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GENERIC FREE RESOLUTIONS II

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1. Introduction. In [**1**], a number of "multilinear" functors $L_p{}^q$, defined for finitely generated free modules, were introduced. They arose as cycles in a generic Koszul complex, and in turn gave rise to a large family of other generic complexes. One of the things we will do in this paper is study some of these new complexes in order to obtain new multilinear functors on free modules which appear as their cycles.

One reason for starting this systematic study is that work on Schubert calculus and Young tableaux, in particular, articles by Lascoux and by Towber [**2**; **4**], indicate a not yet completely understood connection between some of these "multilinear" functors and the more classical representation theory. (For example, our functors $L_p{}^q$ correspond to the irreducible representation belonging to the partition $(p, 1, \ldots, 1)$.)

The functors $L_p{}^q$ arose out of consideration of certain complexes, namely, generic free resolutions of cokernels of the maps

$$\Lambda^p f\colon \Lambda^p F \rightarrow \Lambda^p G,$$

where $f\colon F \rightarrow G$ is a map of free modules. The new multilinear functors introduced in Sections 4, 5, and 6 arise from consideration of complexes resolving the cokernels of the maps

$$\Lambda^p F \otimes \Lambda^q G \rightarrow \Lambda^{p+q} G,$$

which are the composites of

$$\Lambda^p F \otimes \Lambda^q G \rightarrow \Lambda^p G \otimes \Lambda^q G \rightarrow \Lambda^{p+q} G.$$

(In particular, when $p + q = \text{rank } G$, we are attempting to resolve the ideal of $p \times p$ minors of the map $f\colon F \rightarrow G$.) Lascoux has shown [**2**] that certain irreducible representations of $GL(n)$ occur in the minimal resolutions of ideals of low order minors of a matrix. Since minimal resolutions are essentially unique, we have further evidence of a strong connection between the irreducible representations of $GL(n)$ and some of the functors introduced in Sections 4, 5, and 6.

In Section 2, we review those parts of [**1**] that are required for this paper, and in Section 3 we observe what happens when we assume that we are dealing with a graded ring. Using the results of Section 3, we are able to prove the acyclicity of certain free complexes, and thereby obtain new functors of free

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modules $L_{p_1p_2}^{q_1q_2}$, $K_{p_1p_2}^{q_1q_2}$. In fact, we indicate in this section how one might generate a whole sequence of multilinear functors, but we do not study this general procedure here; we will do this in a subsequent article.

In Section 5, we give an explicit construction of a generic resolution of the cokernel of

$$\Lambda^p F \otimes G \to \Lambda^{p+1} G$$

for a map $f: F \to G$, when rank F = rank G. We do this in characteristic zero, since this enables us to make precise a splitting of a certain map. The procedure used in Section 5 suggests a general procedure which we outline (very sketchily) in Section 6. It is here that certain other functors arise; but, except in certain special cases, little can as yet be said about them. Nevertheless, we do use this procedure to construct a resolution of the ideal of $(n-2) \times (n-2)$ minors of an $(n-1) \times n$ matrix, because in this case our functors come up in certain simple exact sequences. This reproduces a result of Poon [**3**], although our construction is so far restricted to the case of characteristic zero.

2. Preliminaries. Throughout this section, rings will be commutative with identity, and free modules will always be of finite rank. If the ring is graded, "commutative" will mean commutative in the graded sense. Thus, if F is a free R-module, and $S(F)$ (resp. $D(F)$) denotes the symmetric (resp. divided power) algebra of F, then we must regard the elements of F as having degree 2 in $S(F)$ (resp. $D(F)$). However, we shall denote by $S_q(F)$ (resp. $D_q(F)$) the elements of degree $2q$ in $S(F)$ (resp. $D(F)$), and thereby return to the classical notation for polynomial rings. As usual, ΛF will denote the exterior algebra of F; in this algebra the elements of F are of degree one.

If F is a free R-module, we define the free R-modules $L_q^p F$ as follows. The identity map $F \to F$ yields an element $c_F \in F \otimes F^*$ which may be considered an element of $SF \otimes \Lambda F^*$. As such, $c_F^2 = 0$. Since SF is an SF-module, and ΛF is a ΛF^*-module (as described in [**1**]), $SF \otimes \Lambda F$ is an $SF \otimes \Lambda F^*$-module. Multiplication by c_F on $SF \otimes \Lambda F$ converts $SF \otimes \Lambda F$ into a complex whose homogeneous components look like

$$(*) \qquad \ldots \longrightarrow S_{q-1}F \otimes \Lambda^p F \xrightarrow{\partial_q^p} S_q F \otimes \Lambda^{p-1} F \xrightarrow{\partial_{q+1}^{p-1}} S_{q+1}F \otimes \Lambda^{p-2} F \longrightarrow S_{q+2}F \otimes \Lambda^{p-3} F \longrightarrow \ldots,$$

and $LF = \{L_q^p F\}$ is defined to be the module of cycles of this complex. In particular,

$$L_q^p F = \operatorname{Ker}\left(S_q F \otimes \Lambda^{p-1} F \xrightarrow{\partial_{q+1}^{p-1}} S_{q+1} F \otimes \Lambda^{p-2} F\right)$$

and, because the complex $(*)$ is acyclic, we also have

$$L_q^p F = \operatorname{Coker}\left(S_{q-2} F \otimes \Lambda^{p+1} F \xrightarrow{\partial_{q-1}^{p+1}} S_{q-1} F \otimes \Lambda^p F\right) \quad q \geqq 1.$$

Notice that

$$(1)\quad \begin{aligned} &L_q^1F = S_qF \quad \text{for all } q\\ &L_1^pF = \Lambda^pF \quad \text{for all } p \neq 0\\ &L_q^0F = L_0^pF = 0 \quad \text{for all } p \neq 1, \text{ and all } q\\ &L_q^pF = 0 \quad \text{for all } p > \text{rank } F.\end{aligned}$$

Also, if rank $F = n$, then

$$(2)\quad L_q^nF \approx S_{q-1}F \otimes \Lambda^nF.$$

(All of this may be found in § 2 of [**1**].)

Letting DF denote the divided power algebra of F, we have $D_qF \approx S_q(F^*)^*$ and $DF \otimes \Lambda F$ is an $SF^* \otimes \Lambda F$-module. Considering the element c_F an element of $SF^* \otimes \Lambda F$, we have the complex

$$(**)\quad \ldots \longrightarrow D_{q+2}F \otimes \Lambda^{p-3}F \xrightarrow{\delta_{q+2}^{p-2}} D_{q+1}F \otimes \Lambda^{p-2}F \xrightarrow{\delta_{q+1}^{p-1}} D_qF \otimes \Lambda^{p-1}F \xrightarrow{\delta_q^p} D_{q-1}F \otimes \Lambda^pF \longrightarrow \ldots$$

which is also acyclic. We define

$$K_q^pF = \text{Coker}\left(D_{q+1}F \otimes \Lambda^{p-2}F \xrightarrow{\delta_{q+1}^{p-1}} D_qF \otimes \Lambda^{p-1}F\right).$$

We therefore also have

$$K_q^pF = \text{Ker}\ (D_{q-1}F \otimes \Lambda^pF \rightarrow D_{q-2}F \otimes \Lambda^{p+1}F) \quad q \geqq 1.$$

and, by dualizing, we see that

$$K_q^pF \approx L_q^p(F^*)^* \quad \text{or} \quad K_q^p(F^*) = (L_q^pF)^*.$$

Corresponding to (1) and (2) we have

$$(1')\quad \begin{aligned} &K_q^1F = D_qF \quad \text{for all } q\\ &K_1^pF = \Lambda^pF \quad \text{for all } p \neq 0\\ &K_q^0F = K_0^pF = 0 \quad \text{for all } p \neq 1 \text{ and all } q\\ &K_q^pF = 0 \quad \text{for all } p > \text{rank } F.\end{aligned}$$

$$(2')\quad K_q^nF \approx D_{q-1}F \otimes \Lambda^nF \quad \text{if } n = \text{rank } F.$$

To show that L_q^pF is free, we showed in [**1**, Proposition 2.5] that

$$L_q^p(F \oplus R) \approx L_q^pF \oplus \Lambda^{p-1}S_{q-1}(F \oplus R)$$

and hence, by induction, L_q^p is free and

$$\text{rank } (L_q^pF) = \binom{n+q-1}{p+q-1}\binom{p+q-2}{q-1}$$

if rank $F = n$.

Similarly, we have

$$K_q^p(F \oplus R) \approx K_q^p F \oplus \Lambda^{p-1}F \otimes D_{q-1}(F \oplus R);$$

$K_q^p F$ is free and its rank is equal to that of $L_q^p F$.

In [**1**] we did not introduce the notation $K_q^p F$ and simply wrote $L_q^p F^*$ to denote $K_q^p(F^*)$. However, it is useful to notice that if R contains the rationals, then the algebra DF is isomorphic to SF and the complex (**) may be replaced by

$$(***) \quad \ldots S_{q+2}F \otimes \Lambda^{p-3}F \rightarrow S_{q+1}F \otimes \Lambda^{p-2}F \rightarrow \ldots$$

where the boundary operator involves the usual partial derivatives $\partial/\partial X_i$ if $X_1, \ldots, X_n$ denotes a basis for F. Thus $K_q^p(F^*)$ may be interpreted as the module of exact p-forms of degree $q - 1$. For this and other reasons, we shall use the functors K_q^p in this paper.

If $\varphi: F \rightarrow G$ is a map of free R-modules, with $m = \text{rank } F$ and $n = \text{rank } G$, we have the complexes introduced in § 3 of [**1**]:

$$\mathbf{L}_q^{p,r}(\varphi): 0 \rightarrow K_{m-r+1}^{r-p}G^* \otimes L_q^m F \xrightarrow{d} K_{m-r}^{r-p}G^* \otimes L_q^{m-1}F \xrightarrow{d} \ldots \xrightarrow{d} K_1^{r-p}G^* \otimes L_q^r F \xrightarrow{d_1} L_q^p F$$

and

$$\mathbf{L}_q^p(\varphi): 0 \longrightarrow K_{m-n}^{n-p+1}G^* \otimes L_q^m F \xrightarrow{d} K_{m-n-1}^{n-p+1}G^* \otimes L_q^{m-1}F \xrightarrow{d} \ldots \xrightarrow{d} K_1^{n-p+1}G^* \otimes L_q^{n+1}F \xrightarrow{d_1} L_q^p F \xrightarrow{L_q^p(\varphi)} L_q^p G.$$

The complex $\mathbf{L}_q^p(\varphi)$ is the complex $\mathbf{L}_q^{p,r}(\varphi)$ augmented by the map $L_q^p(\varphi)$ where $r = n + 1$ and $L_q^p(\varphi)$ is the map induced by φ. Because the maps d, d_1 and $L_p^q(\varphi)$ are described rigorously in [**1**], we will give here only an heuristic description of them.

The map $\varphi: F \rightarrow G$ induces the map $\Lambda\varphi^* : \Lambda G^* \rightarrow \Lambda F^*$ and thus we have the operation of ΛG^* on ΛF. To define a map

$$K_\lambda^\mu G^* \otimes L_q^\nu F \rightarrow K_{\lambda-1}^\mu G^* \otimes L_q^{\nu-1}F,$$

we regard $K_\lambda^\mu G^*$ as a factor module of

$$D_\lambda G^* \otimes \Lambda^{\mu-1}G^*$$

and $L_q^\nu F$ as a submodule of

$$S_q F \otimes \Lambda^{\nu-1}F.$$

Thus we shall represent a "typical" element of $K_\lambda^\mu G^* \otimes L_q^\nu F$ as $\omega \otimes \gamma \otimes H \otimes \alpha$ where $\omega \otimes \gamma \in D_\lambda G^* \otimes \Lambda^{\mu-1}G^*$ and $H \otimes \alpha$ is a sum of elements in $S_q F \otimes \Lambda^{\nu-1}F$. Letting $\epsilon_1, \ldots, \epsilon_n$ be a basis for G, and $\xi_1, \ldots, \xi_n$ the dual basis for G^*, we send the element $\omega \otimes \gamma \otimes H \otimes \alpha$ to $\sum(\partial\omega/\partial\epsilon_i) \otimes \gamma \otimes H \otimes \xi_i(\alpha)$ where

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$\partial/\partial\epsilon_i$ denotes the derivation on DG^* induced by $\epsilon_i \in S(G)$ and $\xi_i(\alpha)$ is the result of operating by $\xi_i \in G^*$ on $\alpha \in \Lambda F$. We thus end up in $D_{\lambda-1}G^* \otimes \Lambda^{\mu-1}G^* \otimes S_qF \otimes \Lambda^{\nu-2}F$ and, heuristically $d(\omega \otimes \gamma \otimes H \otimes \alpha) = \sum(\partial\omega/\partial\epsilon_i) \otimes \gamma \otimes H \otimes \xi_i(\alpha)$.

The map d_1: $K_1{}^sG^* \otimes L_q{}^tF \to L_q^{t-s}F$ is easy to define since $K_1{}^sG^*$ is simply Λ^sG^*. Again representing an element of $L_q{}^tF$ as an element $H \otimes \alpha \in S_qF \otimes \Lambda^{t-1}F$, and taking $\gamma \in \Lambda^sG^*$, we define $d_1(\gamma \otimes H \otimes \alpha) = H \otimes \gamma(\alpha) \in S_qF \otimes \Lambda^{t-s-1}F$.

Finally, the map $L_q{}^p\varphi$: $L_q{}^pF \to L_q{}^pG$ is just that induced by the map

$$S_q(\varphi) \otimes \Lambda^{p-1}\varphi: S_qF \otimes \Lambda^{p-1}F \to S_qG \otimes \Lambda^{p-1}G.$$

With this notation set, we can state the following result of [**1**].

THEOREM 2.1. [**1**, Theorem 3.1]. *Let R be a noetherian ring, and suppose that φ: $F \to G$ is a map between free R-modules of ranks m and n, respectively. Denote by $I_n(\varphi)$ the ideal generated by the minors of φ of order n. If* grade $I_n(\varphi) = m - n + 1$, *then* $\mathbf{L}_q{}^p(\varphi)$ *is a free resolution of* Coker $(L_q{}^p\varphi: L_q{}^pF \to L_q{}^pG)$.

3. The graded case. We now turn to the case where the ring R is graded. Since we shall want R to be strictly commutative, we may suppose that R is zero in odd degrees. However, since we would, in that case, be tempted to divide all the degrees by two, we shall simply write $R = \coprod_{\gamma \geqq 0} R_\gamma$ and assume that R is commutative in the classical sense. The free R-modules we consider will all have the canonical grading, i.e. $F = R \otimes_{R_0} F_0$ where F_0 is a free R_0-module. If $G = R \otimes_{R_0} G_0$, a map φ: $F \to G$ of degree d is given by a map φ_0: $F_0 \to R_d \otimes G_0$.

It is clear that if $F = R \otimes_{R_0} F_0$, then

$$S_R(F) = R \otimes S_{R_0}(F_0)$$
$$\Lambda_R(F) = R \otimes_{R_0} \Lambda_{R_0}(F_0).$$

From this it follows easily that

$$L_q{}^pF = R \otimes L_q{}^pF_0.$$

Given the map φ: $F \to G$ induced by φ_0: $F_0 \to R_d \otimes G_0$, we obtain the maps

$$S(\varphi): S(F) \to S(G) \quad \text{and} \quad \Lambda(\varphi): \Lambda(F) \to \Lambda(G).$$

On the graded components, these maps are:

$$R_\gamma \otimes S_q(F_0) \to F_{\gamma+qd} \otimes S_q(G_0)$$
$$R_\gamma \otimes \Lambda^p(F_0) \to R_{\gamma+pd} \otimes \Lambda^p(G_0).$$

Consequently, the components of the map $L_q{}^p(\varphi)$: $L_q{}^p(F) \to L_q{}^p(G)$ are:

$$R_\gamma \otimes L_q{}^pF_0 \to R_{\gamma+(p+q-1)d} \otimes L_q{}^pG_0.$$

Similarly, we have

$$D_R(F) = R \otimes_{R_0} D_{R_0}(F_0), \quad K_q{}^p(F) = R \otimes K_q{}^p(F_0), \quad \text{and}$$
$$K_q{}^p(\varphi): K_q{}^p(F) \to K_q{}^p(G) \quad \text{has components } R_\gamma \otimes K_q{}^p(F_0) \to R_{\gamma+(p+q-1)d} \otimes K_q{}^p(G_0).$$

We must next transcribe the maps that occur in the complexes $\mathbf{L}_q^{p,r}(\varphi)$ to the graded case. That is, we want to describe the homogeneous components of the maps

$$d: K_\lambda{}^\mu G^* \otimes L_q{}^\nu F \to K_{\lambda-1}^\mu G^* \otimes L_q^{\nu-1} F$$

and

$$d_1: K_1{}^s G^* \otimes L_q{}^t F \to L_q^{t-s} F.$$

It is easy to see that we get:

$$d: R_\gamma \otimes K_\lambda{}^\mu G_0{}^* \otimes L_q{}^\nu F_0 \to R_{\gamma+d} \otimes K_{\lambda-1}^\mu G_0{}^* \otimes L_q^{\nu-1} F_0$$
$$d_1: R_\gamma \otimes K_1{}^s G_0{}^* \otimes L_q{}^t F_0 \to R_{\gamma+sd} \otimes L_q^{t-s} F_0.$$

Taking the grading into account, we see that the complexes $\mathbf{L}_q^{p,r}(\varphi)$ and $\mathbf{L}_q{}^p(\varphi)$ of Section 2 are the direct sums of complexes:

$$\mathbf{L}_q^{p,r}(\varphi)_k: \ldots \xrightarrow{d} R_{k-(r-p+1)d} \otimes K_2^{r-p} G_0{}^* \otimes L_q^{r+1} F_0 \xrightarrow{d} R_{k-(r-p)d} \otimes K_1^{r-p} G_0{}^* \otimes L_q{}^r F_0 \xrightarrow{d_1} R_k \otimes L_q{}^p F_0$$

$$\mathbf{L}_q{}^p(\varphi)_k: \ldots \xrightarrow{d} R_{k-(n+q)d} \otimes K_1^{n-p+1} G_0{}^* \otimes L_q^{n+1} F_0 \xrightarrow{d_1} R_{k-(p+q-1)d} \otimes L_q^p F_0 \to R_k \otimes L_q^p G_0.$$

4. New complexes and modules from old. In this section we will apply Sections 2 and 3 to the following situation.

We let R be a ring, and let F and G be R-modules of ranks m and n, respectively. Denote by $S = \sum S_\nu$ the symmetric algebra $S(F \otimes G^*)$, and by c_G the element in $G^* \otimes G \subset \Lambda G^* \otimes SG$ which is the image of 1 under the map $R \to G^* \otimes G$ corresponding to the identity map of G. Using c_G we define the map

$$\varphi_0: F \to F \otimes G^* \otimes G = S_1 \otimes G$$

to be $1 \otimes c_G$. This defines the map

$$\varphi: S \otimes F \to S \otimes G$$

which is a morphism of free S-modules of degree 1.

If we identify S with the polynomial ring $R[X_{ij}]$ with $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, we see that, with suitable choice of basis, the matrix corresponding to φ is the generic matrix (X_{ij}).

If R is noetherian, we may apply Theorem 2.1 to see that the complexes $\mathbf{L}_q^p(\varphi)$ are acyclic and so, too, are the homogeneous components $\mathbf{L}_q^p(\varphi)_k$, since grade $I_n(\varphi)$ is $m - n + 1$. If R is not noetherian, this is still true as can be easily seen by observing that R is the direct limit of noetherian subrings and noting that the situation is generic.

Consider now the special case when $n = 1$. In that case we need only look at the complexes $\mathbf{L}_q^1(\varphi)_{q+k}$ and, identifying $K_\lambda^1G^*$ with R, we obtain the acyclic complexes

$$\mathbf{B}_{q,k}: \ldots \to S_{k-2}F \otimes L_q^3F \to S_{k-1}F \otimes L_q^2F \to S_kF \otimes S_qF \to S_{q+k}F,$$

with the map m on the extreme right an epi for $k \geqq 0$. In fact, this map is simply the product map in the symmetric algebra of F.

Since the complexes $\mathbf{B}_{q,k}$ are free acyclic complexes, the cycles are projective R-modules. We will shortly see that they are in fact free.

The maps in the complex $\mathbf{B}_{q,k}$ may be described as follows. We saw in [**1**] that LF is an $SF \otimes \Lambda F^*$-module and hence a ΛF^*-module. SF is clearly an SF-module. Thus $SF \otimes LF$ is an $SF \otimes \Lambda F^*$-module. Letting c_F be the element of $F \otimes F^* \subset SF \otimes \Lambda F^*$ analogous to the element c_G described above, the maps $SF \otimes LF \to SF \otimes LF$ in the complex $\mathbf{B}_{q,k}$ are simply multiplication by c_F.

Dualizing $\mathbf{B}_{q,k}$ one sees that the complexes

$$\mathbf{C}_{q,k}: O \to D_{q+k}F \to D_kF \otimes D_qF \to D_{k-1}F \otimes K_q^2F \to D_{k-2}F \otimes K_q^3F \to \ldots$$

are also acyclic, where the map $D_{q+k} \to D_k \otimes D_q$ is the (k, q) component of the diagonal map in the divided power algebra, and the other maps are multiplication by the element $c_F' \in F^* \otimes F \subset SF^* \otimes \Lambda F$.

The map $D_{q+k}F \to D_kF \otimes D_qF$ may also be described as follows. The algebra $SF^* \otimes DF$ is an algebra with divided powers, namely $(x \otimes y)^{(q)} = x^q \otimes y^{(q)}$. In particular, the element $c_F' \in F^* \otimes F$ has divided powers and we may consider $c_F'^{(q)} \in S_qF^* \otimes D_qF$. The map $D_{q+k} \to D_kF \otimes D_qF$ is the composition

$$D_{q+k} = D_{q+k}F \otimes R \xrightarrow{1 \otimes c_F'^{(q)}} D_{q+k}F \otimes S_qF^* \otimes D_qF \xrightarrow{\nu \otimes 1} D_kF \otimes D_qF,$$

where ν: $D_{q+k}F \otimes S_qF^* \to D_kF$ is the operation of SF^* on DF.

For convenience we state the above as

PROPOSITION 4.1. *Let F be a free R-module of rank m. Then for all positive integers q and k, the complexes $\mathbf{B}_{q,k}$ and $\mathbf{C}_{q,k}$ above are acyclic.*

LEMMA 4.2. *Let F be a free R-module of rank m, and let p, q be non-negative integers. Then the complex*

$$\mathbf{D}_{p-q}: 0 \to K_{q+1}^pF \to K_q^pF \otimes F \to \ldots \to K_2^pF \otimes \Lambda^{q-1}F \to \Lambda^pF \otimes \Lambda^qF \to \Lambda^{p+q}F \to 0$$

is exact. The map $\Lambda^p F \otimes \Lambda^q F \to \Lambda^{p+q} F$ is the usual multiplication in ΛF. The other maps are the operation of $c_F' \in SF^* \otimes \Lambda F$.

Proof. The case $p = 1$ is simply the statement that $\mathbf{C}_{1,q}$ is exact, and we now proceed by induction on p. Consider the double complex:

$$\begin{array}{cccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to & D_rF \otimes K_3^{p-r}F \otimes \Lambda^{q-2}F & \to & D_rF \otimes K_2^{p-r}F \otimes \Lambda^{q-1}F & \to & D_rF \otimes \Lambda^{p-r}F \otimes \Lambda^qF & \to & D_rF \otimes \Lambda^{p+q-r}F & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to & D_{r-1}F \otimes K_3^{p-r+1}F \otimes \Lambda^{q-2}F & \to & D_{r-1}F \otimes K_2^{p-r+1} \otimes \Lambda^{q-1}F & \to & D_{r-1}F \otimes \Lambda^{p-r+1}F \otimes \Lambda^qF & \to & D_{r-1}F \otimes \Lambda^{p+q-r+1}F & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to & F \otimes K_3^{p-1}F \otimes \Lambda^{q-2}F & \to & F \otimes K_2^{p-1}F \otimes \Lambda^{q-1}F & \to & F \otimes \Lambda^{p-1}F \otimes \Lambda^qF & \to & F \otimes \Lambda^{p+q-1}F & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to & K_3^pF \otimes \Lambda^{q-2}F & \to & K_2^pF \otimes \Lambda^{q-1}F & \to & \Lambda^pF \otimes \Lambda^qF & \to & \Lambda^{p+q}F & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

The rows of the complex are simply $D_rF \otimes \mathbf{D}_{p-r,q}$, and the columns are $\mathbf{C}_{r,p-1} \otimes \Lambda^{q-r+1}F$. It is easy to check that this is indeed a double complex. By 4.1 we know that the columns are exact, and by induction we have that all but the bottom row are exact. The usual spectral sequence argument yields the exactness of the bottom row, i.e., $\mathbf{D}_{p,q}$ is exact.

Our next step is to consider complexes of the form:

$$\mathbf{D}_{p,q,r}\colon 0 \to K_{p+r}^qF \to K_r^1F \otimes K_p^qF \to \ldots \to K_r^{p-1}F \otimes K_2^qF \to K_r^pF \otimes \Lambda^qF \to K_r^{p+q}F \to 0.$$

The right hand map is just the operation of ΛF on KF. The left hand map is the composition:

$$R \otimes K_{p+r}^qF \xrightarrow{c_F'^{(r)} \otimes 1} D_rF \otimes S_rF^* \otimes K_{p+r}^qF \xrightarrow{1 \otimes \mu} D_rF \otimes K_p^qF = K_r^1F \otimes K_p^qF$$

where $c_F'^{(r)}$ is the rth divided power of c_F' in $D_rF \otimes S_rF^*$, and ν is the operation of SF^* on KF.

The maps in the rest of the complex are given by the operation of $c_F' \in \Lambda F \otimes SF^*$ on $KF \otimes KF$, where we treat the first factor of $KF \otimes KF$ as a ΛF-module, and the second factor as an SF^*-module.

PROPOSITION 4.3. *If F is a free R-module, and p, q, r are non-negative integers, then $\mathbf{D}_{p,q,r}$ is an acyclic complex.*

Proof. In this proposition we proceed by induction on r, the case $r = 0$ being trivial and the case $r = 1$ being Lemma 4.2. Assume, then, that $r \geqq 1$. It is easy to show that the following diagram is commutative:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \to & K_r{}^1F \otimes K_p{}^qF & \to \dots \to & K_r^{p-1}F \otimes K_2{}^qF & \to & K_r{}^pF \otimes \Lambda^qF & \to & K_r^{p+q}F & \to 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & D_{r-1}F \otimes K^q_{p+1}F \to & D_{r-1}F \otimes F \otimes K_p{}^qF & \to \dots \to & D_{r-1}F \otimes \Lambda^{p-1}F \otimes K_2{}^qF & \to & D_{r-1}F \otimes \Lambda^pF \otimes \Lambda^qF & \to & D_{r-1}F \otimes \Lambda^{p+q}F & \to 0 \\
 & \| & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & D_{r-1}F \otimes K^q_{p+1}F \to & K^2_{r-1}F \otimes K_p{}^qF & \to \dots \to & K^p_{r-1}F \otimes K_2{}^qF & \to & K^{p+1}_{r-1}F \otimes \Lambda^qF & \to & K^{p+q+1}_{r-1}F & \to 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

The top and bottom rows are the sequences $\mathbf{D}_{p,q,r}$ and $\mathbf{D}_{p+1,q,r-1}$ respectively with their tails lopped off. The middle row is $D_{r-1}F \otimes \mathbf{D}_{q,p}$. The vertical maps are the inclusion maps (on top) tensored with appropriate KF's, and the canonical surjections, also tensored with KF's. Thus, the columns are exact, the middle row is exact by 4.2, and the bottom row is exact except at the extreme left end, where the homology is $K^q_{p+r}F$ (by our induction hypothesis). It follows, therefore, that the top row is exact except at the extreme left end, and the homology there is also $K^q_{p+r}F$. It only remains to show that the map of $K^q_{p+r}F$ into $K_r{}^1F \otimes K_q{}^pF$ is the one described for the complex $\mathbf{D}_{p,q,r}$. To see this, it suffices to prove that the following diagram is commutative:

$$
\begin{array}{ccccc}
R \otimes K^q_{p+r}F & \xrightarrow{c_F{}'^{(r-1)} \otimes 1} & D_{r-1}F \otimes S_{r-1}F^* \otimes K^q_{p+r}F & \xrightarrow{1 \otimes \nu} & D_{r-1}F \otimes R \otimes K^q_{p+1}F \\
\downarrow{\scriptstyle c_F{}'^{(r)} \otimes 1} & & & & \downarrow{\scriptstyle 1 \otimes c_F{}' \otimes 1} \\
D_rF \otimes S_rF^* \otimes K^q_{p+r}F & & & & D_{r-1}F \otimes F \otimes F^* \otimes K^q_{p+1}F \\
\searrow{\scriptstyle 1 \otimes \nu} & & & & \downarrow{\scriptstyle 1 \otimes 1 \otimes \nu} \\
D_rF \otimes K_p{}^qF & \xrightarrow{1 \otimes c_F{}' \otimes 1} & D_rF \otimes F^* \otimes F \otimes K_p{}^qF & \xrightarrow{\nu \otimes 1 \otimes 1} & D_{r-1}F \otimes F \otimes K_p{}^qF
\end{array}
$$

The proof is probably most easily accomplished by choosing a basis $x_1, \dots, x_m$ for F and the dual basis $\xi_1, \dots, \xi_m$ for F^*. Since $c_F{}'$ is then $\sum x_i \otimes \xi_i$, we see that $c_F{}'^{(k)} = \sum x^{(\zeta)} \otimes \xi^\zeta$ where $x^{(\zeta)}$ means $x_1{}^{(\zeta_1)} \dots x_m{}^{(\zeta_m)}$, ξ^ζ means $\xi_1{}^{\zeta_1} \dots \xi_m{}^{\zeta_m}$, and where $\zeta = (\zeta_1, \dots, \zeta_m)$ runs over all m-tuples of weight k, i.e., $\sum \zeta_i = k$. If one takes a "typical" element $H \otimes \alpha$ of $K^q_{p+r}F$, one obtains the element

$$\sum x^{(\zeta)} \otimes x_i \otimes \xi_i \xi^\zeta(H) \otimes \alpha$$

with ζ's of weight $r - 1$ by going to the right and down in the diagram. Proceeding around the other way, one obtains

$$\sum \xi_i(x^{(\sigma)}) \otimes x_i \otimes \xi^\sigma H \otimes \alpha$$

where the σ's run over m-tuples of weight r. These two terms are clearly the same, and the proposition is proven.

As usual, we may dualize all of the complexes described above, getting a whole family of others. For the sake of completeness we list these also:

$$\mathbf{E}_{p,q,r}: 0 \to L_r^{p+q}F \to L_r{}^pF \otimes \Lambda^qF \to L_r^{p-1}F \otimes L_2{}^qF \to \ldots \to S_rF \otimes L_p{}^qF \to L_{p+r}^qF \to 0.$$

As we have already remarked, the cycles (or boundaries) of the complexes $\mathbf{D}_{p,q,r}$, $\mathbf{E}_{p,q,r}$ are projective R-modules. However, since the functors DF, SF, ΛF all commute with base change, so also do the functors LF and KF. It therefore follows that the cycles of these complexes also commute with base change. If the ring R is the ring of integers, then we see that all these cycles are not only projective but free. If R is any commutative ring, and F a free R-module, then $F = R \otimes_{\mathbf{Z}} F_0$ where F_0 is a free $\mathbf{Z}$-module. Hence we see that all these cycles (or boundaries), are also free R-modules.

Definition. Let F be a free R-module. Define

$$L_{q_1q_2}^{p_1p_2}F = \text{Ker}\,(L_{q_1}^{p_1}F \otimes L^{p_2-1}F \to L_{q_1+1}^{p_1}F \otimes L_{q_2}^{p_2-2}F)$$
$$K_{q_1q_2}^{p_1p_2}F = \text{Coker}\,(K_{q_1+1}^{p_1}F \otimes K_{q_2}^{p_2-2}F \to K_{q_1}^{p_1}F \otimes K_{q_2}^{p_2-1}F).$$

PROPSOITION 4.4. *The modules $L_{q_1q_2}^{p_1p_2}F$ and $K_{q_1q_2}^{p_1p_2}F$ are free R-modules of finite rank.*

It is clear that we may continue the procedure described above to obtain multiply indexed L's and K's. In fact, it is easy to outline the general procedure as follows:

Let Λ and Γ be R-algebras, and let M be a $\Lambda \otimes \Gamma$-module. Suppose we have an element $c \in \Lambda \otimes \Gamma$ such that $c^2 = 0$. Define $L_1(M) = \text{Ker}\,(c\colon M \to M)$. Then $L_1(M)$ is also a $\Lambda \otimes \Gamma$-module. We may therefore consider $L_1(M) \otimes L_1(M)$ a $\Lambda \otimes \Gamma$-module with Λ operating on the first factor, and Γ on the second, and define

$$L_2(M) = \text{Ker}\,(c\colon L_1{}^{(M)} \otimes L_1{}^{(M)} \to L_1{}^{(M)} \otimes L_1{}^{(M)}).$$

Proceeding in this way, we define

$$L_n(M) = \text{Ker}\,(c\colon L_{n-1}(M) \otimes L_{n-1}(M) \to L_{n-1}(M) \otimes L_{n-1}(M)).$$

Letting $\Lambda = \Lambda F^*$, $\Gamma = SF$, $M = \Lambda F \otimes SF$ and $c = c_F$, the first two steps of the procedure above describe our modules $L_q{}^pF$ and $L_{q_1q_2}^{p_1p_2}F$.

Letting $\Lambda = \Lambda F$, $\Gamma = SF^*$, $M = \Lambda F \otimes DF$ and $c = c_F{}'$, the first two steps of the procedure above describe our modules $K_q{}^pF$ and $K_{q_1q_2}^{p_1p_2}F$.

In a subsequent article we shall explore these modules and the complexes of which they form a part. We suspect that the complete sequence of modules $L_{q_1\ldots q_2}^{p_1\ldots p_2}$ describes the irreducible representations of the general linear group (at least in characteristic zero).

We have known for some time that the $L_q{}^pF$'s correspond to the partition

$$(q, \underbrace{1, 1 \ldots, 1}_{p-1}),$$

and it appears that the $L_{q_1q_2}^{p_1p_2}F$'s correspond to sums of certain of the irreducible representations of the general linear group which can be made quite explicit. However, at this point the connection is not completely understood.

5. Some lower order minors. As we mentioned in the introduction, we are interested in finding complexes associated to the lower order minors of a matrix. If $f: F \rightarrow G$ is a map of free modules with m = rank F and n = rank G, we let I_q be the ideal generated by the minors of f of order q, i.e.,

$$I_q = Im(\Lambda^q F \otimes \Lambda^{n-q} G \rightarrow \Lambda^n G).$$

We know that for all $p \leqq \min(n, m)$, the cokernel of

$$f_{p,q}: \Lambda^p F \otimes \Lambda^{n-q} G \rightarrow \Lambda^{n-q+p} G$$

has the same support as R/I_q [**1**], and we also know that the generic height of I_q is $(m - q + 1)(n - q + 1)$. Suppose, then, that we have a canonical way of writing down a free complex

$$X: 0 \longrightarrow X_\alpha \longrightarrow X_{\alpha-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow \Lambda^p F \otimes \Lambda^{n-q} G \xrightarrow{f_{p,q}} \Lambda^{n-q+p} G$$

where $\alpha = (m - q + 1)(n - q + 1)$, and that we want to show it is grade sensitive to the ideal I_q. Then, as pointed out in [**1**], we may first consider the case when f is a generic matrix (X_{ij}) and prove acyclicity there. In order to prove that acyclicity, it suffices to prove it after localization at primes of height less than $(m - q + 1)(n - q + 1)$, in which case I_q blows up to the whole ring. We are therefore reduced to proving that the complex X is acyclic under the assumption that R is a local ring and one of the $q \times q$ submatrices of f is the identity.

To illustrate, suppose $f: F \rightarrow G$ as above, and that $m \leqq n$. We want to get a complex associated to the minors of order m; in fact, we want to resolve the cokernel of

$$f_{p,m} \Lambda^p F \otimes \Lambda^{n-m} G \rightarrow \Lambda^{n-m+p} G$$

for all $p \leqq m$. We write down the complex

$$\begin{aligned}\mathbf{K}(f_{p,m}): 0 &\longrightarrow K_{n-m+1}^p F \longrightarrow K_{n-m}^p F \otimes G \longrightarrow \ldots \\ \ldots &\longrightarrow K_3{}^p F \otimes \Lambda^{n-m-2} G \longrightarrow K_2{}^p F \otimes \Lambda^{n-m-1} G \longrightarrow \Lambda^p F \\ &\otimes \Lambda^{n-m} G \xrightarrow{f_{p,m}} \Lambda^{n-m+p} G\end{aligned}$$

where we regard $KF \otimes \Lambda G$ as an $SF^* \otimes \Lambda F$-module by having SF^* operate on KF and ΛF on ΛG. All the maps (except $f_{p,m}$) are the multiplication by c_F'. (It is easy to check that the composition

$$K_2{}^pF \otimes \Lambda^{n-m-1}G \rightarrow \Lambda^pF \otimes \Lambda^{n-m}G \rightarrow \Lambda^{n-m+p}G$$

is zero.) Notice that this complex is of length $n - m + 1$, which is the height (and grade) of the generic $m \times m$ minors ideal. Therefore, to show that $\mathbf{K}(f_{p,m})$ is grade sensitive to the ideal I_m, we need only show that it is acyclic when R is local and an $m \times m$ submatrix of f is the identity. In this case, by simple change of basis, we may assume that the map f is simply the injection of F as a summand of G, i.e. $G = F \oplus G'$ and $F \rightarrow G$ is the canonical inclusion. Making the identification

$$\Lambda^{n-m}G = \sum \Lambda^qF \otimes \Lambda^{n-m-q}G',$$

we see that the map $f_{p,m}$ is just the direct sum of maps:

$$\sum_{q=0}^{n-m} \Lambda^pF \otimes \Lambda^qF \otimes \Lambda^{n-m-q}G' \rightarrow \sum_{q=0}^{n-m} \Lambda^{p+q}F \otimes \Lambda^{n-m-q}G'.$$

Applying 4.2, we have an exact sequence for each q:

$$0 \rightarrow K_{q+1}^pF \otimes \Lambda^{n-m-q}G' \rightarrow K_q{}^pF \otimes F \otimes \Lambda^{n-m-q}G' \rightarrow \ldots$$
$$\ldots \rightarrow K_2{}^pF \otimes \Lambda^{q-1}F \otimes \Lambda^{n-m-q}G' \rightarrow \Lambda^pF \otimes \Lambda^qF \otimes \Lambda^{n-m-q}G' \rightarrow \Lambda^{p+q}F \otimes \Lambda^{n-m-q}G' \rightarrow 0$$

and this sequence is of length $q + 1$. The sum of these sequences therefore is exact and taking the sum, we see that in dimension l we get

$$\sum_{q=l-1}^{n-m} K_l{}^pF \otimes \Lambda^{q-l+1}F \otimes \Lambda^{n-m-q}G' = K_l{}^pF \otimes \sum_{t=0}^{n-m-l+1} \Lambda^tF \otimes \Lambda^{n-m-l+1-t}G'$$

where $t = q - l + 1$. This is clearly the term $K_l{}^qF \otimes \Lambda^{n-m-l+1}G$ and, since this is the l-dimensional term of the complex $\mathbf{K}(f_{p,m})$, we see that $\mathbf{K}(f_{p,m})$ is acyclic when $F \rightarrow G = F \oplus G'$ is the injection. Consequently we have proven

PROPOSITION 5.1. *Let R be a noetherian ring, F and G free R-modules of ranks m and n respectively, with $m \leqq n$. If f: $F \rightarrow G$ is a map, then $\mathbf{K}(f_{p,m})$ is a free complex which is grade sentitive to the ideal $I_m(f)$ generated by the minors of f of order m. In particular, the homology of $\mathbf{K}(f_{p,m})$ is zero in all positive dimensions if and only if* grade $(I_m(f)) \geqq n - m + 1$.

Suppose now that F' has rank $m + 1$, G has rank n and f': $F' \rightarrow G$ is a map. Then $F' = F \oplus R$, and we may assume that f' is the sum of two maps f: $F \rightarrow G$ and b: $R \rightarrow G$. The problem still is to associate a complex grade sensitive to the ideal $I_m(f')$ of $m \times m$ minors of f'. In this case, the grade of $I_m(f')$ is generically $2(n - m + 1)$ so we would like a complex of that dimension. We see that we have the beginnings of what may be a double complex if we consider:

$$\textbf{(P)}\quad \begin{array}{ccccccccccc}
 & & & & & & & & & & 0 \\
 & & & & & & & & & & \downarrow \\
0 \to K^p_{n-m}F \otimes \Lambda^n G \otimes \Lambda^m F & \to & K^p_{n-m-1}F \otimes \Lambda^{n-1}G^* \otimes \Lambda^m F & \to \dots \to & \Lambda^p F \otimes \Lambda^{m+1}G^* \otimes \Lambda^m F & \to & \Lambda^{m-p+1}G^* \otimes \Lambda^m F & \to & & & \Lambda^{p-1}F \to 0 \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \Lambda^{p-1}F \otimes G \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \vdots \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \Lambda^{p-1}F \otimes \Lambda^{n-m-1}G \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \Lambda^{p-1}F \otimes \Lambda^{n-m}G \\
 & & & & & & & & & & \downarrow\mu \\
0 \to K^p_{n-m+1}F & \to & K^p_{n-m}F \otimes G & \to \dots \to & K_2{}^pF \otimes \Lambda^{n-m-1}G & \to & \Lambda^p F \otimes \Lambda^{n-m}G & \to & & & \Lambda^{n-m+p}G \to 0
\end{array}$$

The bottom row is simply the complex $\mathbf{K}(f_{p,m})$. The map μ is given by $\mu(a_1 \otimes a_2) = f_{p-1,m}(a_1 \otimes a_2) \wedge b$ while the maps

$$\Lambda^{p-1}F \otimes \Lambda^k G \to \Lambda^{p-1}F \otimes \Lambda^{k+1}G$$

are just $a_1 \otimes a_2 \to a_1 \otimes a_2 \wedge b$. The top row is obtained by considering the map f^*: $G^* \to F^*$ and the complex

$$0 \to K^p_{n-m}F \otimes \Lambda^n G \to K^p_{n-m-1}F \otimes \Lambda^{n-1}G^* \to \dots \to K_2{}^pF \otimes \Lambda^{m+1}G^* \to \Lambda^p F \otimes \Lambda^{m+1}G^* \to \Lambda^{m-p+1}G^* \to \Lambda^{m-p+1}F^*$$

of [**1**], which we know is also grade-sensitive to the ideal $I_m(f)$. Tensoring each term of the above complex with $\Lambda^m F$ and identifying $\Lambda^{m-p+1}F^* \otimes \Lambda^m F$ with $\Lambda^{p-1}F$, we get the complex on top. Notice that the column in the diagram has length $n - m + 1$ so that if we can fill in the rectangle suitably we can get a double complex whose total complex will have length $2(n - m + 1)$. Observe, too, that if $I_m(f) = R$, then the top and bottom rows are exact. If we could fill in all the rows acyclically, then the total complex would also be exact and we would have a candidate for a complex $\mathbf{K}(f'_{p,m})$.

What we propose to do in this section is carry out this program in detail for one case. In the next section we will outline the techniques and difficulties encountered in attempting to push the program further.

Let F be a free module of rank $n - 1$, G a free module of rank n, and let f: $F \to G$, b: $R \to G$ be maps. In this case, the diagram (**P**) becomes

$$\textbf{(P}_1\textbf{)}\quad \begin{array}{ccccc}
0 \to \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F & \to & \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F & \to & \Lambda^{p-1}F \to 0 \\
 & & & & \downarrow \\
 & & & & \Lambda^{p-1}F \otimes G \\
 & & & & \downarrow\mu \\
0 \to K_2{}^pF & \xrightarrow{\partial_2} & \Lambda^p F \otimes G & \xrightarrow{\partial_1} & \Lambda^{p+1}G \to 0
\end{array}$$

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for we are attempting to construct a complex $\mathbf{K}(f'_{p,n-1})$ associated to the minors of order $n - 1$ of the map $f'\colon F \oplus R \to G$ determined by f and b.

We now regard the top complex as a complex over the zero module, and we regard

$$0 \to F \xrightarrow{f} G$$

as a complex over the cokernel of f. If we take the tensor product of these two complexes, we obtain a complex over $0 \otimes \operatorname{Coker} f = 0$;

$$(*) \quad 0 \to \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F \otimes F \to \Lambda^{n-p} G^* \otimes \Lambda^{n-1} F \otimes F \otimes \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F \otimes G \to \Lambda^{p-1} F \otimes F \oplus \Lambda^{n-p} G^* \otimes \Lambda^{n-1} F \otimes G \to \Lambda^{p-1} F \otimes G \to 0$$

When we assume that $f\colon F \to G$ is a summand of G, we have the acyclicity of the top (and bottom) row, and $0 \to F \to G$ is a resolution of $\operatorname{Coker} f$, so that the homology of the complex (*) is $\operatorname{Tor}(0, \operatorname{Coker} f) = 0$, i.e. the above tensor product is acyclic.

If we identify the modules $\Lambda^k G$ with $\Lambda^{n-k} G^*$, the bottom row of $(\mathbf{P}_1)$ becomes

$$(1) \quad 0 \to K_2{}^p F \xrightarrow{h} \Lambda^p F \otimes \Lambda^{n-1} G^* \xrightarrow{g} \Lambda^{n-p-1} G^* \to 0$$

where the map g is just the operation of $\Lambda^p F$ on $\Lambda^{n-1} G^*$. To describe the map h more aesthetically, we replace $K_2{}^p F$ by $K_2{}^p F \otimes \Lambda^n G^*$. The map h is then the composition:

$$K_2{}^p F \otimes \Lambda^n G^* \xrightarrow{d \otimes 1} \Lambda^p F \otimes G \otimes \Lambda^n G^* \xrightarrow{1 \otimes \nu} \Lambda^p F \otimes \Lambda^{n-1} G^*$$

where $d\colon K_2{}^p F \to \Lambda^p F \otimes G$ is the map in the bottom row of $(\mathbf{P}_1)$ and $\nu\colon G \otimes \Lambda^n G^* \to \Lambda^{n-1} G^*$ is the isomorphism induced by the operation of G on $\Lambda^n G^*$. Replacing $K_2{}^p F$ in (1) by $K_2{}^p F \otimes \Lambda^n G^*$, and then tensoring the whole complex with $\Lambda^{n-1} F$, we obtain a complex:

$$(**) \quad 0 \to K_2{}^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F \to \Lambda^p F \otimes \Lambda^{n-1} G^* \otimes \Lambda^{n-1} F \to \Lambda^{n-p-1} G^* \otimes \Lambda^{n-1} F \to 0.$$

We now define a map of the complex (**) into the complex (*), which will be a monomorphism. This will make the cokernel acyclic when both (*) and (**) are acyclic.

$$\begin{array}{llllllll}
(**)\colon 0 \to & K_2{}^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F & \xrightarrow{\delta_3} & \Lambda^p F \otimes \Lambda^{n-1} G^* \otimes \Lambda^{n-1} F & \xrightarrow{\delta_2} & \Lambda^{n-p-1} G^* \otimes \Lambda^{n-1} F & \to & 0 \to 0 \\
 & \downarrow \varphi_3 & & \downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \\
(*)\colon 0 \to & \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F \otimes F & \xrightarrow{d_3} & \Lambda^{n-p} G^* \otimes \Lambda^{n-1} F \otimes F & \xrightarrow{d_2} & \Lambda^{p-1} F \otimes F & \xrightarrow{d_1} & \Lambda^{p-1} F \otimes G \to 0 \\
 & & & \oplus\, \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1} F \otimes G & & \oplus\, \Lambda^{n-p} G^* \otimes \Lambda^{n-1} F \otimes G & &
\end{array}$$

The map φ_1 is the sum of two maps

$$\varphi_{11}\colon \Lambda^{n-p-1} G^* \otimes \Lambda^{n-1} F \to \Lambda^{p-1} F \otimes F$$

$$\varphi_{12}\colon \Lambda^{n-p-1} G^* \otimes \Lambda^{n-1} F \to \Lambda^{n-p} G^* \otimes \Lambda^{n-1} F \otimes G.$$

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The first one, φ_{11}, is the composition

$$\Lambda^{n-p-1}G^* \otimes \Lambda^{n-1}F \xrightarrow{1 \otimes \Delta_1} \Lambda^{n-p-1}G^* \otimes \Lambda^{n-2}F \otimes F \xrightarrow{\nu \otimes 1} \Lambda^{p-1}F \otimes F$$

where Δ_1: $\Lambda^{n-1}F \to \Lambda^{n-2}F \otimes F$ is the indicated component of the diagonal map, and ν: $\Lambda^{n-p-1}G^* \otimes \Lambda^{n-2}F \to \Lambda^{p-1}F$ is the operation of ΛG^* on ΛF.

The second map, φ_{12}, is the composition:

$$\Lambda^{n-p-1}G^* \otimes \Lambda^{n-1}F \xrightarrow{1 \otimes c_G \otimes 1} \Lambda^{n-p-1}G^* \otimes G^* \otimes G$$
$$\otimes \Lambda^{n-1}F \xrightarrow{\mu \otimes T} \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes G$$

where c_G: $R \to G^* \otimes G$ is the usual element, μ stands for multiplication in ΛG^* and T: $G \otimes \Lambda^{n-1}F \to \Lambda^{n-1}F \otimes G$ is simply the interchange map.

To define the map φ_2, we define two maps

$$\varphi_{21}: \Lambda^p F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F \to \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes F$$
$$\varphi_{22}: \Lambda^p F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F \to \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F \otimes G.$$

φ_{21} is the composition:

$$\Lambda^p F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F \xrightarrow{\Delta_1 \otimes 1 \otimes 1} \Lambda^{p-1}F \otimes F \otimes \Lambda^{n-1}G^*$$
$$\otimes \Lambda^{n-1}F \xrightarrow{1 \otimes T'} \Lambda^{p-1}F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F$$
$$\otimes F \xrightarrow{\nu \otimes 1 \otimes 1} \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes F$$

where Δ_1 is as before, T': $F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F \to \Lambda^{n-1}G^* \otimes \Lambda^{n-1}F \otimes G$ is just an interchange, and ν: $\Lambda^{p-1}F \otimes \Lambda^{n-1}G^* \to \Lambda^{n-p}G^*$ is the operation of ΛF on ΛG^*.

The map φ_3 is simply the composition:

$$K_2{}^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F \xrightarrow{i \otimes 1 \otimes 1} \Lambda^p F \otimes F \otimes \Lambda^n G^*$$
$$\otimes \Lambda^{n-1}F \xrightarrow{1 \otimes T''} \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F \otimes F$$

where i: $K_2{}^p F \to \Lambda^p F \otimes F$ is the inclusion map, and T'' is the obvious interchange.

Actually, the map φ_1 is $\varphi_{12} - \varphi_{11}$. The map φ_2 is $\varphi_{21} \pm (-1)\ \varphi_{22}$.

We will briefly sketch the proof that these maps do provide a map of complexes. The major thing we shall leave out of the proof is consideration of signs.

To see that $d_1\varphi_1 = 0$, we take

$$\varphi_{12}(\beta \otimes a) - \varphi_{11}(\beta \otimes a) = \sum \beta \wedge \xi_i(a) \otimes x_i - \sum \beta(a_j) \otimes f(a_j')$$

where $\Delta_1(a) = \sum a_j \otimes a_j'$, $\{x_i\}$ and $\{\xi_i\}$ are a basis and dual basis for G, G^* respectively. But

$$\sum \beta \wedge \xi_i(a) = \sum \beta(\xi_i(a_j')a_j)$$

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so that

$$\sum \beta \wedge \xi_i(a) \otimes x_i = \sum \beta(a_j) \otimes \xi_i(a_j')x_i = \sum \beta(a_j) \otimes f(a_j').$$

The essential part of the proof that $d_2\varphi_2 = \varphi_1\delta_2$ is the formula:

$$\alpha_1(\beta)(\alpha_2) = \sum \pm \alpha_{1j} \wedge \beta(\alpha_{1j}' \wedge \alpha_2)$$

where $\Delta(\alpha_1) = \sum \alpha_{1j} \otimes \alpha_{1j}'$, $\alpha_1, \alpha_2 \in \Lambda F$, $\beta \in \Lambda G^*$. This fact can be found in [**1**].

That $d_3\varphi_3 = \varphi_2\delta_3$ is straightforward.

We will now see that φ_1, φ_2, and φ_3 are monomorphisms, actually split monomorphisms.

The map φ_3 is essentially an inclusion; its cokernel is $\Lambda^{p+1}F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F$.

The map φ_{22} is essentially the identification of $\Lambda^{n-1}G^*$ with G, so it is an isomorphism. Therefore φ_2 is a monomorphism whose cokernel is isomorphic to $\Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes F$.

Finally, the map φ_{12} is essentially the formal map

$$\Lambda^{n-p-1}G^* \to \Lambda^{n-p}G^* \otimes G$$

given by multiplication by c_G. If we dualize this map, we obtain

$$\Lambda^{n-p}G \otimes G^* \to \Lambda^{n-p-1}G$$

which is just the operation of G^* on ΛG. If we identify $\Lambda^k G$ with $\Lambda^{n-k}G^*$, the above map is seen to be the split epimorphism

$$\Lambda^p G^* \otimes G^* \to \Lambda^{p+1}G^*$$

whose kernel is $K_2{}^pG^*$. Thus we see that the map φ_{12} is a split mono whose cokernel is isomorphic to $(K_2{}^pG^*)^* \otimes \Lambda^{n-1}F \approx L_2{}^pG \otimes \Lambda^{n-1}F$. Consequently the map φ_1 is a monomorphism whose cokernel is isomorphic to

$$\text{Coker }(\varphi_{12}) \oplus \Lambda^{p-1}F \otimes F.$$

Because the split monomorphism $\Lambda^k G^* \to \Lambda^{k+1}G^* \otimes G$ come up so often, it is convenient to have a notation for the cokernel.

Definition. The cokernel of $\Lambda^k G^* \to \Lambda^{k+1}G^* \otimes G$ is denoted by $T_2^{k+1}G^*$.

$T_2^{k+1}G^*$ is a free module isomorphic to $K_2^{n-(k+1)}G^*$. In this notation, we have:

$$\text{Coker } \varphi_1 = T_2^{n-p}G^* \otimes \Lambda^{n-1}F \oplus \Lambda^{p-1}F \otimes F.$$

Taking the cokernels of the maps φ_i, we obtain the complex:

$$0 \to \Lambda^{p+1}F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F \xrightarrow{d_3'} \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes F$$
$$\xrightarrow{d_2'} \Lambda^{p-1}F \otimes F \oplus T_2^{n-p}G^* \otimes \Lambda^{n-1}F \xrightarrow{d_1'} \Lambda^{p-1}F \otimes G \to 0.$$

As we remarked, this complex is acyclic when the map $f: F \to G$ is split. Since the above modules were identified as the cokernels of φ_i thanks to the splitting of certain canonical morphisms, it is necessary to make explicit the maps d_i' induced by the maps d_i.

Let

$$\sigma: \Lambda^{n-p}G^* \otimes \Lambda^n F \otimes G \to \Lambda^{n-p-1}G^* \otimes \Lambda^n F$$

be a map splitting φ_{12}. In case of characteristic zero, this would just be

$$\sigma(\beta \otimes a_1 \otimes a_2) = \frac{1}{p+1} a_2(\beta) \otimes a_1.$$

Straightforward calculations show:

$$d_1'(\overline{\beta \otimes a_1 \otimes a_2}) = \beta(a_1) \otimes a_2 \pm (1 \otimes f)\Delta_1(\sigma(\beta \otimes a_2)(a_1))$$

$$= \beta(a_1) \otimes a_2 \pm \frac{1}{p+1} \sum a_2(\beta)(a_{1j}) \otimes f a_{1j}' \text{ (in char 0)}$$

where $\Delta_1(a_1) = \sum a_{1j} \otimes a_{1j}' \in \Lambda^{n-2}F \otimes F$

$$d_2'(\beta \otimes a_1 \otimes a_2) = \beta(a_1) \otimes a_2 \pm \overline{\beta \otimes a_1 \otimes f a_2} \pm \Delta_1(\sigma(\beta \otimes a_2)(a_1))$$

$$= \beta(a_1) \otimes a_2 \pm \overline{\beta \otimes a_1 \otimes f a_2} \pm \frac{1}{p+1} \sum a_2(\beta)(a_{1j}) \otimes a_{1j}' \quad \text{(in char 0)}.$$

$$d_3'(a_1 \otimes \beta \otimes a_2) = \sum a_{1j}(\beta) \otimes a_2 \otimes a_{1j}'.$$

Our final step is to fill in the empty spaces in $(\mathbf{P}_1)$. That is, we must now find maps u_1, u_2, v_1, v_2 making the following into a double complex:

$$\text{(Q)}\quad \begin{array}{ccccccccc}
 & & 0 \longrightarrow & \Lambda^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F & \longrightarrow & \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F & \longrightarrow & \Lambda^{p-1}F & \longrightarrow 0 \\
 & & & \downarrow u_2 & & \downarrow u_1 & & \downarrow & \\
0 \longrightarrow & \Lambda^{p+1}F \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F & \xrightarrow{d_3'} & \Lambda^{n-p}G^* \otimes \Lambda^{n-1}F \otimes F & \xrightarrow{d_2'} & \begin{array}{c}\Lambda^{p-1}F \otimes F \\ \oplus\, T_2^{n-p}G^* \otimes \Lambda^{n-1}F\end{array} & \xrightarrow{d_1'} & \Lambda^{p-1}F \otimes G & \\
 & & & \downarrow v_1 & & \downarrow v_2 & & \downarrow & \\
 & & 0 \longrightarrow & K_2{}^p F & \xrightarrow{\partial_2} & \Lambda^p F \otimes G & \xrightarrow{\partial_1} & \Lambda^{p+1}G &
\end{array}$$

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We define the maps as follows:

$$\begin{aligned} u_1(\beta \otimes a) &= \overline{\beta \otimes a \otimes b} \pm \Delta_1(\sigma(\beta \otimes b)(a)) \\ &= \overline{\beta \otimes a \otimes b} \pm \frac{1}{p+1} \sum b(\beta)(a_j) \otimes a_j' \quad \text{(char 0)}. \\ u_2(a_1 \otimes \beta \otimes a_2) &= \sum (b \wedge a_{1j})(\beta) \otimes a_2 \otimes a_{1j}'. \\ v_1(a_1 \otimes a_2) &= a_1 \wedge a_2 \otimes b \\ v_1(\beta \otimes a_1 \otimes a_2) &= b(\beta)(a_2) \otimes a_1 \pm \sigma(\beta \otimes a_1)(a_2) \otimes b \\ &\quad \pm \sum b(\sigma(\beta \otimes a_1))(a_{2j}) \otimes fa_{2j}' \\ &= b(\beta)(a_2) \otimes a_1 \pm \frac{1}{p+1} a_1(\beta)(a_2) \otimes b \\ &\quad \pm \sum (b \wedge a_1)(\beta)(a_{2j}) \otimes fa_{2j}' \quad \text{in char. 0.} \\ v_2(\beta \otimes a_1 \otimes a_2) &= b(\beta)(a_1) \otimes a_2 \pm \sum b(\sigma(\beta \otimes a_2))(a_{1j}) \otimes a_{1j}' \\ &= b(\beta)(a_1) \otimes a_2 \pm \frac{1}{p+1} \sum (b \wedge a_2)(\beta)(a_{1j}) \otimes a_{1j}' \\ &\qquad \text{in char. 0.} \end{aligned}$$

The map v_2 is defined with range $\Lambda^p F \otimes F$, and one must verify that the image is indeed contained in $K_2^p F$.

Although it is easy to check that the maps are well-defined, the commutativity of the diagram **Q** has been checked only in characteristic zero, using the particular splitting indicated. Therefore, from now on, we shall assume characteristic zero, although it is to be hoped that the rest of the arguments in this section will hold for arbitrary rings.

All of the above discussion may be summarized in

THEOREM 5.2 *Let R be a commutative ring containing the rational numbers. Then the diagram* **Q** *with the maps defined as above is a double complex. If the map $f: F \to G$ is split, then the rows of* **Q** *are exact and the total complex consequently is acyclic. If R is noetherian, the total complex of* **Q** *is grade sensitive to the ideal, I'_{n-1}, generated by the minors of order $n - 1$ of the map $f' : F \oplus R \to G$, where $f' = f + b$. The total complex of* **Q** *may be described as follows:*

$$\mathbf{Q}_p' : 0 \to \Lambda^{p+1}F' \otimes \Lambda^n G^* \otimes \Lambda^n F' \xrightarrow{\partial_4} \Lambda^{n-p} G^* \otimes \Lambda^n F' \otimes F' \xrightarrow{\partial_3} K_2^p F' \oplus T_2^{n-p} G^* \otimes \Lambda^n F' \xrightarrow{\partial_2} \Lambda^p F' \otimes G \xrightarrow{\partial_1} \Lambda^{p+1} G$$

∂_1 *is the usual map*;

$\partial_2 : K_2^p F' \to \Lambda^p F' \otimes G$ *is the obvious map*;

$\partial_2 : T_2^{n-p} G^* \otimes \Lambda^n F' \to \Lambda^p F' \otimes G$ *is given by*:

$$\partial_2(\overline{\beta \otimes a_1 \otimes a_2}) = \beta(a_2) \otimes a_1 - \frac{1}{p+1} \sum a_1(\beta)(a_{2j}) \otimes f' a_{2j}';$$

$$\partial_3(\beta \otimes a_1 \otimes a_2) = \beta(\alpha_1) \otimes a_2 + \frac{1}{p+1} \sum a_2(\beta)(a_{1j}) \otimes a_{1j}' + \overline{\beta \otimes a_1 \otimes fa_2}$$

$$\partial_4(a_1 \otimes \beta \otimes a_2) = \sum a_{1j}(\beta) \otimes a_2 \otimes a_{1j}'.$$

Here F' indicates a free R-module of rank n (not $n - 1$), and f': $F \to G$ is a map.

All the assertions, but for the description of the total complex $\mathbf{Q}_p'$ have been proven or at least sketched. The final description of $\mathbf{Q}'$ follows from the observations that

$$\Lambda^k F \oplus R \approx \Lambda^k F \oplus \Lambda^{k-1} F$$

and that

$$K_q^p(F \oplus R) \approx K_q^p F \oplus \Lambda^{p-1} F \otimes D_{q-1}(F \oplus R).$$

6. Some partial results and indications. In the preceding section we started with maps f: $F \to G$ and b: $R \to G$, where G has rank n and F has rank $n - 1$, and succeeded in constructing explicit generic minimal resolutions of the cokernels of the maps $\Lambda^p F' \otimes G \to \Lambda^{p+1} G$, where $F' = F \oplus R$ and f': $F \to G$ is the map $f + b$. We did this, in any event, under the assumption that R contained the rationals and we shall continue to make this assumption throughout this section, although we do not know if this is an essential assumption. We will now indicate how we might try to generalize the procedure used in Section 5 to handle the following situation.

Assume that we have maps f: $F \to G$ and b: $R \to G$ where G is of rank n and F is of rank $n - q$. We want to find minimal generic resolutions of the cokernels of the maps $\Lambda^p F' \otimes \Lambda^q G \to \Lambda^{p+q} G$, where $F' = F \oplus R$ and f': $F' \to G$ is the map $j + b$. We would thereby obtain complexes grade sensitive to the ideal $I_{n-q}(f')$ generated by the minors or order $n - q$ of the $(n - q + 1) \times n$ matrix f'.

The map f: $F \to G$ gives us the following complexes:

$$(\mathbf{B}): 0 \to K_{q+1}^p F \to K_q^p F \otimes G \to \ldots \to K_2^p F \otimes \Lambda^{q-1} G \to \Lambda^p F \otimes \Lambda^q G \to \Lambda^{p+q} G \to 0$$

$$(\mathbf{C}): 0 \to K_q^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-q} F \to \ldots \to K_2^p F \otimes \Lambda^{n-q+2} G^* \otimes \Lambda^{n-q} F \to \Lambda^p F \otimes \Lambda^{n-q+1} G^* \otimes \Lambda^{n-q} F \to \Lambda^{n-q-p+1} G^* \otimes \Lambda^{n-q} F \to \Lambda^{p-1} F \to 0$$

$$(\mathbf{D}^r): 0 \to D_r F \to \ldots \to D_2 F \otimes \Lambda^{r-2} G \to F \otimes \Lambda^{r-1} G \to \Lambda^r G \to 0.$$

We shall also consider the complex

$$(\mathbf{B}'): 0 \to K_{q+1}^p F \otimes \Lambda^n G^* \otimes \Lambda^{n-q} F \to K_q^p F \otimes \Lambda^{n-1} G^* \otimes \Lambda^{n-q} F \to \ldots \to \Lambda^p F \otimes \Lambda^{n-q} G^* \otimes \Lambda^{n-q} F \to \Lambda^{n-(p+q)} G^* \otimes \Lambda^{n-q} F$$

which is the complex $(\mathbf{B})$ with $\Lambda^t G$ replaced by $\Lambda^{n-t} G^*$ and tensored with $\Lambda^{n-q} F$.

The complexes $(\mathbf{B})$ and $(\mathbf{B}')$ we know to be acyclic when f: $F \to G$ is a split injection, and $H_i(\mathbf{D}^r) = 0$ for $i > 0$ when f is a split injection. The complex $\mathbf{C}$ is the complex associated to f^*: $G^* \to F^*$, with the term $\Lambda^{n-q-p+1} F^*$ replaced by

$\Lambda^{p-1}F$ (viz section 5), and tensored with $\Lambda^{n-q}F$ to make the maps compatible with this identification. It, too, is acyclic when f is split. We therefore see that the complexes $\mathbf{C} \otimes \mathbf{D}^r$ and $\mathbf{B}' \otimes \mathbf{D}^r$ are acyclic when f is split.

Notice that $\mathbf{D}^r$ has length r, while $\mathbf{C}$ and $\mathbf{B}'$ have length $q + 1$. Thus $\mathbf{C} \otimes \mathbf{D}^r$ is a complex of length $r + q + 1$, and $\mathbf{B}' \otimes \mathbf{D}^{r-1}$ has length $r + q$. We shall relabel the complexes $\mathbf{B}' \otimes \mathbf{D}^{r-1}$ so that they begin with 0 in dimension 0, and therefore $\mathbf{B}' \otimes \mathbf{D}^{r-1}$ will be a complex of length $r + q + 1$. Our next step is to define a map

$$\varphi^r : \mathbf{B}' \otimes \mathbf{D}^{r-1} \rightarrow \mathbf{C} \otimes \mathbf{D}^r.$$

In degree 0, $\varphi_0{}^r$ is the zero map. In degree 1, we need a map

$$\varphi_1{}^r\colon \Lambda^{n-p-q}G^* \otimes \Lambda^{n-q}F \otimes \Lambda^{r-1}G \rightarrow \Lambda^{p-1}F \otimes F \otimes \Lambda^{r-1}G \oplus \Lambda^{n-p-q+1}G^* \otimes \Lambda^{n-q}F \otimes \Lambda^r G$$

whose composition with the map into $\Lambda^{p-1}F \otimes \Lambda^r G$ is zero.

We define $\varphi_1{}^r$ as the direct sum of the following maps:

$$\varphi_{11}{}^r\colon \Lambda^{n-p-q}G^* \otimes \Lambda^{n-q}F \otimes \Lambda^{r-1}G \rightarrow \Lambda^{p-1}F \otimes F \otimes \Lambda^{r-1}G$$
$$\varphi_{12}{}^r\colon \Lambda^{n-p-q}G^* \otimes \Lambda^{n-q}F \otimes \Lambda^{r-1}G \rightarrow \Lambda^{n-p-q+1}G^* \otimes \Lambda^{n-q}F \otimes \Lambda^r G$$

where

$$\varphi_{11}{}^r(\beta \otimes a_1 \otimes a_2) = \sum \beta(a_{1i}) \otimes a_{1i}{}' \otimes a_2$$
$$\varphi_{12}{}^r(\beta \otimes a_1 \otimes a_2) = \pm \sum \beta \wedge \xi_i \otimes a_1 \otimes x_i \wedge a_2.$$

As in § 5, we have denoted by $\sum a_{1i} \otimes a_{1i}{}'$ the image in $\Lambda^{n-q-1}F \otimes F$ of the diagonal of a_1, and $\{x_i\}$, $\{\xi_i\}$ denote a basis and a dual basis of G and G^*. In fact, the map $\varphi_{12}{}^r$ is a formal map which is used repeatedly in the definition of the map φ^r, so we shall digress to make a formal definition.

We have often considered the element $c_G \in G^* \otimes G$, but usually as an element of the algebra $\Lambda G^* \otimes SG$ or $SG^* \otimes \Lambda G$. This time, however, we shall regard c_G as an element of $\Lambda G^* \otimes \Lambda G$, and we therefore have the map

$$c_G\colon \Lambda^k G^* \otimes \Lambda^l G \rightarrow \Lambda^{k+1}G^* \otimes \Lambda^{l+1}G$$

given by multiplication by c_G.

PROPOSITION 6.1. *For all integers k, l, the cokernel of*

$$c_G\colon \Lambda^k G^* \otimes \Lambda^l G \rightarrow \Lambda^{k+1}G^* \otimes \Lambda^{l+1}G$$

is a free module of finite rank.

Proof. The proof is by induction on rank G, the rank 1 case being trivial. If $G = G' \oplus R$, an analysis of the map shows that its cokernel is the sum of the cokernels of the maps:

$$\Lambda^k G'^* \otimes \Lambda^l G' \rightarrow \Lambda^{k+1}G'^* \otimes \Lambda^{l+1}G'$$
$$\Lambda^k G'^* \otimes \Lambda^{l-1} G' \rightarrow \Lambda^{k+1}G'^* \otimes \Lambda^{l}G'$$
$$\Lambda^{k-1} G'^* \otimes \Lambda^l G' \rightarrow \Lambda^{k}G'^* \otimes \Lambda^{l+1}G'.$$

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We see, therefore, that if we let $a(n; k, l)$ be the rank of the cokernel for a free module of rank n, then

$$a(n + 1; k, l) = a(n; k, l) + a(n; k, l - 1) + a(n; k - 1, l).$$

Since

$$a(n; k, 0) = \binom{n+1}{k+1}(n - k - 1),$$

we can calculate the rank in general.

Definition. We denote by $T_2^{k+1,l+1}G^*$ the cokernel of the map

$$c_G: \Lambda^k G^* \otimes \Lambda^l G \to \Lambda^{k+1} G^* \otimes \Lambda^{l+1} G.$$

Notice that $T_2^{k+1,1}G^*$ is the module $T_2^{k+1}G^*$ defined in Section 5.

To define maps $\varphi_{\nu+1}^r: (\mathbf{B} \otimes \mathbf{D}^{r-1})_{\nu+2} \to (\mathbf{C} \otimes \mathbf{D}^r)_{\nu+1}$, we first note that

$$(\mathbf{B}' \otimes \mathbf{D}^{r-1})_{\nu+1} = \sum_{l+k=\nu} K_l^p F \otimes \Lambda^{n-q+l-1} G^* \otimes \Lambda^{n-q} F \otimes D_k F \otimes \Lambda^{r-k-1} G$$
$$\oplus \Lambda^{n-p-q} G^* \otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu-1} G$$

$$(\mathbf{C} \otimes \mathbf{D}^r)_{\nu+1} = \sum_{l+k=\nu} K_l^p F \otimes \Lambda^{n-q+l} G^* \otimes \Lambda^{n-q} F \otimes D_k F \otimes \Lambda^{r-k} G \oplus \Lambda^{n-q-p+1} G^*$$
$$\otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu} G \oplus \Lambda^{p-1} F \otimes D_{\nu+1} F \otimes \Lambda^{r-\nu-1} G.$$

The map $\varphi_{\nu+1}^r$ is defined to be the sum of maps:

$$\Psi_0: \Lambda^{n-p-q} G^* \otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu-1} G \to \Lambda^{n-q-p+1} G^*$$
$$\otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu} G \oplus \Lambda^{p-1} F \otimes D_{\nu+1} F \otimes \Lambda^{r-\nu-1} G$$
$$\Psi_1: \Lambda^p F \otimes \Lambda^{n-q} G^* \otimes \Lambda^{n-q} F \otimes D_{\nu-1} F \otimes \Lambda^{r-\nu} G \to \Lambda^p F \otimes \Lambda^{n-q+1} G^*$$
$$\otimes \Lambda^{n-q} F \otimes D_{\nu-1} F \otimes \Lambda^{r-\nu+1} G \oplus \Lambda^{n-q-p+1} G^* \otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu} G$$

and, for $l > 1$,

$$\Psi_l: K_l^p F \otimes \Lambda^{n-q+l-1} G^* \otimes \Lambda^{n-q} F \otimes D_k F \otimes \Lambda^{r-k-1} G \to$$
$$\to K_l^p F \otimes \Lambda^{n-q+l} G^* \otimes \Lambda^{n-q} F \otimes D_k F \otimes \Lambda^{r-k} G \oplus K_{l-1}^p F \otimes \Lambda^{n-q+l-1} G^*$$
$$\otimes \Lambda^{n-q} F \otimes D_{k+1} F \otimes \Lambda^{r-k-1} G.$$

For Ψ_0, Ψ_1, and Ψ_l, the first component of each map is just the formal map c_G tensored with the appropriate identity. The map

$$\Lambda^{n-p-q} G^* \otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu-1} G \to \Lambda^{p-1} F \otimes D_{\nu+1} F \otimes \Lambda^{r-\nu-1} G$$

is the composite:

$$\Lambda^{n-p-q} G^* \otimes \Lambda^{n-q} F \otimes D_\nu F \otimes \Lambda^{r-\nu-1} G \to \Lambda^{n-p-q} G^* \otimes \Lambda^{n-q-1} F$$
$$\otimes F \otimes D_\nu F \otimes \Lambda^{r-\nu-1} G \to \Lambda^{p-1} F \otimes D_{\nu+1} F \otimes \Lambda^{r-\nu-1} G,$$

where the left hand map is diagonalization of $\Lambda^{n-q} F$ into $\Lambda^{n-q-1} F \otimes F$, and the second map entails the operation of ΛG^* on ΛF as well as multiplication in DF.

The second component of Ψ_1 is similar, in that one diagonalizes $\Lambda^p F$ to $\Lambda^{p-1}F \otimes F$, operates with $\Lambda^{p-1}F$ on $\Lambda^{n-q}G^*$, and multiplies with F on $D_{\nu-1}F$.

The second component of Ψ_l, for $l > 1$, is purely formal again. This time, we regard $\sum K_l{}^pF$ as an SF^*-module, and DF is clearly a DF-module. Then $c_F \in F^* \otimes F \subset SF^* \otimes DF$ and c_F operates on $K_l{}^pF \otimes D_kF$, carrying it into $K_{l-1}{}^pF \otimes D_{k+1}F$. This multiplication by c_F, tensored with the appropriate identity, is the second component of the map Ψ_l.

Having defined the maps $\varphi_\nu{}^r$, it is not difficult to show that we actually get a map of complexes $\varphi_{p-1}{}^r$: $\mathbf{B}' \otimes \mathbf{D}^{r-1} \to \mathbf{C} \otimes \mathbf{D}^r$. The cokernel of φ^r is therefore a complex starting with $\Lambda^{p-1}F \otimes \Lambda^r G$, which we shall denote by $\mathbf{E}^r$. Moreover, the element $b \in G$ which we are given along with the map f: $F \to G$, defines maps of $\mathbf{D}^k$ into $\mathbf{D}^{k+1}$. Consequently we have maps

$$\mathbf{B}' \otimes \mathbf{D}^{r-1} \to \mathbf{B}' \otimes \mathbf{D}^r \quad \text{and} \quad \mathbf{C} \otimes \mathbf{D}^r \to \mathbf{C} \otimes \mathbf{D}^{r+1}$$

which commute with the maps φ^r. These maps therefore induce maps $\mathbf{E}^{r-1} \to \mathbf{E}^r$ and, since $\mathbf{E}^0 = \mathbf{C}$, we get a double complex:

$$0 \to \mathbf{C} \to \mathbf{E}^1 \to \ldots \to \mathbf{E}^q.$$

It is also possible to define a map $\mathbf{E}^q \to \mathbf{B}$ such that

$$0 \to \mathbf{C} \to \mathbf{E}^1 \to \ldots \to \mathbf{E}^q \to \mathbf{B}$$

is a double complex.

When we take $q = 2$ and $p = n - 2$, the morphisms φ^r are monomorphisms for $r = 1, 2$, so that the complexes $\mathbf{E}^r$ are also acyclic when f is a split monomorphism. In this case, the double complex

$$0 \to C \to \mathbf{E}^1 \to \mathbf{E}$$

looks like this:

$$\begin{array}{llllllllll}
\mathbf{C}: & 0 & \to & K_2^{n-2}F \otimes \Lambda^n G^* \otimes \Lambda^{n-2}F & \to & \Lambda^{n-2}F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-2}F & \to & G^* \otimes \Lambda^{n-2}F & \to & \Lambda^{n-3}F \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{E}^1: 0 \to & \Lambda^{n-2}F \otimes \Lambda^n G^* \otimes \Lambda^{n-2}F \otimes \Lambda^2 F & \to & \Lambda^{n-2}F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-2}F \otimes F & \to & G^* \otimes \Lambda^{n-2}F \otimes F & \to & \Lambda^{n-3}F \otimes F & \to & \Lambda^{n-3}F \otimes G \\
 & & & & & \oplus\ \Lambda^{n-2}F \otimes T_2^{n-1}G^* \otimes \Lambda^{n-2}F & & \oplus\ T_2{}^1G^* \otimes \Lambda^{n-2}F & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{E}^2: & \Lambda^{n-2}F \otimes \Lambda^{n-1}G^* \otimes \Lambda^{n-2}F \otimes D_2F & \to & G^* \otimes \Lambda^{n-2}F \otimes D_2F & \to & \Lambda^{n-3}F \otimes D_2F & \to & \Lambda^{n-3}F \otimes F \otimes G & \to & \Lambda^{n-3}F \otimes \Lambda^2 G \\
 & & & \oplus\ \Lambda^{n-2}F \otimes T_2^{n-1}G^* \otimes \Lambda^{n-2}F \otimes F & & \oplus\ T_2{}^1G^* \otimes \Lambda^{n-2}F \otimes F & & \oplus\ T_2^{12}G^* \otimes \Lambda^{n-2}F & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{B}: & 0 & \to & K_3^{n-2}F & \to & K_2^{n-2}F \otimes G & \to & \Lambda^{n-2}F \otimes \Lambda^2 G & \to & \Lambda^n G
\end{array}$$

Letting $F' = F \oplus R$, the total complex then becomes:

$$\begin{aligned}
0 \to \Lambda^{n-1}F' \otimes \Lambda^2 F' \otimes \Lambda^n G^* \otimes \Lambda^{n-1}F' \to \Lambda^{n-1}F' \otimes \Lambda^{n-1}G^* \\
\otimes \Lambda^{n-1}F' \otimes D_2F' \to \Lambda^{n-1}F' \otimes T_2^{n-1}G^* \otimes \Lambda^{n-1}F' \otimes F' \otimes G^* \oplus \Lambda^{n-1}F' \\
\otimes D_2F' \to K_3^{n-2}F' \oplus T_2{}^1G^* \otimes \Lambda^{n-1}F' \otimes F' \to K_2^{n-2}F' \otimes G \oplus T_2^{12}G^* \\
\otimes \Lambda^{n-1}F' \to \Lambda^{n-2}F' \otimes \Lambda^2 G \to \Lambda^n G.
\end{aligned}$$

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This complex, then, does give a resolution (in characteristic zero) of the ideal of $(n-2) \times (n-2)$ minors of an $(n-1) \times n$ matrix. We were able to write it down because we could explicitly calculate the terms of $\mathbf{E}^r$. The generic acyclicity of the complex results from the acyclicity of $\mathbf{E}^r$, and in general we have not been able to prove this acyclicity for arbitrary q and p. Clearly, more has to be understood about the maps c_F and c_G which are basic to the definition of the maps φ^r. Because the elements c_F and c_G are not nilpotent, their cokernels don't seem to fit naturally into long exact sequences. We hope to investigate all these matters further in a later paper.

Although interest generally focuses on the ideal of $(n-q) \times (n-q)$ minors of a map $f: F \to G$, hence on the cokernel of $\Lambda^{n-q}F \otimes \Lambda^q G \to \Lambda^n G$, it is probably worthwhile to look at all the maps $\Lambda^p F \otimes \Lambda^q G \to \Lambda^{p+q} G$; the supports of all these cokernels (for fixed q) are the same. Moreover, these maps show up in the following context.

In [**1**], we showed that if

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ F & \xrightarrow{f} & G \end{array}$$

is a commutative diagram with rank $F = m$, rank $G = n$, $m \geqq n$, then we had a double complex:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & & \Lambda^{n+1}F & \longrightarrow & F & \xrightarrow{f} & G \\
& & \downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
\cdots \longrightarrow & \Lambda^{n-1}G^* & \otimes \Lambda^{n+1}F & \longrightarrow & \Lambda^2 F & \xrightarrow{\Lambda^2 f} & \Lambda^2 G \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & G^* & \otimes \Lambda^{n+1}F & \longrightarrow & \Lambda^n F & \xrightarrow{\Lambda^n f} & \Lambda^n G
\end{array}$$

whose rows are the complexes associated to the maps $\Lambda^p f: \Lambda^p F \to \Lambda^p G$. The total complex is grade sensitive to the ideal generated by the $n \times n$ minors of f and by b^*. When $R = k[X_1, \ldots, X_m]$ and $b(1) = (F_1, \ldots, F_n)$ where F_i are forms generating a complete intersection, we may choose f to be the Jacobian matrix $(\partial F_i/\partial X_j)$. In characteristic zero, we therefore get a complex grade sensitive to the singular locus of the complete intersection $(F_1, \ldots, F_n)$.

Suppose, now, that $(F_1, \ldots, F_n)$ generate a variety of codimension $n - q$. Then the singular locus of $(F_1, \ldots, F_n)$ is generated by $F_1, \ldots, F_n$ together with the minors of $\partial F_i/\partial X_j$ of order $n - q$. In analogy with the case of complete intersections (where $q = 0$), we consider:

$$\begin{array}{ccc}
F \otimes \Lambda^q G & \rightarrow & \Lambda^{q+1} G \\
{\scriptstyle a \otimes 1} \downarrow & & \downarrow {\scriptstyle b} \\
\Lambda^2 F \otimes \Lambda^q G & \rightarrow & \Lambda^{q+2} G \\
 & & \downarrow \\
 & & \cdot \\
 & & \cdot \\
 & & \cdot \\
 & & \downarrow \\
\Lambda^{n-q} F \otimes \Lambda^q F & \rightarrow & \Lambda^n G
\end{array}$$

and we would like to find an extension of this diagram to obtain a double complex grade sensitive to the singular locus of $(F_1, \ldots, F_n)$. In this case, we are looking for complexes over the cokernels of the maps

$$\Lambda^p F \otimes \Lambda^q G \rightarrow \Lambda^{p+q} G.$$

Here, we are not treating the case of a generic matrix, for we are assuming that the minors of order $n - q + 1$ of the Jacobian are contained in the ideal generated by $(F_1, \ldots, F_n)$. Nevertheless, the interest in the above maps for all p persists.

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