P 85. Denote by $f_1(n)$ the number of Abelian groups of order n and by $f_2(n)$ the number of semi-simple rings of order n (see I. G. Connell, this Bulletin 7(1964), 23-34. Prove that

$$\overline{\text{Lim}} \log f_i(n)/(\log n/\log\log n) = k_i, \quad i = 1, 2,$$

and determine the constants k_i.

P. Erdős

SOLUTIONS

P 6. (Conjecture). If $a_1 < a_2 < \ldots$ is a sequence of positive integers with $a_n/a_{n+1} \to 1$, and if for every d, every residue class (mod d) is representable as the sum of distinct a's, then at most a finite number of positive integers are not representable as the sum of distinct a's.

P. Erdős

In its present generality the conjecture is false; this is shown by an example due to J. W. S. Cassels, On the representation of integers as the sums of distinct summands taken from a fixed set, Acta Sci. Math. Szeged 21(1960), 111-124 (Math. Rev. 24(1962), A103). See also P. Erdős, On the representation of large integers as sums of distinct summands taken from a fixed set, Acta Arith. 7(1961/62), 345-354 (Math. Rev. 26(1963), no. 2387).

P 27. Prove that

$$\sum_{n=1}^{\infty} \sum_{\substack{d \mid F_n \\ d>1}} d^{-1/2} < \infty, \quad F_n = 2^{2^n} + 1.$$

P. Erdős

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Partial solution by the proposer.

We outline the proof of

$$(1) \qquad S = \sum_{\substack{d|F_n \\ d>1}} d^{-1/2} < c/n \ ;$$

this is not strong enough to solve the original problem. (I probably made a mistake in estimating (1) when I posed the problem.) If $F_n \equiv 0 \pmod p$ then the exponent of 2 mod p is 2^{n+1}, and therefore $p \equiv 1 (\mathrm{mod}\ 2^{n+1})$. If the prime factors of F_n are $q_1 < q_2 < \ldots$, then a straightforward application of Brun's method gives

$$(2) \qquad q_i > C_1 \, 2^{n+1} \, i \log i \ ,$$

whence $S < \prod_{i \leq 2^n/n} (1 + p_i^{-1/2})$

$$< \exp\Big(\sum_{i \leq 2^n/n} p_i^{-1/2} \Big) < 2 \sum_{i < 2^n/n} p_i^{-1/2}$$

$$< 2 . 2^{-(n+1)/2} \sum_{i < 2^n/n} (C_1 i \log i)^{-1/2} < c/n \ ,$$

which is (1). It is easy to see that (1) implies

$$\sum_{n=1}^{\infty} \sum_{\substack{d|F_n \\ d>1}} d^{-1/2} (\log\log d)^{-1-\epsilon} < \infty$$

for all $\epsilon > 0$.

P 46. Given infinitely many points in the plane such that

(a) the distance between any two of them is greater than 1,

(b) for infinitely many n, there are more than cn^2 points in the circle $|z| \Gamma n$.

Show that for any $\epsilon > 0$ there is a line through the origin which comes closer than ϵ to infinitely many of the points.

P. Erdős

Solution by the proposer.

A simple argument shows that our second condition implies the existence of a sequence $n_1 < n_2 < \ldots, n_k \to \infty$, so that the number of our points z_i satisfying $n_k < |z_i| < n_k + \frac{1}{2}$ is greater than $c_1 n_k$ (c_1 is an absolute constant which depends only on c).

Project the points z_i onto the circle $|z| = n_k$. Thus we obtain the points $w_i^{(k)}$, $1 \leq i \leq \ell_k$, $\ell_k \geq c_1 n_k$. Clearly, the distance between any two $w_i^{(k)}$ is at least $\frac{1}{2}$. Denote now by $S_k(\epsilon)$ the set of those points on the circle $|z| = n_k$ whose distance from at least one of the points $w_i^{(k)}$ is less than $\frac{\epsilon}{2}$. Clearly for ϵ small enough $S_k(\epsilon)$ consists of ℓ_k disjoint arcs of length $> \epsilon$. Thus the measure of $S_k(\epsilon)$ is greater than $c_1 \epsilon n_k$. Project finally $S_k(\epsilon)$ onto the unit circle. This gives a set $E_k(\epsilon)$ on the unit circle of measure $> c_1 \epsilon$. Therefore by a well known theorem in measure theory there is a point z, $|z| = 1$ which is contained in infinitely many $E_k(\epsilon)$.

By our construction it is evident that the line connecting z with the origin comes closer than ϵ to infinitely many of our points z_i.

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I would like to make three remarks.

1. Since I can not give a reference to "the well known theorem" I give the proof. Put

$$F_k(\epsilon) = \bigcup_{\ell > k} E_k(\epsilon) .$$

Clearly $F_{k+1}(\epsilon) \subset F_k(\epsilon)$ and each $F_k(\epsilon)$ has measure $> c_1\epsilon$. Since by a well known theorem of Lebesgue a descending sequence of bounded sets of measure $> \alpha$ have a non-empty intersection, our theorem is proved.

2. The theorem is best possible in the following sense: Let $f(n) \to \infty$ as slowly as we wish. Then there is a set of points z_k any two of which have distance > 1, the number of $|z_k| < n$ is $> \frac{cn^2}{f(n)}$ for every n, and nevertheless if L is any line then for every A there are only a finite number of z's at distance $< A$ from L.

The construction is very simple. Let $g(n)/n \to 0$ but sufficiently slowly. Let the z's be the points $(2u, 2v)$ where

$$\log 2u < 2v < \frac{2u}{g(2u)} .$$

A simple argument shows that the z's have all the required properties.

3. I now prove that the assumption of P 46 implies that there is a line through the origin L so that for every ϵ there are infinitely many z_i closer than ϵ to L.

In view of the proof of P 46 it will suffice to show the following.

LEMMA. Let $n_1 < n_2 < \dots$ be an infinite sequence of numbers tending to infinity. Let $w_i^{(k)}$, $1 \le i \le \ell_k$, $\ell_k \ge c_1 n_k$ be a sequence of points on $|z| = 1$ satisfying

$$|w_{i_1}^{(k)} - w_{i_2}^{(k)}| > \frac{1}{2n_k} .$$

Denote by $S_\epsilon(k)$ the set of points on $|z| = 1$ for which at least one $w_{i\ell}^{(k)}$ satisfies $|z - w_i^{(k)}| < \epsilon / n_k$ (then $m(S)$ denotes the Lebesgue measure of S):

$$(1) \qquad m\Big(\bigcup_{\ell > k} S_\epsilon(\ell)\Big) \geq c_1 .$$

To see this it will suffice to show that to every $\eta > 0$ there is a $K = K(\eta)$ so that

$$(2) \qquad m\Big(\bigcup_{k < \ell \leq K} S_\epsilon(\ell)\Big) > c_1 - \eta .$$

If (2) would be false put

$$(3) \qquad \lim_{K=\infty} m\Big(\bigcup_{k<\ell\leq K} (S_\epsilon(\ell)\Big) = D < c_1 .$$

Then if $K = K(\eta)$ is large enough

$$(4) \qquad m\Big(\bigcup_{k<\ell\leq K} S_\epsilon(\ell)\Big) > D - \eta .$$

The set $S_\epsilon(\ell)$ consists of a finite number of disjoint arcs the sum of whose lengths is between $D - \eta$ and D. Let now n_r be very large (compared to n_K of (4)). By assumption there are at least $c_1 n_r$ points $w_i^{(r)}$, and since n_r is large compared to n_K a simple argument shows that there are less than Dn_r of the $w_i^{(r)}$ in $\bigcup_{k<\ell\leq K} S_\epsilon(\ell)$. But then there are at least $(c_1 - D)n_r$ points $w_i^{(r)}$ not in $\bigcup_{k<\ell\leq K} S_\epsilon(\ell)$ and therefore the measure of those points in $S_\epsilon(r)$ which is not contained in $\bigcup_{k<\ell\leq K} S_\epsilon(\ell)$ is greater than $(c_1 - D)\epsilon/2$. Thus by (4)

$$m\Big(\bigcup_{k<\ell\leq r} S_\epsilon(\ell)\Big) > D - \eta + (c_1 - D)\epsilon/2 > D$$

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if $\eta = \eta(\epsilon)$ is sufficiently small. But this contradicts (3) and hence our Lemma is proved.

I was led to P 46 by problem 93, p. 39 of the book of Hadwiger and Debrunner "Combinatorial Geometry of the Plane".

P 74. Let $f(n)$ denote the number of (associative) rings with n elements. Show that f is multiplicative, that is, if the g.c.d. $(m,n) = 1$, then $f(mn) = f(m)f(n)$. Similarly if f enumerates the rings with 1, the commutative rings, or the commutative rings with 1.

I. G. Connell, McGill University

Solution by H. Gonshor, Rutgers University.

Let us first consider the case of Abelian groups. Then if G is a group of order m and H is a group of order n, $G \oplus H$ is a group of order mn. Thus the direct sum operation maps pairs of groups of orders m and n onto groups of order mn. Since m is prime to n it is clear from the fundamental theorem for finite Abelian groups that this map is onto. Furthermore if $K = G \oplus H$ where G is of order m and H is of order N then G is exactly the set of all elements of K whose order divides m. Thus K determines G and H uniquely, hence the mapping is one-one. This shows that f is multiplicative.

The extension to various types of rings is essentially a corollary. The direct sum operation maps pairs of rings of orders m and n into rings of order mn. The one-oneness is a fortiori true (modulo logical quibling, i.e. one should think in terms of the internal direct sum). To prove that the map is onto we consider a ring of order mn and express it as a direct sum of groups of order m and n. It suffices to show that the components are ideals. Let $K = G \oplus H$. By symmetry it is enough to show that G is a right ideal. This is trivial in view of the above characterization of G, for $ma = 0 \Rightarrow m(ax) = (ma)x = 0$. Thus $a \in G \Rightarrow ax \in G$.

Now in order to complete the solution to the problem as stated one need only remark that the properties of commutativity, associativity, and possession of an identity are preserved by taking direct sums and direct summands. (Incidentally this takes care of 8 possible cases rather than the 4 listed in the problem.)