J. Austral. Math. Soc. 25 (Series A) (1978), 195–200

ITERATIVE CRITERIA FOR BOUNDS ON THE GROWTH OF POSITIVE SOLUTIONS OF A DELAY DIFFERENTIAL EQUATION

RAYMOND D. TERRY

(Received 9 March 1976; revised 20 April 1977)
Communicated by N. S. Trudinger

Abstract

Following Terry (*Pacific J. Math.* 52 (1974), 269–282), the positive solutions of equation (E): $D^n[r(t)\, D^n y(t)] + a(t) f[y(\sigma(t))] = 0$ are classified according to types B_j. We denote

$$y_i(t) = D^i y(t) \text{ for } i = 0, \ldots, n-1;$$

$$y_i(t) = D^{i-n}[r(t)\, D^n y(t)] \text{ for } i = n, \ldots, 2n-1.$$

A necessary condition is given for a B_k-solution $y(t)$ of (E) to satisfy $y_{2k}(t) \geq m(t) > 0$. In the case $m(t) = C > 0$, we obtain a sufficient condition for all solutions of (E) to be oscillatory.

Subject classification (Amer. Math. Soc. (MOS) 1970): primary 34 C 10; secondary 34 C 15, 34 K 05, 34 K 15, 34 K 20, 34 K 25.

In this paper a number of results are presented concerning the possible rate of growth of nonoscillatory solutions of a functional differential equation of even order. We let $R = (-\infty, \infty)$, $R_0 = [0, \infty)$, $R^* = (0, \infty)$ and consider the equation

$$D^n[r(t)\, D^n y(t)] + a(t) f[y(\sigma(t))] = 0, \tag{1}$$

where $f(u)$ is a nondecreasing function in $C[R, R]$,

$$a(t) \in C[R_0, R^*], \quad r(t) \in C\{R_0, [m, M]\}, \quad m > 0,$$

$$\sigma(t) \in C[R_0, R^*], \quad uf(u) > 0 \quad \text{for} \quad u \neq 0, \quad \sigma(t) \leq t \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) = +\infty.$$

In a special case, the main result will yield a criterion for the oscillation of all solutions of (1). When $r(t) \equiv 1$ and $n = 1$, the main result and its corollary will reduce to Theorems 3 and 4, respectively, of Burton and Grimmer (1972).

A solution $y(t)$ of (1), or of the equation (7) below, is said to be *oscillatory* on $[a, \infty)$ if for each $\alpha > a$ there is a $\beta > \alpha$ such that $y(\beta) = 0$. Following Terry (1974), we define auxiliary functions $y_j(t)$ by

$$y_j(t) = \begin{cases} D^j y(t), & j = 0, \ldots, n-1, \\ D^{j-n}[r(t)\, Dy(t)], & j = n, \ldots, 2n-1. \end{cases} \tag{2}$$

A solution $y(t)$ of (1) is of *type* B_k on $[T_0, \infty)$ if for $t \geqslant T_0$, $y_j(t) > 0$ for $j = 0, \ldots, 2k+1$ and $(-1)^{j+1} y_j(t) > 0$ for $j = 2k+2, \ldots, 2n-1$. Since $\lim_{t\to\infty} \sigma(t) = +\infty$, there is a $T_1 > T_0$ such that $\sigma(t) \geqslant T_0$ for $t \geqslant T_1$. As shown in Terry (1974), a positive solution $y(t)$ of (1) is necessarily of type B_k for some $k = 0, \ldots, n-1$. Moreover, the following lemmas have been established.

LEMMA 1. *Let $y(t)$ be a solution of* (1) *of type B_k on $[T_0, \infty)$. Then there exist constants $N_{j,j-1} > 0$ such that*

$$(3) \qquad \begin{aligned} (t-T_1)\, y_j(t) &\leqslant N_{j,j-1}\, y_{j-1}(t), \quad t \geqslant T_1, \\ t y_j(t) &\leqslant 2N_{j,j-1}\, y_{j-1}(t), \quad t \geqslant 2T_1. \end{aligned}$$

LEMMA 2. *Let $y(t)$ be a solution of* (1) *of type B_k on $[T_0, \infty)$. Let $2k+1 \geqslant r \geqslant s$. Then there exist constants $N_{r,s} > 0$ such that*

$$(t-T_1)^{r-s}\, y_r(t) \leqslant N_{r,s}\, y_s(t), \quad t \geqslant T_1$$

and

$$t^{r-s}\, y_r(t) \leqslant 2^{r-s} N_{r,s}\, y_s(t), \quad t \geqslant 2T_1.$$

It is clear that the $N_{r,s}$ may be defined in terms of the $N_{j,j-1}$. Specifically,

$$N_{r,s} = \prod_{j=s+1}^{r} N_{j,j-1}.$$

Estimates for the $N_{j,j-1}$ may be found in Terry (1974); those for the $N_{r,s}$ are in Terry (1975). We let $M_0 = m$ if $y_n(t) < 0$, $M_0 = M$ if $y_n(t) > 0$, $\omega_k = (2n-2k-1)!$ if $2k \geqslant n$, $\omega_k = M_0(2n-2k-1)!$ if $2k < n$, $\gamma_k = 2^{2k} \omega_k N_{2k}$, where $N_{2k} = N_{2k,0}$. In addition to this notation, we introduce the oscillation transform $I_{T,s}$ defined by

$$I_{T,s}[y(u)] = \int_T^s (u-T)^{2n-2k-1}\, a(u)\, f[\gamma_k^{-1}(\sigma(u))^{2k}\, y(\sigma(u))]\, du.$$

Repeated applications of the oscillation transform will be indicated in the sequel by standard notation for the composite of two functions, that is,

$$(I_{T_2,s_2} \circ I_{T_1,s_1})(f) = I_{T_2,s_2}[I_{T_1,s_1}(f)].$$

The product symbol $\prod_{i=1}^{n} I_{T_i,s_i}$ will be used, where appropriate, to represent multiple composition, not ordinary multiplication. In terms of this notation we may state the main result of this paper.

THEOREM 1. *Let $m(t) \in C[R_0, R^*]$. Suppose that there is a positive integer N such that any finite sequence $\{T_{i+1}\}_{i=0}^{N}$ with $0 \leqslant T_1$ and $T_i < T_{i+1}$*

$$(4) \qquad \int_{T_{N+1}}^{\infty} a(s_N)\, f\left[N_{2k}^{-1}(\sigma(s_N))^{2k} \left(\prod_{j=0}^{N-1} I_{T_{N-j},\sigma(s_{N-j})}(\omega_k\, m(s_0)) \right) \right] ds_N = +\infty.$$

Then there is no solution $y(t)$ of (1) *of type B_k for which $y_{2k}(t) \geqslant m(t)$ for large t.*

PROOF. We argue by way of contradiction and suppose that $y(t)$ is a solution of (1) of type B_k on $[T_0, \infty)$. If $k \geqslant n/2$, we multiply (1) by $(s-T_1)^{2n-2k-1}$ and integrate by parts from T_1 to t to obtain

$$\text{(5a)}\quad \int_{T_1}^{t} (s-T_1)^{2n-2k-1} D^n[r(s)\, D^n y(s)]\, ds = R_1(t)-(2n-2k-1)!\,[y_{2k}(s)]_{T_1}^{t},$$

where

$$R_1(s) = (s-T_1)^{2n-2k-1} y_{2n-1}(s) - \sum_{j=2}^{2n-2k-1} (-1)^j (2n-2k-1)_{j-1} (s-T_1)^{2n-2k-j} y_{2n-j}(s)$$

and $(n)_k = n(n-1)\ldots(n-k+1)$. If $k < n/2$, we proceed as above, pausing momentarily at the stage where $r(s)\, D^n y(s)$ appears undifferentiated to change the equality to an inequality using $m \leqslant r(s) \leqslant M$. In this case we obtain

$$\text{(5b)}\quad \int_{T_1}^{t} (s-T_1)^{2n-2k-1} D^n[r(s)\, D^n y(s)]\, ds \geqslant R_2(t) - M_0(2n-2k-1)!\,[y_{2k}(s)]_{T1}^{t},$$

where

$$R_2(s) = (s-T_1)^{2n-2k-1} y_{2n-1}(s) - \sum_{j=2}^{n} (-1)^j (2n-2k-1)_{j-1} (s-T_1)^{2n-2k-j} y_{2n-j}(s)$$
$$- M_0 \sum_{j=n+1}^{2n-2k-1} (-1)^j (2n-2k-1)_{j-1} (s-T_1)^{2n-2k-j} y_{2n-j}(s).$$

When $r(t) \equiv 1$, the two expressions coincide. See Ladas (1971) for another application in this case. We note that $\omega_k y_{2k}(T_1)$ and each of the component terms of $R_i(t)$ are positive. Omitting them, it follows that

$$\text{(5c)}\qquad \omega_k y_{2k}(t) \geqslant \int_{T_1}^{t} (s-T_1)^{2n-2k-1} a(s) f[y(\sigma(s))]\, ds.$$

Since $y(t)$ is of type B_k on $[T_0, \infty)$, $t^{2k} y_{2k}(t) \leqslant 2^{2k} N_{2k} y(t)$ for $t \geqslant 2T_1$, where $N_{2k} = N_{2k,0}$. Moreover, since $\lim_{t\to\infty} \sigma(t) = +\infty$, there is a $T_{11} > 2T_1$ such that $\sigma(t) \geqslant 2T_1$ whenever $t \geqslant T_{11}$. Thus, for $t \geqslant T_{11}$ the following chain of inequalities hold:

$$\begin{aligned} y(\sigma(t)) &\geqslant 2^{-2k} N_{2k}^{-1} (\sigma(t))^{2k} y_{2k}(\sigma(t)) \\ &\geqslant 2^{-2k} N_{2k}^{-1} (\sigma(t))^{2k} m(\sigma(t)) \\ &= 2^{-2k} N_{2k}^{-1} \omega_k^{-1} (\sigma(t))^{2k} \omega_k m(\sigma(t)) \\ &= \gamma_k^{-1} (\sigma(t))^{2k} \omega_k m(\sigma(t)). \end{aligned}$$

Since $f(u)$ is a nondecreasing function of u,

$$f[y(\sigma(s))] \geqslant f[\gamma_k^{-1} (\sigma(s))^{2k} \omega_k m(\sigma(s))].$$

Multiplication of this inequality by $(s-T_1)^{2n-2k-1} a(s)$ preserves the inequality as does integration over the interval $[T_1, t]$. From (5c)

$$\omega_k y_{2k}(s) \geqslant \int_{T_{11}}^{s} (s_0 - T_1)^{2n-2k-1} a(s_0) f[\gamma_k^{-1}(\sigma(s_0))^{2k} \omega_k m(\sigma(s_0))] \, ds_0;$$

that is,

$$\text{(5d)} \qquad y_{2k}(s) \geqslant \omega_k^{-1} I_{T_{11},s}(\omega_k m(s_0)), \quad s \geqslant T_{11}.$$

Since $\lim_{t\to\infty} \sigma(t) = +\infty$, there is a $T_2 > T_{11}$ such that $\sigma(s_1) \geqslant T_{11}$ for $s_1 > T_2$. Thus, we may let $s = \sigma(s_1)$ in (5d) so that

$$y_{2k}(\sigma(s_1)) \geqslant \omega_k^{-1} I_{T_{11},\sigma(s_1)}(\omega_k m(s_0)).$$

Multiplying this by $2^{-2k} N_{2k}^{-1}(\sigma(s_1))^{2k}$,

$$y(\sigma(s_1)) \geqslant \gamma_k^{-1}(\sigma(s_1))^{2k} I_{T_{11},\sigma(s_1)}(\omega_k m(s_0)).$$

Since (5c) holds with t replaced by s, s replaced by s_1, and T_1 replaced by T_2,

$$\begin{aligned} \omega_k y_{2k}(s) &\geqslant \int_{T_2}^{s} (s_1 - T_2)^{2n-2k-1} a(s_1) f[y(\sigma(s_1))] \, ds_1 \\ &\geqslant \int_{T_2}^{s} (s_1 - T_2)^{2n-2k-1} a(s_1) f[\gamma_k^{-1}(\sigma(s_1))^{2k} I_{T_{11},\sigma(s1)}(\omega_k m(s_0))] \, ds_1 \\ &= I_{T_2,s}[I_{T_{11},\sigma(s_1)}(\omega_k m(s_0))]. \end{aligned}$$

Since $\lim_{t\to\infty} \sigma(t) = +\infty$, there is a $T_3 > T_2$ such that $\sigma(s_2) \geqslant T_2$ for $s_2 > T_3$. Thus, we may let $s = \sigma(s_2)$ in the above expression to obtain

$$\omega_k y_{2k}(\sigma(s_2)) \geqslant I_{T_2,\sigma(s_2)}[I_{T_{11},\sigma}(s_1)(\omega_k m(s_0))].$$

Proceeding in this way, it follows that there exist $T_2, \ldots, T_N$ such that for $i = 2, \ldots, N-1$, $T_{i+1} > T_i$, $\sigma(s_i) \geqslant T_i$ and

$$\omega_k y_{2k}(\sigma(s_i)) \geqslant \prod_{j=0}^{i-2} I_{T_{i-j},\sigma(s_{i-j})}[I_{T_{11},\sigma(s_1)}(\omega_k m(s_0))].$$

In particular, for $i = N$,

$$\omega_k y_{2k}(\sigma(s_N)) \geqslant \prod_{j=0}^{N-2} I_{T_{N-j},\sigma(s_{N-j})}[I_{T_{11},\sigma(s_1)}(\omega_k m(s_0))].$$

As in previous computations,

$$\begin{aligned} y(\sigma(s_N)) &\geqslant 2^{-2k} N_{2k}^{-1}(\sigma(s_N))^{2k} y_{2k}(\sigma(s_N)) \\ &\geqslant \gamma_k^{-1}(\sigma(s_N))^{2k} \prod_{j=0}^{N-2} I_{T_{N-j},\sigma(s_{N-j})}[I_{T_{11},\sigma(s_1)}(\omega_k m(s_0))]. \end{aligned} \qquad (6)$$

An integration of (1) from T_{N+1} to t yields

$$y_{2n-1}(T_{N+1}) - y_{2n-1}(t) = \int_{T_{N+1}}^{t} a(s_N) f[y(\sigma(s_N))] \, ds_N;$$

that is,

$$y_{2n-1}(t) = y_{2n-1}(T_{N+1}) - \int_{T_{N+1}}^{t} a(s_N) f[y(\sigma(s_N))]\, ds_N$$

so that

$$\lim_{t\to\infty} y_{2n-1}(t) = y_{2n-1}(T_{N+1}) - \int_{T_{N+1}}^{\infty} a(s_N) f[y(\sigma(s_N))]\, ds_N.$$

An application of (6) and the integral condition in the statement of the theorem shows that $\lim_{t\to\infty} y_{2n-1}(t) = -\infty$. Since

$$y_{2n-1}(t) < 0 \quad \text{and} \quad Dy_{2n-1}(t) = -a(t) f[y(\sigma(t))] < 0,$$

it follows that $y_j(t) < 0$ for $j = 0, \ldots, 2n-2$, contradicting the fact that $y(t)$ is of type B_k in addition to the hypothesis that $y_{2k}(t) \geqslant m(t) > 0$.

REMARK 1. When $N = 0$, the multiple integral of (4) reduces to a single integral. Even in this case the result is new.

REMARK 2. When $n = 1$, $k = 0$, $m(t) > 0$, we may choose $N_{2k} =$ as discussed in Terry (1976). Moreover, for $r(t) \equiv 1$, $m = M = 1$ so that $M_0 = 1$, $(2n-2k-1)! = 1$, $\omega_k = 1$ and $\gamma_k = 1$.

$$I_{T_1,s_1}[y(u)] = \int_{T_1}^{s_1} (s_0 - T_1)\, a(s_0) f[y(\sigma(s_0))]\, ds_0.$$

The integral condition (4) reduces to

$$\int_{T_{N+1}}^{\infty} a(s_N) f[I_{T_N,\sigma(s_N)}(\ldots(I_{T_{11},\sigma(s_1)}(m(s_0)))\ldots)]\, ds_N = +\infty,$$

which is a variant of the hypothesis of Theorem 3 of Burton and Grimmer (1972). The conclusion here is that there are no B_0-solutions $y(t)$ of

$$y''(t) + a(t) f[y(\sigma(t))] = 0$$

such that $y(t) \geqslant m(t) > 0$, which is the conclusion of Theorem 3 of Burton and Grimmer (1972).

REMARK 3. Suppose we define $\bar{\gamma}_k = 2^{2k} \bar{\omega}_k N_{2k}$, where

$$\bar{\omega}_k = \begin{cases} 2^{2n-2k-1}(2n-2k-1)!, & k \geqslant n/2, \\ 2^{2n-2k-1} M_0 (2n-2k-1)!, & k < n/2, \end{cases}$$

and let $\bar{I}_{T_1,s_1}$ be defined in the same manner as I_{T_1,s_1} with the exceptions that γ_k is replaced by $\bar{\gamma}_k$ and $(s_0 - T_1)^{2n-2k-1}$ is replaced by $s_0^{2n-2k-1}$. Then

$$y_{2k}(s) \geqslant \bar{\omega}_k^{-1} \bar{I}_{T_1,s_1}(\bar{\omega}_k m(s_0)).$$

or $n = 1$ and $k = 0$

$$\bar{I}_{T_1,s_1}[y(u)] = \int_{T_1}^{s_1} s_0\, a(s_0)\, f[\tfrac{1}{2} y(\sigma(s_0))]\, ds_0.$$

This time the hypothesis of the theorem is the same as that of Theorem 3 of Burton and Grimmer (1972) except for the factor $\frac{1}{2}$ appearing in the integrand of $\bar{I}_{T_1,s_1}$. The conclusions are identical.

REMARK 4. When $k = 0$ and $m(t) = C > 0$, the conclusion is that there are no B_0-solutions $y(t)$ of (1) such that $y(t) \geqslant C > 0$. However, a B_k-solution $y(t)$ of (1) satisfies $y(t) > 0$ and $y'(t) > 0$. Thus, if (4) holds for all constant functions $m(t)$, the conclusion of Theorem 1 may be strengthened to exclude all positive non-oscillatory solutions of (1). When $n = 1$ and $r(t) \equiv 1$, the above statement is formalized in Theorem 4 of Burton and Grimmer (1972).

REMARK 5. The lemmas, the theorem and the above remarks hold for the more general equation

(7) $$D^{2n-i}[r(t)\, D^i y(t)] + a(t)\, f[y(\sigma(t))] = 0$$

provided we redefine the $y_j(t)$ as follows:

$$y_j(t) = \begin{cases} D^j y(t), & j = 0, \ldots, i-1, \\ D^{j-i}[r(t)\, D^i y(t)], & j = i, \ldots, 2n-1. \end{cases}$$

The details of this are left to the reader.

REFERENCES

T. Burton and R. Grimmer (1972), "Oscillatory solutions of $x''(t) + a(t) f[x(g(t))] = 0$", *Delay and Functional Differential Equations and their Applications*, 335–342 (Academic Press, New York).

G. Ladas (1971), "On principal solutions of nonlinear differential equations", *J. Math. Anal. Appl.* **36**, 103–109.

R. D. Terry (1974), "Oscillatory properties of a delay differential equation of even order", *Pacific J. Math.* **52**, 269–282.

R. D. Terry (1975), "Some oscillation criteria for delay differential equations of even order", *SIAM J. Appl. Math.* **28**, 319–334.

R. D. Terry (1976), "Oscillatory and asymptotic properties of homogeneous and non-homogeneous delay differential equations of even order", *J. Austral. Math. Soc.* **22** (Ser. A), 282–304.

California Polytechnic State University
San Luis Obispo, California 93407
U.S.A.