B_1(\xi,i)}R(y)d\xi dy\\
&-d\sum_{i=1}^{2}J(i)\int_{M-i}^{M}\int_M^{y+i}e^{B_2(\xi,i)}R(y)d\xi dy-d\sum_{i=1}^{2}J(i)\int_M^0\int_y^{y+i}e^{B_2(\xi,i)}R(y)d\xi dy\\
&+d\sum_{i=1}^{2}J(i)\int_{-i}^{0}\int_y^{y+i}e^{B_2(\xi,i)}R(y)d\xi dy-d\sum_{i=1}^{2}J(i)\int_{-i}^{0}\int_y^{0}e^{B_2(\xi,i)}R(y)d\xi dy\\
&-b_0\int_0^{\tau}f(s)\int_{M-cs}^{M}\int_M^{y+cs}e^{B_3(\xi,s)}R(y)d\xi dyds-b_0\int_0^{\tau}f(s)\int_M^0\int_y^{y+cs}e^{B_3(\xi,s)}R(y)d\xi dyds\\
&+b_0\int_0^{\tau}f(s)\int_{-cs}^{0}\int_y^{y+cs}e^{B_3(\xi,s)}R(y)d\xi dyds-b_0\int_0^{\tau}f(s)\int_{-cs}^{0}\int_y^{0}e^{B_3(\xi,s)}R(y)d\xi dyds. \qquad (21)
\end{aligned}$$

Let $\varepsilon>0$ be small enough, by translation if necessary, we may assume that $\phi(\xi)<\lambda_1+\varepsilon$ for $\xi<\theta$ and $\lambda_1+\varepsilon\le\phi(\xi)$ for $\xi\ge\theta$, where $\theta=2+c\tau$. Note that there exists some constant $K>0$ such that $|R(\xi)|\le K$ for all $\xi\in\mathbb{R}$. Then we can obtain

$$\begin{aligned}
& d\sum_{i=-2}^{-1}J(i)\int_M^{M-i}\int_M^{y}e^{B_1(\xi,i)}R(y)d\xi dy\\
\le& dK\sum_{i=-2}^{-1}J(i)\int_M^{M-i}\int_M^{y}e^{-i(\lambda_1+\varepsilon)}d\xi dy=\frac{1}{2}dK\sum_{i=-2}^{-1}J(i)i^2e^{-i(\lambda_1+\varepsilon)}<+\infty. \qquad (22)
\end{aligned}$$

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It is also easy to verify that

$$d\sum_{i=-2}^{-1} J(i)\int_0^{-i}\int_{y+i}^{0} e^{B_1(\xi,i)}R(y)d\xi dy < +\infty, \quad d\sum_{i=1}^{2} J(i)\int_{-i}^{0}\int_{y}^{y+i} e^{B_2(\xi,i)}R(y)d\xi dy < +\infty,$$

$$b_0\int_0^{\tau} f(s)\int_{-cs}^{0}\int_{y}^{y+cs} e^{B_3(\xi,s)}R(y)d\xi dy ds < +\infty.$$

Then it follows from (21) that

$$\int_M^0 \Gamma(y)R(y)dy \le \mathcal{O}(1), \tag{23}$$

where $\mathcal{O}(1)$ is uniformly bounded and

$$\Gamma(y) := c - d\sum_{i=-2}^{-1} J(i)\int_{y+i}^{y} e^{B_1(\xi,i)}d\xi + d\sum_{i=1}^{2} J(i)\int_{y}^{y+i} e^{B_2(\xi,i)}d\xi + b_0\int_0^{\tau} f(s)\int_y^{y+cs} e^{B_3(\xi,s)}d\xi ds.$$

Note that

$$\begin{aligned}\Gamma(y) \ge & c + d\sum_{i=-2}^{-1} J(i)ie^{-i(\lambda_1+\varepsilon)} + d\sum_{i=1}^{2} J(i)ie^{-i(\lambda_1+\varepsilon)} + b_0\int_0^{\tau} f(s)cse^{-cs(\lambda_1+\epsilon)}ds\\ =& -\frac{\partial}{\partial\lambda}\Psi(d,b_0,b,c,\lambda_1+\varepsilon) > 0,\end{aligned}$$

for small $\varepsilon > 0$. Let $M \to -\infty$ in (23), we have

$$-\frac{\partial}{\partial\lambda}\Psi(d,b_0,b,c,\lambda_1+\varepsilon)\int_{-\infty}^{0} R(y)dy \le \mathcal{O}(1).$$

Thus, $\int_{-\infty}^{0} R(y)dy < +\infty$, that is $\int_{-\infty}^{0} (\phi(\xi) - \phi(-\infty))d\xi < +\infty$. By the same way, we can obtain that $\int_0^{+\infty} (\phi(+\infty) - \phi(\xi))d\xi < +\infty$. The proof is complete. □

Based on Lemma 2.3, we can represent solutions of (10) explicitly in the following lemma.

Lemma 2.4. *Assume that $\phi(\xi)$ is a solution of (10), then $\phi(\xi)$ takes the form*

$$\phi(\xi) = \frac{l\lambda_1 e^{\lambda_1\xi} + (1-l)\lambda_2 e^{\lambda_2\xi}}{le^{\lambda_1\xi} + (1-l)e^{\lambda_2\xi}}, \quad \text{for } \xi \in \mathbb{R}, \tag{24}$$

with some $l \in [0, 1]$, where λ_1, λ_2 are two real roots of $\Psi(d, b_0, b, c, \cdot) = 0$. Specially, when $l \ne 0$ or 1, then $\phi(\xi)$ is a non-constant solution of (10) which is strictly increasing on $\mathbb{R}$.

Proof. If $\phi(\xi)$ is constant solution of (10), by Lemma 2.3, we have $\phi(\xi) = \lambda_1$ or $\phi(\xi) = \lambda_2$ for $\xi \in \mathbb{R}$, i.e., (24) holds with $l = 1$ or 0 respectively. Note that λ_1 may equal to λ_2.

If $\phi(\xi)$ is a non-constant solution of (10), according to Lemma 2.3, we have $\lambda_1 = \phi(-\infty) < \phi(\xi) < \phi(+\infty) = \lambda_2$. Then we consider the functions

$$w(\xi) := e^{\int_0^{\xi}\phi(z)dz}, w_1(\xi) := le^{\lambda_1\xi} \text{ and } w_2(\xi) := w(\xi) - w_1(\xi), \text{ with } l := e^{-\int_{-\infty}^{0}(\phi(z)-\lambda_1)dz} \in (0, 1).$$

It's easy to verify that

$$w'(\xi) = \phi(\xi)w(\xi), \quad w_2(0) = 1 - l \quad\text{and}\quad w_2(\xi)e^{-\lambda_1\xi} = e^{\int_0^{\xi}(\phi(z)-\lambda_1)dz} - e^{-\int_{-\infty}^{0}(\phi(z)-\lambda_1)dz}. \tag{25}$$

Note that $\int_0^{\xi} (\phi(z) - \lambda_1)dz > -\int_{-\infty}^{0} (\phi(z) - \lambda_1)dz$ for any $\xi \in \mathbb{R}$. We have

$$w_2(\xi)e^{-\lambda_1\xi} > 0 \;\text{ for } \xi \in \mathbb{R} \;\text{ and }\; w_2(\xi)e^{-\lambda_1\xi} \to 0 \;\text{ as } \xi \to -\infty. \tag{26}$$

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Based on (26), we further consider the function $\varphi(\xi) := w_2'(\xi)/w_2(\xi)$. By simple computations, $\varphi(\xi)$ satisfies the equation (10). We claim that $\varphi(\xi)$ is a constant solution of (10). If false, that is $\varphi(\xi)$ is not a constant function, by Lemma 2.3, we have

$$\int_{-\infty}^{0} (\varphi(\xi)-\lambda_1))d\xi < \infty \quad \text{and} \quad \ln[w_2(\xi)e^{-\lambda_1\xi}] - \ln[w_2(0)] = \int_0^{\xi} \frac{(w_2(z)e^{-\lambda_1 z})'}{(w_2(z)e^{-\lambda_1 z})} dz = \int_0^{\xi} [\varphi(z)-\lambda_1]dz.$$

As $\xi \to -\infty$, it follows that

$$\lim_{\xi\to-\infty} \ln[w_2(\xi)e^{-\lambda_1\xi}] < \infty,$$

which contradicts to (26). Hence, by Lemma 2.3, $\varphi(\xi)$ is a constant function which equals to λ_1 or λ_2 for all $\xi \in \mathbb{R}$. If $\varphi(\xi) = \lambda_1$, we have $w(\xi) = ae^{\lambda_1\xi}$ for some constant a. Then $\phi(\xi)$ is constant function, which gives a contradiction. Therefore $\varphi(\xi) = \lambda_2$. In view of the definition of $\varphi(\xi)$, we have

$$w_2(\xi) = w_2(0)e^{\lambda_2\xi} = (1-l)e^{\lambda_2\xi} \quad \text{and} \quad w(\xi) = le^{\lambda_1\xi} + (1-l)e^{\lambda_2\xi}.$$

Since $w'(\xi) = \phi(\xi)w(\xi)$, the solution form (24) holds obviously. According to (24), it is easy to verify that $\phi(\xi)$ is strictly increasing on $\mathbb{R}$. The proof is complete. □

Remark 1. *From the proof of Lemma 2.4, it can be easily seen that the conclusion of Lemma 2.3 (1) also holds for more general form of (10) by replacing the constant b as any continuous function $b(\xi)$ whose limits $b_\pm := b(\pm\infty)$ exist. That is, if $\phi(\xi)$ is a solution of (10) with b replaced by such continuous function $b(\xi)$, then $\phi(\pm\infty)$ exist and $\Psi(d, b_0, b_\pm, c, \phi(\pm\infty)) = 0$.*

By Lemma 2.4, we have the following results on solutions of the linearized equation (8).

Proposition 2.5. *Suppose that $\varphi(\xi)$ is a nonnegative solution of the linear equation (8), then*

$$\varphi(\xi) = C_1e^{\lambda_1\xi} + C_2e^{\lambda_2\xi}, \quad \textit{for some constants } C_1,\ C_2, \tag{27}$$

where λ_1, λ_2 are two real roots of the characteristic equation $\Psi(d, b_0, b, c, \cdot) = 0$.

Proof. The result can be proved in the following two cases.

Case 1: $\varphi(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. In this case, it can be easily deduced from (8) that $\varphi(\xi_0 \pm 1) = 0$. An induction argument shows that $\varphi(\xi_0 \pm k) = 0$ for any $k \in \mathbb{N}_+$. On the other hand, by (8), one can see that

$$c\varphi'(\xi) + |b|\varphi(\xi) \geq 0, \quad \text{for } \xi \in \mathbb{R},$$

which implies that $\varphi(\xi)e^{\frac{|b|}{c}\xi}$ is non-decreasing on $\mathbb{R}$. Thus, it follows that $\varphi(\xi) = 0$ on $\mathbb{R}$.

Case 2: $\varphi(\xi) > 0$ for all $\xi \in \mathbb{R}$. Dividing the equation (8) by $\varphi(\xi)$, we have

$$cq(\xi) = d\sum_{i\in\tilde{\Omega}} J(i)e^{\int_\xi^{\xi-i} q(y)dy} + b_0\int_0^{\tau} f(s)e^{\int_\xi^{\xi-cs} q(y)dy}ds + b, \quad \text{for } \xi \in \mathbb{R}. \tag{28}$$

where $q(\xi) := \varphi'(\xi)/\varphi(\xi)$. By Lemma 2.4, $q(\xi)$ admits the form

$$q(\xi) = \frac{\varphi'(\xi)}{\varphi(\xi)} = \frac{l\lambda_1e^{\lambda_1\xi} + (1-l)\lambda_2e^{\lambda_2\xi}}{le^{\lambda_1\xi} + (1-l)e^{\lambda_2\xi}}, \quad l \in [0, 1].$$

Integrating the above equality gives

$$\varphi(\xi) = p(le^{\lambda_1\xi} + (1-l)e^{\lambda_2\xi})$$

for some $p > 0$. Hence, (27) holds by letting $C_1 = pl$ and $C_2 = p(1-l)$. □

https://doi.org/10.1017/S0956792524000846 Published online by Cambridge University Press

3. Priori estimates on wave profiles and wave speeds

To establish some priori estimates on positive travelling waves and wave speeds, we first provide the necessary condition for the existence of positive travelling waves that satisfy (6).

Lemma 3.1. *If $(S(\xi), I(\xi))$ is a positive solution of (5) satisfying (6), then $R_0 > 1$ and $c > 0$.*

Proof. Suppose the assertion is false, that is (5) admits a positive solution $(S(\xi), I(\xi))$ satisfying (6) for $R_0 \leq 1$ and some $c \in \mathbb{R}$. We first claim that $\int_{\mathbb{R}} I(\xi)d\xi < +\infty$. Since $I(\pm\infty) = 0$, there exists $K > 0$ such that $I(\xi) \leq K$ for $\xi \in \mathbb{R}$. Integrating the first equation of (5) from y to x for any $x, y \in \mathbb{R}$, we have

$$
\begin{aligned}
\Big|\int_y^x \beta S(\xi)\int_0^\tau f(s)I(\xi - cs)dsd\xi\Big| \leq & d_S\Big|\sum_{i\in\tilde{\Omega}} J_1(i)i\int_y^x\int_0^1 S'(\xi - ti)dtd\xi\Big| + |c(S(x) - S(y))| \\
\leq & d_S\Big|\sum_{i\in\tilde{\Omega}} J_1(i)i\int_0^1 (S(x - ti) - S(y - ti))\, dt\Big| + |c(S(x) - S(y))| \\
\leq & \big(2d_S\sum_{i\in\tilde{\Omega}} J_1(i)|i| + 2|c|\big)S_0.
\end{aligned}
$$

Then, it follows from the second equation of (5) that

$$
\begin{aligned}
\Big|\gamma\int_y^x I(\xi)d\xi\Big| \leq & d_I\Big|\sum_{i\in\tilde{\Omega}} J_2(i)\int_y^x (I(\xi - i) - I(\xi))\, d\xi\Big| + \Big|\int_y^x \beta S(\xi)\int_0^\tau f(s)I(\xi - cs)dsd\xi\Big| \\
& + |c(I(x) - I(y))| \\
\leq & \big(2d_I\sum_{i\in\tilde{\Omega}} J_2(i)|i| + 2|c|\big)K + \big(2d_S\sum_{i\in\tilde{\Omega}} J_1(i)|i| + 2|c|\big)S_0.
\end{aligned}
$$

Thus, it follows that $\int_{\mathbb{R}} I(\xi)d\xi < +\infty$ by the arbitrariness of x and y.

By the second equation of (5), we have

$$
\begin{aligned}
\gamma\int_{\mathbb{R}} I(\xi)d\xi \leq & d_I\sum_{i\in\tilde{\Omega}} J_2(i)\int_{\mathbb{R}} (I(\xi - i) - I(\xi))\, d\xi + \beta S_0\int_0^\tau f(s)\int_{\mathbb{R}} I(\xi - cs)d\xi ds \\
= & d_I\sum_{i\in\tilde{\Omega}} J_2(i)\int_{\mathbb{R}} (I(\xi - i) - I(\xi))\, d\xi + \beta S_0\int_{\mathbb{R}} I(\xi)d\xi = \beta S_0\int_{\mathbb{R}} I(\xi)d\xi.
\end{aligned}
\tag{29}
$$

Note that $R_0 = \beta S_0/\gamma$. If $R_0 < 1$, by (29), we have

$$
\int_{\mathbb{R}} I(\xi)d\xi \leq \frac{\beta S_0}{\gamma}\int_{\mathbb{R}} I(\xi)d\xi < \int_{\mathbb{R}} I(\xi)d\xi, \tag{30}
$$

which gives a contradiction. If $R_0 = 1$, the second equation of (5) also gives

$$
\begin{aligned}
0 = & d_I\sum_{i\in\tilde{\Omega}} J_2(i)\int_{\mathbb{R}} (I(\xi - i) - I(\xi))\, d\xi + \int_{\mathbb{R}} \beta S(\xi)\int_0^\tau f(s)I(\xi - cs)dsd\xi - \gamma\int_{\mathbb{R}} I(\xi)d\xi \\
= & d_I\sum_{i\in\tilde{\Omega}} J_2(i)\int_{\mathbb{R}} (I(\xi - i) - I(\xi))\, d\xi + \beta\int_{\mathbb{R}}\int_0^\tau f(s)S(\xi + cs)I(\xi)dsd\xi - \beta S_0\int_{\mathbb{R}} I(\xi)d\xi \\
= & \beta\int_{\mathbb{R}} I(\xi)\int_0^\tau f(s)\,(S(\xi + cs) - S_0)\, dsd\xi.
\end{aligned}
\tag{31}
$$

Since $0 < S(\xi) \leq S_0$ and $I(\xi) > 0$ on $\mathbb{R}$, (31) implies that $S(\xi) \equiv S_0$ on $\mathbb{R}$, which contradicts to $S(+\infty) = S_\infty < S_0$. Therefore, $R_0 > 1$.

Next, we show that $c>0$. Suppose, by contradiction, that $(S(\xi), I(\xi))$ is a positive solution of (5) satisfying (6) with $c\le 0$. Since $R_0>1$ and $S(-\infty)=S_0$, there exists a $\hat{\xi}<0$ such that

$$S(\xi)\int_0^\tau f(s)I(\xi-cs)ds>\frac{\beta S_0+\gamma}{2\beta}\int_0^\tau f(s)I(\xi-cs)ds,\ \text{for}\ \xi<\hat{\xi}.$$

Then, for $\xi<\hat{\xi}$, we have

$$\begin{aligned}cI'(\xi)=&d_I\sum_{i\in\tilde{\Omega}}J_2(i)\left(I(\xi-i)-I(\xi)\right)+\beta S(\xi)\int_0^\tau f(s)I(\xi-cs)ds-\gamma I(\xi)\\ >&d_I\sum_{i\in\tilde{\Omega}}J_2(i)\left(I(\xi-i)-I(\xi)\right)+\frac{\beta S_0+\gamma}{2}\int_0^\tau f(s)\left(I(\xi-cs)-I(\xi)\right)ds+\frac{\beta S_0-\gamma}{2}I(\xi).\end{aligned}$$

Integrating the above inequality over $(-\infty,\xi)$ with $\xi<\hat{\xi}$, one can obtain

$$\frac{\beta S_0-\gamma}{2}\Theta(\xi)<cI(\xi)-d_I\sum_{i\in\tilde{\Omega}}J_2(i)\left(\Theta(\xi-i)-\Theta(\xi)\right)+\frac{\beta S_0+\gamma}{2}\int_0^\tau f(s)\left(\Theta(\xi)-\Theta(\xi-cs)\right)ds,$$

where $\Theta(\xi):=\displaystyle\int_{-\infty}^{\xi}I(y)dy>0$. Since

$$\begin{aligned}\int_{-\infty}^{\xi}\int_0^\tau f(s)(\Theta(\eta)-\Theta(\eta-cs))dsd\eta&=\int_{-\infty}^{\xi}\int_0^\tau f(s)cs\int_0^1\Theta'(\eta-tcs)dtdsd\eta\\&=\int_0^\tau f(s)cs\int_0^1\Theta(\xi-tcs)dtds\le 0,\\ \sum_{i\in\tilde{\Omega}}J_2(i)\int_{-\infty}^{\xi}\big(\Theta(\eta-i)-\Theta(\eta)\big)d\eta&=\sum_{i\in\tilde{\Omega}}J_2(i)\int_{\xi}^{\xi-i}\Theta(\eta)d\eta\\&=\sum_{i=1}^{2}J_2(i)\Big(\int_{\xi}^{\xi+i}\Theta(\eta)d\eta-\int_{\xi-i}^{\xi}\Theta(\eta)d\eta\Big)>0,\end{aligned}$$

it follows that

$$0<\frac{\beta S_0-\gamma}{2}\int_{-\infty}^{\xi}\Theta(\eta)d\eta<c\Theta(\xi)\le 0,\ \text{for}\ \xi<\hat{\xi},$$

which also gives a contradiction. Hence the assertion of the lemma holds. □

In addition, we have the following limiting results of the wave profiles.

Lemma 3.2. *Assume that $(S(\xi), I(\xi))$ is a solution of (5) with $c>0$, and $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence satisfying $\lim\limits_{j\to+\infty}I(\xi_j)=+\infty$, then $\lim\limits_{j\to+\infty}S(\xi_j)=0$.*

Proof. Let's prove the result by using the contradiction argument. Suppose, for some constant $\epsilon>0$, there exists a subsequence of $\{\xi_j\}_{j\in\mathbb{N}}$, still written as $\{\xi_j\}_{j\in\mathbb{N}}$, such that $S(\xi_j)\ge\epsilon$ for each $j\in\mathbb{N}$. Let $d_1=d_S\sum_{i\in\tilde{\Omega}}J_1(i)$ and $d_2=d_I\sum_{i\in\tilde{\Omega}}J_2(i)$. By the first equation of (5), we have $S'(\xi)\le d_1S_0/c$ which implies $S(\xi)\ge\epsilon/2$ for $\xi\in[\xi_j-\xi_0,\xi_j]$, where $\xi_0:=c\epsilon/(2d_1S_0)$. Note that

$$cI'(\xi)+(d_2+\gamma)I(\xi)=d_I\sum_{i\in\tilde{\Omega}}J_2(i)I(\xi-i)+\beta S(\xi)\int_0^\tau f(s)I(\xi-cs)ds>0.$$

So, $I(\xi)e^{\frac{d_2+\gamma}{c}\xi}$ is strictly increasing in $\mathbb{R}$. Then,

$$\int_0^\tau f(s)I(\xi-cs)ds<I(\xi)\int_0^\tau f(s)e^{(d_2+\gamma)s}ds\ \text{ and }\ \frac{I(\xi-x)}{I(\xi)}<e^{\frac{(d_2+\gamma)x}{c}},\ \text{for}\ x>0. \tag{32}$$

Moreover, for any $p \in \tilde{\Omega}$ with $p > 0$, we have

$$cI'(\xi) + (d_2 + \gamma)I(\xi) > d_I J_2(p) I(\xi + p),$$

which implies

$$\frac{d}{d\xi}\left(e^{\frac{(d_2+\gamma)\xi}{c}} I(\xi)\right) > \frac{d_I J_2(p)}{c} e^{\frac{(d_2+\gamma)\xi}{c}} I(\xi + p). \tag{33}$$

Since $I(\xi)e^{\frac{d_2+\gamma}{c}\xi}$ is strictly increasing, integrating (33) from $\xi - \dfrac{p}{2}$ to ξ gives

$$e^{\frac{(d_2+\gamma)\xi}{c}} I(\xi) - e^{\frac{(d_2+\gamma)(\xi-\frac{p}{2})}{c}} I(\xi - \frac{p}{2}) > \frac{d_I J_2(p)}{c} \int_{\xi-\frac{p}{2}}^{\xi} e^{\frac{(d_2+\gamma)\eta}{c}} I(\eta + p) d\eta > \frac{d_I J_2(p) p}{2c} e^{\frac{(d_2+\gamma)(\xi-\frac{p}{2})}{c}} I(\xi + \frac{p}{2}).$$

It follows that

$$\frac{I(\xi + \frac{p}{2})}{I(\xi)} < \frac{2c}{d_I J_2(p) p} e^{\frac{(d_2+\gamma)p}{2c}} \text{ and } \frac{I(\xi + p)}{I(\xi)} = \frac{I(\xi + p)}{I(\xi + \frac{p}{2})} \frac{I(\xi + \frac{p}{2})}{I(\xi)} < \frac{4c^2}{(d_I J_2(p) p)^2} e^{\frac{(d_2+\gamma)p}{c}}. \tag{34}$$

Dividing the second equation of (5) by $I(\xi)$, and using (32) and (34), we have

$$\frac{I'(\xi)}{I(\xi)} = \frac{d_I}{c} \sum_{i\in\tilde{\Omega}} J_2(i) \frac{I(\xi - i)}{I(\xi)} - \frac{d_2}{c} + \frac{\beta}{c} S(\xi) \int_0^{\tau} f(s) \frac{I(\xi - cs)}{I(\xi)} ds - \frac{\gamma}{c} < +\infty. \tag{35}$$

Thus, there exists a constant $\rho > 0$ such that $|I'(\xi)/I(\xi)| \le \rho$, and hence

$$\frac{I(\xi_j)}{I(\xi)} = e^{\int_{\xi}^{\xi_j} \frac{I'(\eta)}{I(\eta)} d\eta} \le e^{\rho(c\tau+\xi_0)}, \text{ for } \xi \in [\xi_j - \xi_0 - c\tau, \xi_j].$$

Then it follows that

$$\min_{\xi\in[\xi_j-\xi_0-c\tau,\xi_j]} I(\xi) \ge I(\xi_j) e^{-\rho(c\tau+\xi_0)} \to +\infty \text{ as } j \to +\infty.$$

Furthermore, by the first equation of (5), one can see

$$\max_{\xi\in[\xi_j-\xi_0,\xi_j]} S'(\xi) \le \frac{d_1 S_0}{c} - \frac{\beta\epsilon}{2c} \min_{\xi\in[\xi_j-\xi_0,\xi_j]} \int_0^{\tau} f(s) I(\xi - cs) ds \le \frac{d_1 S_0}{c} - \frac{\beta\epsilon}{2c} I(\xi_j) e^{-\rho(c\tau+\xi_0)} \to -\infty$$

as $j \to +\infty$. Let j be sufficiently large such that $S'(\xi) \le -2S_0/\xi_0$ for $\xi \in [\xi_j - \xi_0, \xi_j]$, then

$$S(\xi_j) - S(\xi_j - \xi_0) \le -2S_0,$$

which is impossible since $0 < S(\xi) \le S_0$ on $\mathbb{R}$. Therefore, $S(\xi_j) \to 0$ as $j \to +\infty$. □

We further establish the following limiting lemma, which is crucial for proving the existence of travelling wave solutions with minimal speed.

Lemma 3.3. *Assume $R_0 > 1$. Let $(S_k(\xi), I_k(\xi))$ $(k \in \mathbb{N})$ be the positive solutions of (5) satisfying (6) with $c = c_k$ such that $\lim_{k\to+\infty} c_k = c_0 > 0$. Then (5) admits a positive solution $(S(\xi), I(\xi))$ satisfying (6) with $c = c_0$.*

Proof. Without loss of generality, we assume $\{c_k\}_{k=1}^{+\infty}$ is a strictly decreasing sequence with $\lim_{k\to+\infty} c_k = c_0 > 0$. Let $(S_k(\xi), I_k(\xi))$ be a positive solution of (5) satisfying (6) for $c = c_k$. We first claim that the sequence $\{I_k(\xi)\}_{k\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}$. If not, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} I_k(z_k) = +\infty$. Then, it follows from Lemma 3.2 that $\lim_{k\to+\infty} S_k(z_k) = 0$. Since $I_k(\pm\infty) = 0$ and $I_k(\xi) > 0$, without loss of generality, we may assume that $I_k(z_k) = \max_{\xi\in\mathbb{R}} I_k(\xi)$. By the second equation of (5), we have

$$\begin{aligned} 0 = c_k I_k'(z_k) =& d_I \sum_{i\in\tilde{\Omega}} J_2(i) \left(I_k(z_k - i) - I_k(z_k)\right) + \beta S_k(z_k) \int_0^{\tau} f(s) I_k(z_k - c_k s) ds - \gamma I_k(z_k) \\ \le& \beta S_k(z_k) \int_0^{\tau} f(s) I_k(z_k - c_k s) ds - \gamma I_k(z_k) < \left(\beta S_k(z_k) - \gamma\right) I_k(z_k). \end{aligned} \tag{36}$$

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However, the inequality (36) contradicts to the properties $\lim_{k\to+\infty} I_k(z_k) = +\infty$ and $\lim_{k\to+\infty} S_k(z_k) = 0$, when k is large enough. Therefore, $\{I_k(\xi)\}_{k\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}$.

Since $\{S_k(\xi)\}_{k\in\mathbb{N}}$ and $\{I_k(\xi)\}_{k\in\mathbb{N}}$ are uniformly bounded on $\mathbb{R}$, it follows from (5) that $||S_k||_{C^2(\mathbb{R})}$ and $||I_k||_{C^2(\mathbb{R})}$ are both uniformly bounded on $\mathbb{R}$. Then, by Arzela-Ascoli theorem, there exists subsequences of $\{S_k(\xi)\}_{k\in\mathbb{N}}$ and $\{I_k(\xi)\}_{k\in\mathbb{N}}$, still written as $\{S_k(\xi)\}_{k\in\mathbb{N}}$ and $\{I_k(\xi)\}_{k\in\mathbb{N}}$, such that $S_k(\xi)\to S(\xi)$ and $I_k(\xi)\to I(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $k\to+\infty$. By Lebesgue dominated convergence theorem, it is easy to see the limiting function $(S(\xi), I(\xi))$ is a solution of (5) with $c = c_0$, i.e., satisfies the following system

$$\begin{cases} c_0S'(\xi) = d_S \sum_{i\in\tilde{\Omega}} J_1(i)\,(S(\xi - i) - S(\xi)) - \beta S(\xi)\int_0^\tau f(s)I(\xi - c_0s)ds, \\ c_0I'(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\,(I(\xi - i) - I(\xi)) + \beta S(\xi)\int_0^\tau f(s)I(\xi - c_0s)ds - \gamma I(\xi). \end{cases} \tag{37}$$

Since $\{I_k\}_{k\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}$, there exists a constant $I_0 > 0$ such that $I(\xi)\le I_0$ on $\mathbb{R}$. Hence, the solution $(S(\xi), I(\xi))$ satisfies $0\le S(\xi)\le S_0$ and $0\le I(\xi)\le I_0$ for all $\xi\in\mathbb{R}$.

Next, we claim that $S(\xi)$ and $I(\xi)$ are non-trivial, and they satisfy the asymptotic boundary conditions of (6).

Claim 1: $I(\pm\infty) = 0$.

Integrating the first equation of (37) from y to x for any $x, y\in\mathbb{R}$, we have

$$\begin{aligned} \Big|\int_y^x \beta S(\xi)\int_0^\tau f(s)I(\xi - c_0s)dsd\xi\Big| &\le d_S\Big|\sum_{i\in\tilde{\Omega}} J_1(i)i\int_y^x\int_0^1 S'(\xi - ti)dtd\xi\Big| + c_0|S(x) - S(y)| \\ &\le d_S\Big|\sum_{i\in\tilde{\Omega}} J_1(i)i\int_0^1 (S(x - ti) - S(y - ti))\,dt\Big| + c_0\,|S(x) - S(y)| \\ &\le \Big(2d_S\sum_{i\in\tilde{\Omega}} J_1(i)|i| + 2c_0\Big)S_0. \end{aligned} \tag{38}$$

Thus, by the second equation of (37), we can obtain

$$\begin{aligned} \Big|\gamma\int_y^x I(\xi)d\xi\Big| \le{}& d_I\Big|\sum_{i\in\tilde{\Omega}} J_2(i)\int_y^x (I(\xi - i) - I(\xi))\,d\xi\Big| + \Big|\int_y^x \beta S(\xi)\int_0^\tau f(s)I(\xi - c_0s)dsd\xi\Big| \\ &+ c_0|I(x) - I(y)| \\ \le{}& \Big(2d_I\sum_{i\in\tilde{\Omega}} J_2(i)|i| + 2c_0\Big)I_0 + \Big(2d_S\sum_{i\in\tilde{\Omega}} J_1(i)|i| + 2c_0\Big)S_0. \end{aligned} \tag{39}$$

Due to the arbitrariness of x and y, it follows from (39) that $\int_{\mathbb{R}} I(\xi)d\xi < +\infty$. In addition, it is easy to see from (37) that $I'(\xi)$ is bounded on $\mathbb{R}$. Therefore, we have $I(\pm\infty) = 0$.

Claim 2: $S(\xi) > 0$ on $\mathbb{R}$ and $S(-\infty) = S_0$.

Assume that $S(\xi_0) = 0$ for some $\xi_0\in\mathbb{R}$. From the first equation of (37), we have $S(\xi_0\pm 1) = 0$, and inductively that $S(\xi_0\pm k) = 0$ for any $k\in\mathbb{N}$. Moreover, the first equation of (37) gives

$$c_0S'(\xi) + \delta S(\xi)\ge 0,\ \text{ for } \xi\in\mathbb{R},$$

where $\delta = (d_S \sum_{i\in\tilde{\Omega}} J_1(i) + \beta I_0)$, which implies that $S(\xi)e^{\frac{\delta}{c_0}\xi}$ is non-decreasing over $\mathbb{R}$. Then it follows that $S(\xi) \equiv 0$ on $\mathbb{R}$, and hence, $S_k(\xi) \to 0$ as $k \to +\infty$ for any $\xi \in \mathbb{R}$. On the other hand, let's write

$$\sum_{i\in\tilde{\Omega}} J_1(i) \int_{-\infty}^{0} (S_k(\xi - i) - S_k(\xi))\, d\xi = -\sum_{i\in\tilde{\Omega}} J_1(i) i \int_{-\infty}^{0} \int_{0}^{1} S_k'(\xi - ti) dt d\xi$$

$$= -\sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} \int_{-\infty}^{0} S_k'(\xi - ti) d\xi dt = \sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} (S_0 - S_k(-ti))\, dt = -\sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} S_k(-ti) dt,$$

then the first equation of (5) gives

$$c_k(S_0 - S_k(0)) = \int_{-\infty}^{0} \beta S_k(\xi) \int_{0}^{\tau} f(s) I_k(\xi - c_k s) ds d\xi + d_S \sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} S_k(-ti) dt. \tag{40}$$

As $k \to +\infty$, it follows from (40) that $c_0 S_0 = 0$, which leads to a contradiction. Hence $S(\xi) > 0$ on $\mathbb{R}$.

To prove $S(-\infty) = S_0$, it is sufficient to show that $\underline{S} := \liminf_{\xi \to -\infty} S(\xi) = S_0$. Suppose that $\underline{S} < S_0$, then there exists a sequence $\{\zeta_q\}_{q\in\mathbb{N}}$ such

$$\lim_{q\to+\infty} \zeta_q = -\infty \quad \text{and} \quad \lim_{q\to+\infty} S(\zeta_q) = \underline{S}.$$

Denote

$$\tilde{S}_q(\xi) := S(\xi + \zeta_q) \ \text{ and } \ \tilde{I}_q(\xi) := I(\xi + \zeta_q), \ \text{ for } \xi \in \mathbb{R}.$$

Obviously, $\lim_{q\to+\infty} \tilde{I}_q(\xi) = 0$ locally uniformly on $\mathbb{R}$. Since $||\tilde{S}_q||_{C^2(\mathbb{R})}$ is uniformly bounded on $\mathbb{R}$, there exists a subsequence of $\{\tilde{S}_q(\xi)\}_{q\in\mathbb{N}}$, still denote as $\{\tilde{S}_q(\xi)\}_{q\in\mathbb{N}}$, such that $\tilde{S}_q(\xi) \to \tilde{S}(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $q \to +\infty$. Hence, the first equation of (37) gives

$$c_0 \tilde{S}'(\xi) = d_S \sum_{i\in\tilde{\Omega}} J_1(i) \big(\tilde{S}(\xi - i) - \tilde{S}(\xi)\big). \tag{41}$$

It is easy to note that zero is the root of the characteristic equation of (41). Since $\tilde{S}(\xi)$ is nonnegative and bounded with $\tilde{S}(0) = \underline{S}$, it follows that $\tilde{S}(\xi) = \underline{S}$ on $\mathbb{R}$ according to Proposition 2.5. Hence, we get that $\tilde{S}_q(\xi) \to \underline{S}$ in $C^1_{loc}(\mathbb{R})$ as $q \to +\infty$. Since

$$c_k S_k'(\xi) = d_S \sum_{i\in\tilde{\Omega}} J_1(i) \left(S_k(\xi - i) - S_k(\xi)\right) - \beta S_k(\xi) \int_{0}^{\tau} f(s) I_k(\xi - c_k s) ds, \tag{42}$$

we have

$$c_k(S_k(\zeta_q) - S_0) = d_S \sum_{i\in\tilde{\Omega}} J_1(i) \int_{-\infty}^{\zeta_q} (S_k(\xi - i) - S_k(\xi))\, d\xi - \beta \int_{-\infty}^{\zeta_q} S_k(\xi) \int_{0}^{\tau} f(s) I_k(\xi - c_k s) ds d\xi.$$

Note that

$$\sum_{i\in\tilde{\Omega}} J_1(i) \int_{-\infty}^{\zeta_q} (S_k(\xi - i) - S_k(\xi))\, d\xi = -\sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} S_k(\zeta_q - ti) dt.$$

As $k \to +\infty$, one can see

$$c_0(S(\zeta_q) - S_0) = -d_S \sum_{i\in\tilde{\Omega}} J_1(i) i \int_{0}^{1} S(\zeta_q - ti) dt - \beta \int_{-\infty}^{\zeta_q} S(\xi) \int_{0}^{\tau} f(s) I(\xi - c_0 s) ds d\xi. \tag{43}$$

As $q \to +\infty$, it follows that $c_0(\underline{S} - S_0) = 0$, which contradicts to $\underline{S} < S_0$. Hence, $S(-\infty) = S_0$.

Claim 3: $I(\xi) > 0$ on $\mathbb{R}$.

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Assume that $I(\eta)=0$ for some $\eta \in \mathbb{R}$, then it is easy to see from the second equation of (37) that $I(\xi)\equiv 0$ on $\mathbb{R}$. Then, the first equation of (37) gives

$$c_0 S'(\xi) = d_S \sum_{i\in\tilde{\Omega}} J_1(i)\left(S(\xi-i)-S(\xi)\right). \tag{44}$$

Since $S(\xi)$ is nonnegative and bounded, by Proposition 2.5, it follows that $S(\xi)$ is a constant function, i.e., $S(\xi)\equiv\widehat{S}$ for some constant $\widehat{S}\in[0,S_0]$. Note that

$$I_k(\pm\infty)=0 \quad\text{and}\quad \int_{-\infty}^{+\infty}\int_0^{\tau} f(s)I_k(\xi-c_ks)ds d\xi = \int_{-\infty}^{+\infty} I_k(\xi)d\xi.$$

By the second equation of (5), we have

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \beta S_k(\xi)\int_0^{\tau} f(s)I_k(\xi-c_ks)dsd\xi - \gamma\int_{-\infty}^{+\infty} I_k(\xi)d\xi \\
&= \int_{-\infty}^{+\infty} \beta S_k(\xi)\int_0^{\tau} f(s)I_k(\xi-c_ks)dsd\xi - \gamma\int_{-\infty}^{+\infty}\int_0^{\tau} f(s)I_k(\xi-c_ks)dsd\xi \\
&= \int_{-\infty}^{+\infty}\left(\beta S_k(\xi)-\gamma\right)\int_0^{\tau} f(s)I_k(\xi-c_ks)dsd\xi.
\end{aligned} \tag{45}$$

Note that $\int_0^{\tau} f(s)I_k(\xi-c_ks)ds>0$ for any $\xi\in\mathbb{R}$. There exists some η_k such that $\gamma=\beta S_k(\eta_k)$. Since $(S_k(\xi),I_k(\xi))$ is translation invariant, we may assume $\eta_k=0$. As $k\to+\infty$, we have

$$\gamma=\beta S_k(0)\to\beta S(0)=\beta\widehat{S},$$

which implies $\widehat{S}=\gamma/\beta$. Since $R_0=\beta S_0/\gamma>1$, then $S(\xi)\equiv\widehat{S}<S_0$, which contradicts to $S(-\infty)=S_0$. So, $I(\xi)>0$ on $\mathbb{R}$.

Claim 4: $S(+\infty)=S_\infty<S_0$.

We first show that $\bar{\alpha}:=\liminf_{\xi\to+\infty} S(\xi)=\limsup_{\xi\to+\infty} S(\xi)=:\hat{\alpha}$. If $\bar{\alpha}<\hat{\alpha}$, one can find two sequences $\{\xi_k\}_{k\in\mathbb{N}}$ and $\{\eta_k\}_{k\in\mathbb{N}}$, with $\xi_k<\eta_k$, $\lim_{k\to+\infty}\xi_k=+\infty$ and $\lim_{k\to+\infty}\eta_k=+\infty$, such that

$$\lim_{k\to+\infty} S(\xi_k)=\bar{\alpha},\ S'(\xi_k)=0; \text{ and } \lim_{k\to+\infty} S(\eta_k)=\hat{\alpha},\ S'(\eta_k)=0.$$

Denote

$$\hat{S}_k(\xi):=S(\xi+\xi_k) \quad\text{and}\quad \hat{I}_k(\xi):=I(\xi+\xi_k),\ \text{for } \xi\in\mathbb{R}.$$

Obviously, $\hat{I}_k(\xi)\to 0$ locally uniformly on $\mathbb{R}$ as $k\to+\infty$. Since $||\hat{S}_k||_{C^2(\mathbb{R})}$ is uniformly bounded on $\mathbb{R}$, there exists a subsequence of $\{\hat{S}_k(\xi)\}_{k\in\mathbb{N}}$, still written as $\{\hat{S}_k(\xi)\}_{k\in\mathbb{N}}$, such that $\hat{S}_k(\xi)\to\hat{S}(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $k\to+\infty$. Hence, the first equation of (37) yields

$$c_0\hat{S}'(\xi)=d_S\sum_{i\in\tilde{\Omega}} J_1(i)\big(\hat{S}(\xi-i)-\hat{S}(\xi)\big). \tag{46}$$

By Proposition 2.5, it follows that $\hat{S}(\xi)$ is constant function. Since $\hat{S}(0)=\bar{\alpha}$, we have $\hat{S}(\xi)\equiv\bar{\alpha}$ on $\mathbb{R}$. Hence, we get that $S(\xi+\xi_k)\to\bar{\alpha}$ in $C^1_{loc}(\mathbb{R})$ as $k\to+\infty$. By the same way, we also obtain that

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$\lim_{k\to+\infty} S(\xi+\eta_k)=\hat{\alpha}$ in $C^1_{loc}(\mathbb{R})$. Then, integrating the first equation of (37) from ξ_k to η_k, we have

$$\begin{aligned}c_0(\hat{\alpha}-\bar{\alpha})=&\lim_{k\to+\infty} d_S\sum_{i\in\tilde{\Omega}}J_1(i)\int_{\xi_k}^{\eta_k}(S(\xi-i)-S(\xi))\,d\xi-\lim_{k\to+\infty}\beta\int_{\xi_k}^{\eta_k}S(\xi)\int_0^\tau f(s)I(\xi-c_0s)dsd\xi\\ \le&\lim_{k\to+\infty} d_S\sum_{i\in\tilde{\Omega}}J_1(i)\int_{\xi_k}^{\eta_k}(S(\xi-i)-S(\xi))\,d\xi\\ =&d_S\lim_{k\to+\infty}\sum_{i\in\tilde{\Omega}}J_1(i)i\int_0^1(S(\xi_k-ti)-S(\eta_k-ti))\,dt=0,\end{aligned}$$

which contradicts to $\bar{\alpha}<\hat{\alpha}$. Thus, $\hat{\alpha}=\bar{\alpha}$, and then $S_\infty:=S(+\infty)$ exists.

Next, we further show that $S_\infty<S_0$ by proving $\liminf\limits_{\xi\to+\infty}S(\xi)<S_0$. Suppose, by contradiction, that $\liminf\limits_{\xi\to+\infty}S(\xi)=S_0$, then $S(+\infty)=S_\infty=S_0$ and the first equation of (37) gives

$$\begin{aligned}0=&d_S\sum_{i\in\tilde{\Omega}}J_1(i)\int_{-\infty}^{+\infty}(S(\xi-i)-S(\xi))\,d\xi-\beta\int_{-\infty}^{+\infty}S(\xi)\int_0^\tau f(s)I(\xi-c_0s)dsd\xi\\ =&-d_S\sum_{i\in\tilde{\Omega}}J_1(i)i\int_{-\infty}^{+\infty}\int_0^1 S'(\xi-ti)dtd\xi-\beta\int_{-\infty}^{+\infty}S(\xi)\int_0^\tau f(s)I(\xi-c_0s)dsd\xi\\ =&-d_S\sum_{i\in\tilde{\Omega}}J_1(i)i\int_0^1\int_{-\infty}^{+\infty}S'(\xi-ti)d\xi dt-\beta\int_{-\infty}^{+\infty}S(\xi)\int_0^\tau f(s)I(\xi-c_0s)dsd\xi\\ =&-\beta\int_{-\infty}^{+\infty}S(\xi)\int_0^\tau f(s)I(\xi-c_0s)dsd\xi<0,\end{aligned}$$

which gives a contradiction. Thus, $S_\infty<S_0$. The proof is complete. □

4. Proof of the main results

To prove the existence result of travelling wave solutions by using the truncation method, we first establish the existence of solutions for the profile system (6) over large finite domains.

Lemma 4.1. *Assume $R_0>1$. For any $c>c^*$ and large $\mathcal{X}>0$, the bounded domain problem*

$$\begin{cases} cS'(\xi)=d_S\sum_{i\in\tilde{\Omega}}J_1(i)\,(S(\xi-i)-S(\xi))-\beta S(\xi)\int_0^\tau f(s)I(\xi-cs)ds,\\ cI'(\xi)=d_I\sum_{i\in\tilde{\Omega}}J_2(i)\,(I(\xi-i)-I(\xi))+\beta S(\xi)\int_0^\tau f(s)I(\xi-cs)ds-\gamma I(\xi),\end{cases}\quad \text{for } \xi\in(-\mathcal{X},\mathcal{X}) \tag{47}$$

has a solution $(S(\xi),I(\xi))$ satisfying $0\le S_-(\xi)\le S(\xi)\le S_0$ and $0\le I_-(\xi)\le I(\xi)\le I_+(\xi)$ with

$$S_-(\xi):=\max\{S_0-\delta e^{\nu\xi},0\},\ I_-(\xi):=\max\{e^{\lambda_1\xi}(1-Ke^{\varsigma\xi}),0\}\ \text{ and } I_+(\xi):=e^{\lambda_1\xi},$$

when $\nu\in(0,\lambda_1)$, $\varsigma\in(0,\min\{\nu,\lambda_2-\lambda_1\})$ are small enough; and $\delta>S_0$, $K>1$ are sufficiently large.

Proof. The idea of proof is similar to that of [28, Proposition 3.1]. However, due to the consideration of distributed latent period, the computations are more complicated than those of [28].

Firstly, for large $Y>0$, we define the set

$$\begin{aligned}\Phi_Y:=&\big\{(\omega(\cdot),\sigma(\cdot))\in C([-Y,Y],\mathbb{R}^2):S_-(\xi)\le\omega(\xi)\le S_0,\ I_-(\xi)\le\sigma(\xi)\le I_+(\xi),\ \xi\in[-Y,Y]\big\}\\ &\cap\big\{(\omega(\cdot),\sigma(\cdot))\in C([-Y,Y],\mathbb{R}^2):\omega(-Y)=S_-(-Y),\ \sigma(-Y)=I_-(-Y)\big\}.\end{aligned}$$

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Obviously, Φ_Y is a closed and convex set. In addition, we extend any function $(\tilde{\Omega}(\cdot), \sigma(\cdot))$ to $(\tilde{\omega}(\cdot), \tilde{\sigma}(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2)$ in the way

$$(\tilde{\omega}(\xi), \tilde{\sigma}(\xi)) = \begin{cases} (S_-(\xi), I_-(\xi)), & \text{for } \xi < -Y; \\ (\omega(\xi), \sigma(\xi)), & \text{for } |\xi| \leq Y; \\ (\omega(Y), \sigma(Y)), & \text{for } \xi > Y. \end{cases}$$

For the convenience of statement, without loss of generality, we next assume that $\sum_{i\in\tilde{\Omega}} J_k(i) = 1$ ($k = 1, 2$). Define an operator Γ on Φ_Y by

$$\Gamma := (\Gamma_1, \Gamma_2):(\omega(\cdot), \sigma(\cdot)) \in \Phi_Y \to (S(\cdot), I(\cdot)) \in C^1([-Y, Y]),$$

where $(S(\xi), I(\xi))$ satisfies the following initial value problem of ODE:

$$cS'(\xi) = -(d_S + l)S(\xi) + d_S \sum_{i\in\tilde{\Omega}} J_1(i)\tilde{\omega}(\xi - i) + l\omega(\xi) - \beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds, \tag{48}$$

$$cI'(\xi) = -(d_2 + \gamma)I(\xi) + d_I \sum_{i\in\tilde{\Omega}} J_2(i)\tilde{\sigma}(\xi - i) + \beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds, \tag{49}$$

$$S(-Y) = S_-(-Y), \ \ I(-Y) = I_-(-Y), \tag{50}$$

where l is a constant satisfying $l \geq \beta e^{\lambda_1 Y} \int_0^\tau f(s)e^{-c\lambda_1 s}ds$. Then we claim that Γ is completely continuous which maps from Φ_Y to Φ_Y.

We first show that $\Gamma[\Phi_Y] \subseteq \Phi_Y$, i.e.,

$$S_-(\xi) \leq \Gamma_1[\omega, \sigma](\xi) \leq S_0 \ \text{ and } \ I_-(\xi) \leq \Gamma_2[\omega, \sigma](\xi) \leq I_+(\xi), \ \text{ for } \xi \in [-Y, Y]. \tag{51}$$

It is easy to see that 0 is a lower solution of (48). By comparison principle, we have

$$\Gamma_1[\omega, \sigma](\xi) \geq 0, \ \text{ for } \xi \in [-Y, Y].$$

Since $0 \leq \tilde{\omega}(\xi) \leq S_0$ on $\mathbb{R}$, for $\xi \in [-Y, Y]$, one can obtain

$$\begin{aligned} &-(d_S + l)S_0 + d_S \sum_{i\in\tilde{\Omega}} J_1(i)\tilde{\omega}(\xi - i) + l\omega(\xi) - \beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds \\ &\leq -\beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds \leq 0, \end{aligned}$$

which implies that S_0 is an upper solution of (48). Then, by comparison principle again,

$$\Gamma_1[\omega, \sigma](\xi) \leq S_0, \ \text{ for } \xi \in [-Y, Y].$$

Hence, the left part of (51) holds.

Now we prove the right part of (51). According to the choice of l, it is easy to see that

$$h(\omega) := l\omega(\xi) - \beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds$$

is non-decreasing in ω. Then, for $\xi \in [-Y, \frac{1}{\nu} \ln \frac{S_0}{\delta})$, one can see $S_-(\xi) = S_0 - \delta e^{\nu\xi}$ and

$$\begin{aligned} &cS_-'(\xi) - d_S \sum_{i\in\tilde{\Omega}} J_1(i)\tilde{\omega}(\xi - i) + (d_S + l)S_-(\xi) - l\omega(\xi) + \beta\omega(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds \\ \leq &cS_-'(\xi) - d_S \sum_{i\in\tilde{\Omega}} J_1(i)S_-(\xi - i) + (d_S + l)S_-(\xi) - lS_-(\xi) + \beta S_-(\xi) \int_0^\tau f(s)\tilde{\sigma}(\xi - cs)ds \\ \leq &cS_-'(\xi) - d_S \sum_{i\in\tilde{\Omega}} J_1(i)S_-(\xi - i) + d_S S_-(\xi) + \beta S_-(\xi) \int_0^\tau f(s)I_+(\xi - cs)ds. \end{aligned} \tag{52}$$

Since $I_+(\xi)=e^{\lambda_1\xi}$ and $S_-(\xi)\geq S_0-\delta e^{\nu\xi}$ for $\xi\in\mathbb{R}$, direct calculation gives

$$cS_-'(\xi)-d_S\sum_{i\in\tilde{\Omega}}J_1(i)S_-(\xi-i)+d_SS_-(\xi)+\beta S_-(\xi)\int_0^\tau f(s)I_+(\xi-cs)ds\leq 0,$$

for small ν and large δ. Then it follows from (52) that $S_-(\xi)$ is a lower solution of (48). By comparison principle, one can obtain

$$S_-(\xi)\leq\Gamma_1[\omega,\sigma](\xi),\ \text{for}\ \xi\in[-Y,\frac{1}{\nu}\ln\frac{S_0}{\delta}).$$

On the other hand, when $\xi\in[\frac{1}{\nu}\ln\frac{S_0}{\delta},Y]$, we have $S_-(\xi)=0$. Thus,

$$S_-(\xi)\leq\Gamma_1[\omega,\sigma](\xi)\leq S_0,\quad\text{for}\ \xi\in[-Y,Y].$$

Similarly, we can obtain

$$I_-(\xi)\leq\Gamma_2[\omega,\sigma](\xi)\leq I_+(\xi),\ \text{for}\ \xi\in[-Y,Y].$$

Hence, the inequalities of (51) hold.

Next, we show that $\Gamma[\cdot,\cdot]$ is continuous on Φ_Y. Assume that $(\omega_i(\xi),\sigma_i(\xi))\in\Phi_Y(i=1,2)$ and $\Gamma_2[\omega_i,\sigma_i](\xi)=I_i(\xi)(i=1,2)$ for $\xi\in[-Y,Y]$, one can verify that

$$c(I_1'(\xi)-I_2'(\xi))+(d_I+\gamma)(I_1(\xi)-I_2(\xi))=h(\xi),$$

where

$$h(\xi):=d_I\sum_{i\in\tilde{\Omega}}J_2(i)\left(\tilde{\sigma}_1(\xi-i)-\tilde{\sigma}_2(\xi-i)\right)+$$
$$\beta\Big(\omega_1(\xi)\int_0^\tau f(s)\tilde{\sigma}_1(\xi-cs)ds-\omega_2(\xi)\int_0^\tau f(s)\tilde{\sigma}_2(\xi-cs)ds\Big).$$

Thus,

$$I_1(\xi)-I_2(\xi)=\frac{1}{c}\int_{-Y}^{\xi}e^{\frac{d_I+\gamma}{c}(x-\xi)}h(x)dx,\tag{53}$$

Note that for $x\in[-Y,Y]$, one has

$$|\tilde{\sigma}_1(x-i)-\tilde{\sigma}_2(x-i)|\leq\max_{x\in[-Y,Y]}|\sigma_1(x)-\sigma_2(x)|$$

and

$$\left|\omega_1(x)\int_0^\tau f(s)\tilde{\sigma}_1(x-cs)ds-\omega_2(x)\int_0^\tau f(s)\tilde{\sigma}_2(x-cs)ds\right|$$
$$\leq|\omega_1(x)-\omega_2(x)|\int_0^\tau f(s)\tilde{\sigma}_1(x-cs)ds+\omega_2(x)\int_0^\tau f(s)\left|\tilde{\sigma}_1(x-cs)-\tilde{\sigma}_2(x-cs)\right|ds$$
$$\leq e^{\lambda_1Y}\max_{x\in[-Y,Y]}|\omega_1(x)-\omega_2(x)|+S_0\max_{x\in[-Y,Y]}|\sigma_1(x)-\sigma_2(x)|\,.$$

Then, for $x\in[-Y,Y]$, it follows that

$$|h(x)|\leq\beta e^{\lambda_1Y}\max_{x\in[-Y,Y]}|\omega_1(x)-\omega_2(x)|+(\beta S_0+d_I)\max_{x\in[-Y,Y]}|\sigma_1(x)-\sigma_2(x)|\,.\tag{54}$$

Combine (53) and (54), we have

$$|I_1(\xi)-I_2(\xi)|\leq\frac{1}{c}\int_{-Y}^{\xi}e^{\frac{d_I+\gamma}{c}(x-\xi)}|h(x)|dx\leq\frac{\kappa}{d_I+\gamma}(1-e^{-\frac{d_I+\gamma}{c}(Y+\xi)})\leq\frac{\kappa}{d_I+\gamma},$$

where

$$\kappa:=\ \beta e^{\lambda_1Y}\max_{x\in[-Y,Y]}|\omega_1(x)-\omega_2(x)|+(\beta S_0+d_I)\max_{x\in[-Y,Y]}|\sigma_1(x)-\sigma_2(x)|\,.$$

Thus, Γ_2 is continuous on Φ_Y. Similarly, Γ_1 is continuous on Φ_Y. Furthermore, in view of (48) and (49), $S'(\xi)$ and $I'(\xi)$ are bounded on $[-Y, Y]$. Then, it can be deduced by the Arzela-Ascoli theorem that Γ is compact. So, Γ is completely continuous which maps from Φ_Y to Φ_Y.

Finally, by Schauder's fixed point theorem, one can see that there exists a fixed point $(S(\xi), I(\xi)) \in \Phi_Y$ such that

$$(S(\xi), I(\xi)) = \Gamma[S, I](\xi), \text{ for } \xi \in [-Y, Y].$$

Clearly, $(S(\xi), I(\xi))$ satisfies (47) with $\mathcal{X} = Y - 2 - c\tau$,

$$0 \le S_-(\xi) \le S(\xi) \le S_0 \text{ and } 0 \le I_-(\xi) \le I(\xi) \le I_+(\xi).$$

The proof is complete. □

Based on the previous lemmas, we are ready to prove the main results.

Proof of Theorem 1.1.

(1) Let $\{\mathcal{X}_m\}_{m\in\mathbb{N}}$ be an increasing sequence satisfying $\mathcal{X}_m \to +\infty$ as $m \to +\infty$. According to Lemma 4.1, when m is large enough (say $m \ge m_0 \gg 1$), we denote $(S_m(\xi), I_m(\xi))$ as the solution of (47) over $[-\mathcal{X}_m, \mathcal{X}_m]$. Note that $\{S(\xi)\}_{m\ge m_0}$ and $\{I_m(\xi)\}_{m\ge m_0}$ are uniformly bounded on $[-\mathcal{X}_{m_0}, \mathcal{X}_{m_0}]$, it follows from (47) that $\{S'_m(\xi)\}_{m\ge m_0}$ and $\{I'_m(\xi)\}_{m\ge m_0}$ are uniformly bounded on $[-\mathcal{X}_{m_0} + a, \mathcal{X}_{m_0} - a]$, where $a = 2 + c\tau$. Then, for any $\xi_1, \eta_1 \in [-\mathcal{X}_{m_0} + 2a, \mathcal{X}_{m_0} - 2a]$, we have

$$\begin{aligned}
&|S'_m(\xi_1) - S'_m(\eta_1)|\\
\le &\frac{d_S}{c}\sum_{i\in\tilde{\Omega}} J_1(i)\,|S_m(\xi_1 - i) - S_m(\eta_1 - i)| + \frac{d_S}{c}|S_m(\xi_1) - S_m(\eta_1)|\\
&+ \frac{\beta}{c}\left|S_m(\xi_1)\int_0^\tau f(s)I_m(\xi_1 - cs)ds - S_m(\eta_1)\int_0^\tau f(s)I_m(\eta_1 - cs)ds\right|\\
\le &\frac{d_S}{c}\sum_{i\in\tilde{\Omega}} J_1(i)\Big|S_m(\xi_1 - i) - S_m(\eta_1 - i)\Big| + \frac{d_S}{c}|S_m(\xi_1) - S_m(\eta_1)|\\
&+ \frac{\beta}{c}\,|S_m(\xi_1) - S_m(\eta_1)|\int_0^\tau f(s)I_m(\xi_1 - cs)ds + \frac{\beta S_0}{c}\int_0^\tau f(s)\,|I_m(\xi_1 - cs) - I_m(\eta_1 - cs)|\,ds,
\end{aligned}$$

which implies that $\{S'_m(\xi)\}_{m\ge m_0}$ and $\{I'_m(\xi)\}_{m\ge m_0}$ are equicontinuous on $[-\mathcal{X}_{m_0} + 2a, \mathcal{X}_{m_0} - 2a]$. Moreover, for any compact set Λ of $\mathbb{R}$, there is some $q_0 \in \mathbb{N}_+$ such that $\Lambda \subset [-\mathcal{X}_m + 2a, \mathcal{X}_m - 2a]$ for any $m \ge q_0$. Then, it follows from the Arzela-Ascoli theorem that there exists a subsequence $\{(S_{m_k}(\xi), I_{m_k}(\xi))\}_{m_k\ge m_0}$ of $\{(S_m(\xi), I_m(\xi))\}_{m\ge m_0}$ such that $S_{m_k}(\xi) \to S(\xi)$ and $I_{m_k}(\xi) \to I(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $k \to +\infty$. It is clear that $(S(\xi), I(\xi))$ is a solution of (5) that satisfies

$$S_-(\xi) \le S(\xi) \le S_0 \text{ and } I_-(\xi) \le I(\xi) \le I_+(\xi), \text{ for } \xi \in \mathbb{R}.$$

Furthermore, we claim that $S(\xi) > 0$ and $I(\xi) > 0$ on $\mathbb{R}$. Suppose that $S(\overline{\eta}) = 0$ for some $\overline{\eta} \in \mathbb{R}$, then $S'(\overline{\eta}) = 0$ and the first equation of (5) implies that $S(\overline{\eta} \pm 1) = 0$. By induction argument, we have $S(\overline{\eta} \pm k) = 0$ for any $k \in \mathbb{N}$. Let k be sufficiently large such that $\overline{\eta} - k < \frac{1}{\nu}(\ln S_0 - \ln \delta)$, one can see $S(\overline{\eta} - k) \ge S_-(\overline{\eta} - k) > 0$, which gives a contradiction. Hence, $S(\xi) > 0$ on $\mathbb{R}$. Similarly, one can obtain $I(\xi) > 0$ on $\mathbb{R}$.

Next, we show that the positive solution $(S(\xi), I(\xi))$ satisfies the condition (6). According to the facts $S_-(-\infty) = S_0$ and $S_-(\xi) \le S(\xi) \le S_0$ for $\xi \in \mathbb{R}$, it is clear that $S(-\infty) = S_0$. To prove $I(\pm\infty) = 0$ and $S(+\infty) < S_0$, we first show that $I(\xi)$ is bounded on $\mathbb{R}$. Suppose that $\limsup_{\xi\to+\infty} I(\xi) = +\infty$, if $\sigma := \liminf_{\xi\to+\infty} I(\xi) < +\infty$ then there exists a sequence $\{s_j\}_{j\in\mathbb{N}}$ satisfying $s_j \to +\infty$ such that $I(s_j) \to \sigma$ as

https://doi.org/10.1017/S0956792524000846 Published online by Cambridge University Press

$j \to +\infty$. Without loss of generality, we may assume that $I(s_j) < \sigma + 1$ for $j \in \mathbb{N}$. Given any j, one can find $\xi_j \in [s_j, s_{j+1}]$ such that

$$I(\xi_j) = \max\{I(\xi) | \xi \in [s_j, s_{j+1}]\}.$$

Clearly, $\lim_{j\to+\infty} I(\xi_j) = +\infty$. Then, by Lemma 3.2, we have $\lim_{j\to+\infty} S(\xi_j) = 0$. Without loss of generality, we may assume that $I(\xi_j) \geq (1+\sigma)e^{2C_0}$ for $j \in \mathbb{N}$, where $C_0 := \sup_{\xi\in\mathbb{R}} |I'(\xi)/I(\xi)|$ (cf. (35)). Then we have

$$\frac{I(\xi_j)}{I(\xi)} = e^{\int_\xi^{\xi_j} \frac{I'(\eta)}{I(\eta)} d\eta} \leq e^{2C_0}, \text{ for } \xi \in [\xi_j - 2, \xi_j + 2].$$

Thus,

$$I(\xi) \geq I(\xi_j)e^{-2C_0} \geq \sigma + 1, \text{ for } \xi \in [\xi_j - 2, \xi_j + 2]. \tag{55}$$

It then follows that $[\xi_j - 2, \xi_j + 2] \subset [s_j, s_{j+1}]$. In fact, if $\xi_j - 2 < s_j$ and (or) $\xi_j + 2 > s_{j+1}$, then (55) contradicts to $I(s_j) < \sigma + 1$. Hence, $[\xi_j - 2, \xi_j + 2] \subset [s_j, s_{j+1}]$. Then, by the second equation of (5) and (32), we have

$$\begin{aligned} 0 = cI'(\xi_j) =& d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(I(\xi_j - i) - I(\xi_j)\big) + \beta S(\xi_j) \int_0^\tau f(s)I(\xi_j - cs)ds - \gamma I(\xi_j) \\ \leq& \beta S(\xi_j) \int_0^\tau f(s)I(\xi_j - cs)ds - \gamma I(\xi_j) < \Big(\beta S(\xi_j) \int_0^\tau f(s)e^{(d_I+\gamma)s}ds - \gamma\Big)I(\xi_j), \end{aligned}$$

which is impossible since $I(\xi_j) \to +\infty$ and $S(\xi_j) \to 0$ as $j \to +\infty$. Thus, we have $I(+\infty) = +\infty$. It then follows from Lemma 3.2 that $S(+\infty) = 0$.

On the other hand, dividing the second equation of (5) by $I(\xi)$, we have

$$c\zeta(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(e^{\int_\xi^{\xi-i}\zeta(x)dx} - 1\big) + \beta S(\xi) \int_0^\tau f(s)\frac{I(\xi - cs)}{I(\xi)}ds - \gamma,$$

where $\zeta(\xi) := I'(\xi)/I(\xi)$. Since $S(+\infty) = 0$ and

$$\int_0^\tau f(s)\frac{I(\xi - cs)}{I(\xi)}ds < \int_0^\tau f(s)e^{(d_I+\gamma)s}ds < +\infty, \text{ for } \xi \in \mathbb{R},$$

by Remark 1, it follows that the limit $\lambda_3 := \zeta(+\infty)$ exists, and it is a real root of the characteristic equation

$$\Delta(\lambda) := d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(e^{-\lambda i} - 1\big) - c\lambda - \gamma = 0. \tag{56}$$

Note that $\lambda_3 = \zeta(+\infty) \geq 0$ due to $I(+\infty) = +\infty$. Moreover, $\Delta(\lambda)$ is convex with $\Delta(0) = -\gamma < 0$. Thus, it follows that λ_3 is the unique positive real root of the characteristic equation (56). On other hand, it is clear that

$$\Delta(\lambda_k) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(e^{-i\lambda_k} - 1\big) - c\lambda_k - \gamma = -\beta S_0 \int_0^\tau f(s)e^{-cs\lambda_k}ds < 0,$$

where λ_k $(k = 1, 2)$ is the positive real root of $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$. It follows that $\lambda_1 < \lambda_2 < \lambda_3$. Since $\lim_{\xi\to+\infty} I'(\xi)/I(\xi) = \lambda_3 > \frac{\lambda_2+\lambda_3}{2}$, there is some sufficiently large constant $X > 0$ and some constant $C_1(X) > 0$ such that

$$I(\xi) \geq C_1(X)e^{\frac{(\lambda_2+\lambda_3)\xi}{2}}, \text{ for } \xi \geq X,$$

which contradicts to the fact $I(\xi) \leq e^{\lambda_1\xi}$ on $\mathbb{R}$. Therefore, $\limsup_{\xi\to+\infty} I(\xi) < \infty$, i.e., $I(\xi)$ is bounded on $\mathbb{R}$. By the boundedness of $I(\xi)$, following the same proof procedure as Lemma 3.3, we can obtain that $I(\pm\infty) = 0$ and $S(+\infty) < S_0$.

Now we show the existence of travelling wave with speed $c = c^*$. Let $\{c_k\}_{k\in\mathbb{N}} \subset (c^*, 2c^*)$ be a strictly decreasing sequence satisfying $c_k \to c^*$ as $k \to \infty$. According to the above proof, we denote $(S_k(\xi), I_k(\xi))$ as the solutions of (5) satisfying (6) with $c = c_k$. Then, as a consequence of Lemma 3.3, we can obtain that (5) admits a solution $(S(\xi), I(\xi))$ satisfying (6) with wave speed $c = c^*$.

(2) By Lemma 3.1, we only need to show that (5) has no positive solution satisfying (6) for $c \in (0, c^*)$. Assume that (5) admits a positive solution $(S(\xi), I(\xi))$ satisfying (6) with $c \in (0, c^*)$. Let $\{\overline{\xi}_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $\overline{\xi}_n \to -\infty$ as $n \to +\infty$, we consider the functions

$$\overline{S}_n(\xi) := S(\overline{\xi}_n + \xi) \;\text{ and }\; \overline{I}_n(\xi) := I(\overline{\xi}_n + \xi)/I(\overline{\xi}_n), \;\text{ for } \xi \in \mathbb{R}.$$

Obviously, $\lim_{n\to+\infty} \overline{S}_n(\xi) = S_0$ locally uniformly on $\mathbb{R}$, and $(\overline{S}_n(\xi), \overline{I}_n(\xi))$ satisfies the equation

$$c\overline{I}_n'(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(\overline{I}_n(\xi - i) - \overline{I}_n(\xi)\big) + \beta\overline{S}_n(\xi)\int_0^\tau f(s)\overline{I}_n(\xi - cs)ds - \gamma\overline{I}_n(\xi). \tag{57}$$

Denote $E(\xi) := I'(\xi)/I(\xi)$, which is bounded on $\mathbb{R}$ (cf.(35)). Since

$$\overline{I}_n(\xi) = e^{\int_{\overline{\xi}_n}^{\overline{\xi}_n+\xi} E(y)dy},$$

it follows that $\overline{I}_n(\xi)$ is locally uniformly bounded on $\mathbb{R}$. Further, by (57), $\overline{I}_n'(\xi)$ and $\overline{I}_n''(\xi)$ are locally uniformly bounded on $\mathbb{R}$. Thus, there exists a subsequence of $\{\overline{I}_n(\xi)\}_{n\in\mathbb{N}}$, still written as $\{\overline{I}_n(\xi)\}_{n\in\mathbb{N}}$, such that $\overline{I}_n(\xi) \to \overline{I}(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$. It follows from (57) that $\overline{I}(\xi)$ satisfies

$$c\overline{I}'(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)\big(\overline{I}(\xi - i) - \overline{I}(\xi)\big) + \beta S_0 \int_0^\tau f(s)\overline{I}(\xi - cs)ds - \gamma\overline{I}(\xi). \tag{58}$$

It is clear that $\overline{I}(0) = 1$ and $\overline{I}(\xi) \geq 0$ on $\mathbb{R}$. We claim that $\overline{I}(\xi) > 0$ on $\mathbb{R}$. If the claim is false, then there exists some $\overline{\xi}_0 \in \mathbb{R}$ such that $\overline{I}(\overline{\xi}_0) = 0$ and $\overline{I}'(\overline{\xi}_0) = 0$. It can be further deduced from (58) that $\overline{I}(\xi) \equiv 0$ on $\mathbb{R}$, which is impossible since $\overline{I}(0) = 1$. Thus, $\overline{I}(\xi) > 0$ on $\mathbb{R}$.

Finally, we define $\overline{E}(\xi) := \overline{I}'(\xi)/\overline{I}(\xi)$ for $\xi \in \mathbb{R}$. By (58), $\overline{E}(\xi)$ satisfies

$$c\overline{E}(\xi) = d_I \sum_{i\in\tilde{\Omega}} J_2(i)e^{\int_\xi^{\xi-i}\overline{E}(y)dy} + \beta S_0 \int_0^\tau f(s)e^{\int_\xi^{\xi-cs}\overline{E}(y)dy}ds - (\gamma + d_I \sum_{i\in\tilde{\Omega}} J_2(i)). \tag{59}$$

According to Lemma 2.3, the limits $\overline{E}(\pm\infty)$ exist, which are real roots of the characteristic equation $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$. However, by the definition of c^*, $\Psi(d_I, \beta S_0, -d_I \sum_{i\in\tilde{\Omega}} J_2(i) - \gamma, c, \lambda) = 0$ has no nonnegative real roots for $0 < c < c^*$. This gives a contradiction. The proof of Theorem 1.1 is complete.

Acknowledgements. We are very grateful to the anonymous referees for their careful reading and helpful suggestions which led to an improvement of our original manuscript.

Funding statement. The first author was partially supported by the NNSF of China (No. 12071182), the Fundamental Research Funds for the Central Universities.

The third author was partially supported by the NSTC and NCTS of Taiwan (No. NSTC 112-2115-M-008-005-MY2).

Competing interests. No conflict of interest exists in the submission of this manuscript.

References

[1] Allen, L. J. S., Bolker, B. M., Lou, Y. & Nevai, A. L. (2007) Asymptotic profiles of the steady states for an SIS epidemic patch model. *SIAM J. Appl. Math.* **67**(5), 1283–1309.

[2] Bai, Z. & Zhang, S. L. (2015) Traveling waves of a diffusive SIR epidemic model with a class of nonlinear incidence rates and distributed delay. *Commun. Nonlinear Sci. Numer. Simul.* **22**(1-3), 1370–1381.

[3] Beretta, E., Hara, T., Ma, W. & Takeuchi, Y. (2001) Global asymptotic stability of an SIR epidemic model with distributed time delay. *Nonlinear Anal.* **47**(6), 4107–4115.

[4] Brown, K. J. & Carr, J. (1977) Deterministic epidemic waves of critical velocity. *Math. Proc. Camb. Philos. Soc.* **81**(3), 431–433.

[5] Chen, X. F. & Guo, J.-S. (2003) Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. *Math. Ann.* **326**(1), 123–146.

[6] Chen, Y. Y., Guo, J.-S. & Hamel, F. (2017) Traveling waves for a lattice dynamical system arising in a diffusive endemic model. *Nonlinearity* **30**(6), 2334–2359.

[7] Connell McCluskey, C. (2010) Complete global stability for an SIR epidemic model with delay-distributed or discrete. *Nonlinear Analysis: RWA* **11**, 55–59.

[8] Cui, R., Lam, K. Y. & Lou, Y. (2017) Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments. *J. Differ. Equ.* **263**(4), 2343–2373.

[9] Ducrot, A., Magal, P. & Ruan, S. G. (2010) Travelling wave solutions in multi-group age-structured epidemic models. *Arch. Ration. Mech. Anal* **195**(1), 311–331.

[10] Fang, J. & Zhao, X.-Q. (2011) Monotone wavefronts of the nonlocal Fisher-KPP equation. *Nonlinearity* **24**(11), 3043–3054.

[11] Fu, S.-C., Guo, J.-S. & Wu, C.-C. (2016) Traveling wave solutions for a discrete diffusive epidemic model. *J. Nonlinear Convex Anal* **17**, 1739–1751.

[12] Guo, J.-S. & Lin, Y.-C. (2012) Traveling wave solution for a lattice dynamical system with convolution type nonlinearity. *Discrete Contin. Dyn. Syst* **32**(1), 101–124.

[13] Hu, H. J. & Zou, X. F. (2021) Traveling waves of a diffusive SIR epidemic model with general nonlinear incidence and infinitely distributed latency but without demography. *Nonlinear Anal. Real World Appl.* **58**, 103224.

[14] Hsu, C.-H. & Yang, T.-S. (2013) Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models. *Nonlinearity* **26**(1), 121–139.

[15] Lam, K. Y., Wang, X. & Zhang, T. (2018) Traveling waves for a class of diffusive disease-transmission models with network structures. *SIAM J. Math. Anal.* **50**(6), 5719–5748.

[16] Lou, Y. & Salako, R. (2023) Mathematical analysis of the dynamics of some reaction-diffusion models for infectious diseases. *J. Differ. Equ.* **370**, 424–469.

[17] Ruan, S. G. & Xiao, D. M. (2004) Stability of steady states and existence of traveling waves in a vector disease model. *Proc. Roy. Soc. Edinburgh Sect. A* **134**(5), 991–1011.

[18] San, X. F., Wang, Z. C. & Feng, Z. S. (2020) Spreading speed and traveling waves for an epidemic model in a periodic patchy environment. *Commun. Nonlinear Sci. Numer. Simul.* **90**, 105387.

[19] Thieme, H. R. & Zhao, X.-Q. (2003) Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. *J. Differ. Equ.* **195**(2), 430–470.

[20] Wang, W. & Zhao, X.-Q. (2004) An epidemic model in a patchy environment. *Math. Biosci.* **190**(1), 97–112.

[21] Wang, J. B., Qiao, S. X. & Wu, C. F. (2022) Wave phenomena in a compartmental epidemic model with nonlocal dispersal and relapse. *Discrete Contin. Dyn. Syst. Ser. B* **27**(5), 2635–2660.

[22] Wang, X. J., Lin, G. & Ruan, S. G. (2022) Spatial propagation in a within-host viral infection model. *Stud. Appl. Math.* **149**(1), 43–75.

[23] Wu, C.-C. (2017) Existence of traveling waves with the critical speed for a discrete diffusive epidemic model. *J. Differ. Equ.* **262**(1), 272–282.

[24] Wu, S. L., Zhao, H. Q., Zhang, X. & Hsu, C.-H. (2023) Spatial dynamics for a time-periodic epidemic model in discrete media. *J. Differ. Equ.* **374**, 699–736.

[25] Xu, Z. & Xiao, D. M. (2018) Uniqueness of epidemic waves in a host-vector disease model. *Proc. Am. Math. Soc.* **146**(9), 3875–3886.

[26] Xu, Z. & Xiao, D. M. (2014) Minimal wave speed and uniqueness of traveling waves for a nonlocal diffusion population model with spatio-temporal delays. *Differ. Integral Equ.* **27**, 1073–1106.

[27] Xu, Z. (2021) Global stability of travelling waves for a class of monostable epidemic models. *Commun. Nonlinear Sci. Numer. Simul.* **95**, 105595.

[28] Xu, Z., Tan, T. & Hsu, C.-H. (2023) Spatial propagation for an epidemic model in a patchy environment. *J. Dyn. Differ. Equ.* doi: 10.1007/s10884-023-10284-0.

[29] Xu, Z. (2018) Asymptotic speeds of spread for a nonlocal diffusion equation. *J. Dyn. Differ. Equ.* **30**(2), 473–499.

[30] Yang, F. Y. & Li, W. T. (2018) Traveling waves in a nonlocal dispersal SIR model with critical wave speed. *J. Math. Anal. Appl.* **458**(2), 1131–1146.

[31] Yang, F. Y., Li, W. T. & Wang, Z. C. (2013) Traveling waves in a nonlocal dispersal Kermack-McKendrick epidemic model. *Discrete Contin. Dyn. Syst. Ser. B* **18**(7), 1969–1993.

[32] Zhang, Q. & Wu, S. L. (2019) Wave propagation of a discrete SIR epidemic model with a saturated incidence rate. *Int. J. Biomath* **12**(03), 1950029.

[33] Zhang, L., Wang, Z. C. & Zhao, X.-Q. (2020) Time periodic traveling wave solutions for a Kermack-Mckendrick epidemic model with diffusion and seasonality. *J. Evol. Equ.* **20**(3), 1029–1059.

[34] Zhang, R., Wang, J. L. & Liu, S. Q. (2021) Traveling wave solutions for a class of discrete diffusive SIR epidemic model. *J. Nonlinear Science* **31**(1), 1–33.

[35] Zhao, X.-Q. & Xiao, D. M. (2006) The asymptotic speed of spread and traveling waves for a vector disease model. *J. Dyn. Differ. Equ.* **18**(4), 1001–1019.

[36] Zhou, J. B., Song, L. Y. & Wei, J. D. (2020) Mixed types of waves in a discrete diffusive epidemic model with nonlinear incidence and time delay. *J. Differ. Equ.* **268**(8), 4491–4524.
[37] Zhou, J. B., Li, J. H., Wei, J. D. & Tian, L. X. (2022) Wave propagation in a diffusive SAIV epidemic model with time delays. *European J. Appl. Math.* **33**(4), 674–700.
[38] Zhou, J. L., Yang, Y. & Hsu, C.-H. (2024) Propagation dynamics for a spatial discrete virus model with HIV viral load and 2-LTR dynamics. *Z. Angew. Math. Phys.* **75**(4), 22.

Cite this article: Xu Z., Tan T. and Hsu C.-H. Propagation dynamics for an epidemic patch model with variable incubation period. *European Journal of Applied Mathematics*, https://doi.org/10.1017/S0956792524000846