



A Morita Cancellation Problem

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Abstract. We study a Morita-equivalent version of the Zariski cancellation problem.

1 Introduction

An algebra A is called *cancellative* if any algebra isomorphism $A[t] \cong B[t]$ of polynomial algebras for some algebra B implies that A is isomorphic to B . The famous Zariski Cancellation Problem (ZCP) asks

Is the commutative polynomial ring $\mathbb{k}[x_1, \dots, x_n]$ over a field \mathbb{k} cancellative for $n \geq 1$?

See [Kr, BZ1, Gu3]. There is a long history of studying the cancellation property of affine commutative domains. For example, $\mathbb{k}[x_1]$ is cancellative by a result of Abhyankar, Eakin, and Heinzer in 1972 [AEH], while $\mathbb{k}[x_1, x_2]$ is cancellative by a result of Fujita in 1979 [Fu] and Miyanishi and Sugie in 1980 [MS] in characteristic zero, and by a result of Russell in 1981 [Ru] in positive characteristic. The ZCP for $n \geq 3$ has been open for many years. One remarkable achievement in this research area is a result of Gupta in 2014 [Gu1, Gu2], which settled the ZCP negatively in positive characteristic for $n \geq 3$. The ZCP in characteristic zero remains open for $n \geq 3$.

The ZCP (especially in dimension two) is closely related to the Automorphism Problem, the Characterization Problem, the Linearization Problem, the Embedding Problem, and the Jacobian Conjecture; see [Kr, EH, Gu3, BZ1] for history, partial results and references concerning the cancellation problem.

The ZCP for noncommutative algebras was introduced in [BZ1] and further investigated in [LWZ]. During the last few years, several researchers have been making significant contributions to the cancellation problem in the noncommutative setting and related topics; see, for example, [BZ1, BZ2, BY, CPWZ1, CPWZ2, CYZ1, CYZ2, Ga, GKM, GWY, LY, LWZ, LMZ, NTY, Ta1, Ta2, WZ].

The first goal of this paper is the introduction of a new cancellation property for noncommutative algebras. Let \mathbb{k} be a base field; in the sequel, everything is over \mathbb{k} . For any algebra A , let $M(A)$ denote the category of right A -modules.

Received by the editors May 8, 2018; revised January 3, 2019.

Published online on Cambridge Core January 29, 2019.

D.-M. Lu was partially supported by the National Natural Science Foundation of China (Grant No. 11671351). Q.-S. Wu was partially supported by the National Natural Science Foundation of China (Grant No. 11771085 and Key Project No. 11331006). J.J. Zhang was partially supported by the US National Science Foundation (Nos. DMS-1402863 and DMS-1700825).

AMS subject classification: 16P99, 16W99.

Keywords: Zariski cancellation problem, Morita cancellation problem.

Definition 1.1 An algebra A is called *Morita cancellative* if for any algebra B ,

$$M(A[t]) \text{ is equivalent to } M(B[t])$$

implies that

$$M(A) \text{ is equivalent to } M(B).$$

This Morita version of the cancellation property is one of the natural generalizations of the original Zariski cancellation property when we study noncommutative algebras. Another generalization involves the derived category of modules. Let $D(A)$ denote the derived category of right A -modules for an algebra A .

Definition 1.2 An algebra A is called *derived cancellative* if for any algebra B ,

$$D(A[t]) \text{ is triangulated equivalent to } D(B[t])$$

implies that

$$D(A) \text{ is triangulated equivalent to } D(B).$$

We will show [Theorem 1.7] that if Z is a commutative domain, then

$$Z \text{ is Morita cancellative if and only if } Z \text{ is cancellative}$$

and

$$Z \text{ is derived cancellative if and only if } Z \text{ is cancellative.}$$

In general, when A is noncommutative, the relationships between these three different versions of cancellation property are not clear. Lemma 2.4 (together with Example 2.5) provides noncommutative algebras that are neither cancellative, nor Morita cancellative, nor derived cancellative. We will introduce some general methods to handle the Morita cancellation problems for noncommutative algebras.

The second aim of the paper is to show several classes of algebras are Morita (or derived) cancellative. First, we generalize a result of [LWZ, Theorem 0.2].

Theorem 1.3 Suppose A is strongly Hopfian (Definition 4.2) and the center of A is artinian. Then A is Morita cancellative.

Note that left (or right) noetherian algebras and locally finite \mathbb{N} -graded algebras are strongly Hopfian [Example 4.5]. So Theorem 1.3 covers a large class of algebras. The following are consequences of the above theorem; see also [LWZ, Corollary 0.3 and Theorem 0.4] for comparison.

Theorem 1.4 Let A be a left (or right) noetherian algebra such that its center is artinian. Then A is Morita cancellative. As a consequence, every finite dimensional algebra over a base field \mathbb{k} is Morita cancellative.

For non-noetherian algebras we have the following theorem.

Theorem 1.5 For every finite quiver Q , the path algebra $\mathbb{k}Q$ is Morita cancellative.

Recall from [BZ1, Theorem 0.5] that, if A is an affine domain of GK-dimension two over an algebraically closed field of characteristic zero and A is not commutative, then A is cancellative. It is well-known that, in contrast, noncommutative affine prime (non-domain) algebras of GK-dimension two need not be cancellative [LWZ, Example 1.3(5)] and that commutative affine domains of GK-dimension two need not be cancellative, by examples of Hochster [Ho] and Danielewski [Da]; see Example 2.5(i) and (ii). For GK-dimension one, a classical result of Abhyankar, Eakin, and Heinzer [AEH, Theorem 3.3] says that every affine commutative domain of GK-dimension one is cancellative. Recently, it was proved that every affine prime \mathbb{k} -algebra of GK-dimension one is cancellative. Next we add another result in low GK-dimension.

Theorem 1.6 *Let \mathbb{k} be algebraically closed. Then every affine prime \mathbb{k} -algebra of GK-dimension one is Morita cancellative.*

We are mainly dealing with the Morita cancellation property in this paper, but occasionally, we have some results concerning the derived cancellation property, such as the next result.

Theorem 1.7 (Corollary 8.2) *Let Z be a commutative domain. Then Z is cancellative if and only if Z is Morita cancellative, if and only if Z is derived cancellative.*

A question in [LWZ, Question 5.4(3)] asks if the Sklyanin algebras are cancellative. We partially answer this question.

Corollary 1.8 (Example 6.10(2)) *Let A be a non-PI Sklyanin algebra of global dimension three. Then A is both cancellative and Morita cancellative.*

The paper is organized as follows. Section 2 contains definitions, known examples, and preliminaries. In Sections 3 and 4, we introduce the Morita version of the retractable and detectable properties. In Section 5, we prove Theorems 1.3 and 1.4. Theorems 1.6 and 1.7 are proved in Section 6 and Section 7, respectively. The derived cancellation property is briefly studied in Section 8. Section 8 also contains some comments, remarks, and examples.

2 Definitions and Preliminaries

Some definitions and examples are copied from [BZ1, LWZ]. First, we recall a classical definition. Let $A[t]$ (or $A[s]$) be the polynomial algebra over A by adding one central indeterminate.

Definition 2.1 Let A be an algebra.

(i) We call A *cancellative* if any algebra isomorphism $A[t] \cong B[s]$ implies that $A \cong B$.

(ii) We call A *strongly cancellative* if, for each $n \geq 1$, any algebra isomorphism

$$A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

implies that $A \cong B$.

The following are two new cancellation properties that we will study in this paper.

Definition 2.2 Let A be an algebra.

(i) We call A *m-cancellative* if any equivalence of abelian categories $M(A[t]) \cong M(B[s])$ implies that $M(A) \cong M(B)$.

(ii) We call A *strongly m-cancellative* if, for each $n \geq 1$, any equivalence of abelian categories

$$M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$$

implies that $M(A) \cong M(B)$.

The letter m here stands for the word “Morita”.

Definition 2.3 Let A be an algebra.

(i) We call A *d-cancellative* if any equivalence of triangulated categories

$$D(A[t]) \cong D(B[s])$$

implies that $D(A) \cong D(B)$.

(ii) We call A *strongly d-cancellative* if, for each $n \geq 1$, any equivalence of triangulated categories

$$D(A[t_1, \dots, t_n]) \cong D(B[s_1, \dots, s_n])$$

implies that $D(A) \cong D(B)$.

The letter d here stands for the word “derived”.

Let $A[t]$ denote the polynomial algebra $A[t_1, \dots, t_n]$ and $A[s]$ the polynomial algebra $A[s_1, \dots, s_n]$ for an integer n (that is not specified) when no confusion occurs.

Lemma 2.4 Let A be a commutative algebra that is not (strongly) cancellative. Let B be an algebra with center $Z(B) = \mathbb{k}$. Then $A \otimes B$ is neither (strongly) cancellative, nor (strongly) m -cancellative, nor (strongly) d -cancellative.

Proof Since A is not (strongly) cancellative, there is a commutative algebra C such that A is not isomorphic to C , but $A[t_1, \dots, t_n] \cong C[s_1, \dots, s_n]$ for $n = 1$ (or some $n \geq 1$). Then $A \otimes B[t] \cong C \otimes B[s]$. As a consequence, we obtain that

$$M(A \otimes B[t]) \cong M(C \otimes B[s]) \quad \text{and} \quad D(A \otimes B[t]) \cong D(C \otimes B[s]).$$

Since the center $Z(A \otimes B) = A$ is not isomorphic to $Z(C \otimes B) = C$, we obtain that $M(A \otimes B) \not\cong M(C \otimes B)$ and that $D(A \otimes B) \not\cong D(C \otimes B)$. Therefore, the assertions follow. ■

Next we give some precise examples of non-cancellative commutative algebras. The above lemma gives an easy way of producing non-cancellative noncommutative algebras.

Example 2.5 (i) Let \mathbb{k} be the field of real numbers \mathbb{R} . Hochster showed that $\mathbb{k}[P, Q, X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ is not cancellative [Ho].

(ii) The following example is due to Danielewski [Da]. Let $n \geq 1$ and let B_n be the coordinate ring of the surface $x^n y = z^2 - 1$ over $\mathbb{k} := \mathbb{C}$. Then $B_i \not\cong B_j$ if $i \neq j$,

but $B_i[t] \cong B_j[s]$ for all $i, j \geq 1$; see [Fi, Wi] for more details. Therefore, all the B_n 's are not cancellative.

(iii) Suppose $\text{char } \mathbb{k} > 0$. Gupta showed that $\mathbb{k}[x_1, \dots, x_n]$ is not cancellative for every $n \geq 3$ [Gu1, Gu2].

As a consequence of Lemma 2.4 (by taking $B = \mathbb{k}$), the algebras above are neither m-cancellative nor d-cancellative.

We also need to recall higher derivations and Makar–Limanov invariants.

Definition 2.6 Let A be an algebra.

(i) [HS] A *higher derivation* (or *Hasse–Schmidt derivation*) on A is a sequence of \mathbb{k} -linear endomorphisms $\partial := \{\partial_i\}_{i=0}^\infty$ such that:

$$\partial_0 = \text{id}_A \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^n \partial_i(a) \partial_{n-i}(b)$$

for all $a, b \in A$ and all $n \geq 0$. The collection of all higher derivations on A is denoted by $\text{Der}^H(A)$.

(ii) A higher derivation is called *locally nilpotent* if

- (a) given any $a \in A$ there exists $n \geq 1$ such that $\partial_i(a) = 0$ for all $i \geq n$,
- (b) the map

$$G_{\partial, t} : A[t] \longrightarrow A[t]$$

defined by

$$a \longmapsto \sum_{i=0}^{\infty} \partial_i(a) t^i \text{ for all } a \in A \text{ and } t \longmapsto t$$

is an algebra automorphism of $A[t]$.

(iii) For any $\partial \in \text{Der}^H(A)$, the kernel of ∂ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$

(iv) The set of locally nilpotent higher derivations is denoted by $\text{LND}^H(A)$. Given a nonzero element $d \in A$, let

$$\text{LND}_d^H(A) = \{\partial \in \text{LND}^H(A) \mid d \in \ker \partial\}.$$

Note that (a) in part (ii) of the above definition implies that the map $G_{\partial, t}$ defined in (b) is an algebra endomorphism. It is not clear to us whether $G_{\partial, t}$ is automatically an automorphism. However, by [BZ1, Lemma 2.2(2)], when ∂ is an iterative higher derivation, $G_{\partial, t}$ is automatically an automorphism.

It is easy to see that $1 \in \ker \partial$ for all higher derivations ∂ . Hence, $\text{LND}_1^H(A) = \text{LND}^H(A)$. We generalize the original definition of the Makar–Limanov invariant [Mak].

Definition 2.7 Let A be an algebra and d a nonzero element in A .

(i) The *Makar–Limanov $_d^H$ invariant* of A is defined to be

$$(E1.7.1) \quad ML_d^H(A) := \bigcap_{\delta \in \text{LND}_d^H(A)} \ker(\delta).$$

- (ii) We say that A is LND_d^H -rigid if $ML_d^H(A) = A$.
- (iii) A is called *strongly* LND_d^H -rigid if $ML_d^H(A[t_1, \dots, t_n]) = A$, for all $n \geq 1$.
- (iv) The *Makar–Limanov* center of A is defined to be

$$ML_{d,Z}^H(A) = ML_d^H(A) \cap Z(A).$$

- (v) A is called *strongly* $\text{LND}_{d,Z}^H$ -rigid if $ML_{d,Z}^H(A[t_1, \dots, t_n]) = Z(A)$, for all $n \geq 1$.

3 Morita Invariant Properties and the \mathcal{P} -discriminant

In this section we will recall some well-known facts about Morita equivalence. Two algebras A and B are Morita equivalent if their right module categories $M(A)$ and $M(B)$ are equivalent. We list some properties concerning Morita theory.

Lemma 3.1 ([AF, Ch. 6]) *Let A and B be two algebras that are Morita equivalent.*

- (i) *There is an (A, B) -bimodule Ω that is invertible, namely, $\Omega \otimes_B \Omega^\vee \cong A$ and $\Omega^\vee \otimes_A \Omega \cong B$ as bimodules, where $\Omega^\vee := \text{Hom}_B(\Omega_B, B_B)$.*
- (ii) *The bimodule Ω induces naturally algebra isomorphisms $A \cong \text{End}(\Omega_B)$ and $B^{\text{op}} \cong \text{End}({}_A\Omega)$.*
- (iii) *Further, $Z(A) \cong \text{Hom}_{(A,B)}(\Omega, \Omega) \cong Z(B)$, which induces an isomorphism*

$$(E2.1.1) \quad \omega : Z(A) \longrightarrow Z(B)$$

such that, for each $x \in Z(A)$, the left multiplication of x on Ω equals the right multiplication of $\omega(x)$ on Ω .

- (iv) *By using ω to identify the center $Z = Z(A)$ of A with the center of B , both A and B are central Z -algebras. In this case, both Ω and Ω^\vee are central Z -modules.*

- (v) *Let ω be given as in (E2.1.1). Then, for any ideal I of $Z(A)$, A/IA and $B/\omega(I)B$ are Morita equivalent.*

- (vi) [AF, Ex.9, p.267] *Let A, B, T be K -algebra for some commutative ring K . Then $A \otimes_K T$ and $B \otimes_K T$ are Morita equivalent.*

Morita equivalences have been studied extensively for decades. A ring theoretic property is called a *Morita invariant* if it is preserved by Morita equivalences.

Example 3.2 The following properties are Morita invariants:

- (i) being simple (resp., semisimple);
- (ii) being right (or left) noetherian, right (or left) artinian;
- (iii) having global dimension d (Krull dimension d , GK-dimension d , etc);
- (iv) being a full matrix algebra $M_n(\mathbb{k})$ for some n , when \mathbb{k} is algebraically closed;
- (v) being an Azumaya algebra [Sc, Theorem 4];
- (vi) being quasi-Frobenius;
- (vii) being prime, semiprime, right (or left) primitive, semiprimitive;
- (viii) being semilocal;
- (ix) being primitive, but not simple;
- (x) being noetherian, but not artinian;
- (xi) the center being \mathbb{k} ;
- (xii) being projective over its center.

Let R be a commutative algebra, $\text{Spec } R$ be the prime spectrum of R and $\text{MaxSpec}(R) := \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } R\}$ be the maximal spectrum of R . For any $S \subseteq \text{Spec } R$, $I(S)$ is the ideal of R vanishing on S , namely,

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

For any algebra A , A^\times denotes the set of invertible elements in A .

A property \mathcal{P} considered in the following means a property defined on a class of algebras that is an invariant under algebra isomorphisms.

Definition 3.3 Let A be an algebra, $Z = Z(A)$ be the center of A . Let \mathcal{P} be a property defined for \mathbb{k} -algebras (not necessarily a Morita invariant).

(i) The \mathcal{P} -locus of A is defined to be

$$L_{\mathcal{P}}(A) := \{\mathfrak{m} \in \text{MaxSpec}(Z) \mid A/\mathfrak{m}A \text{ has property } \mathcal{P}\}.$$

(ii) The \mathcal{P} -discriminant set of A is defined to be

$$D_{\mathcal{P}}(A) := \text{MaxSpec}(Z) \setminus L_{\mathcal{P}}(A).$$

(iii) The \mathcal{P} -discriminant ideal of A is defined to be

$$I_{\mathcal{P}}(A) := I(D_{\mathcal{P}}(A)) \subseteq Z.$$

(iv) If $I_{\mathcal{P}}(A)$ is a principal ideal of Z generated by $d \in Z$, then d is called the \mathcal{P} -discriminant of A , denoted by $d_{\mathcal{P}}(A)$. In this case $d_{\mathcal{P}}(A)$ is unique up to an element in Z^\times .

(v) Let \mathcal{C} be a class of algebras over \mathbb{k} . We say that \mathcal{P} is \mathcal{C} -stable if for every algebra A in \mathcal{C} and every $n \geq 1$,

$$I_{\mathcal{P}}(A \otimes \mathbb{k}[t_1, \dots, t_n]) = I_{\mathcal{P}}(A) \otimes \mathbb{k}[t_1, \dots, t_n]$$

as an ideal of $Z \otimes \mathbb{k}[t_1, \dots, t_n]$. If \mathcal{C} is a singleton $\{A\}$, we simply call \mathcal{P} A -stable. If \mathcal{C} is the whole collection of \mathbb{k} -algebras with the center affine over \mathbb{k} , we simply call \mathcal{P} stable.

In general, neither $L_{\mathcal{P}}(A)$ nor $D_{\mathcal{P}}(A)$ is a subscheme of $\text{Spec } Z(A)$.

Example 3.4 Suppose $\mathbb{k} = \mathbb{C}$. Let A be the universal enveloping algebra of the simple Lie algebra sl_2 . It is well known that $Z(A) = \mathbb{k}[Q]$, where $Q = 2(e f + f e) + h^2$.

Let \mathcal{S} be the property of being simple. Then $D_{\mathcal{S}}(A)$ is the set of integer points of the form $\{n^2 + 2n \mid n \in \mathbb{N}\}$ inside the $\text{MaxSpec } \mathbb{k}[Q]$; see [Di] or [Sm, p. 98]. In this case, the \mathcal{S} -discriminant ideal of A is the zero ideal of $\mathbb{k}[Q]$ and the \mathcal{S} -discriminant of A is the element $0 \in \mathbb{k}[Q]$.

Note from [Di] or [Sm, p. 98] that for each $c = n^2 + 2n$, $A/(Q - c)A$ has a unique proper two-sided ideal M_c and M_c is of codimension $(n+1)^2$. Let \mathcal{P}_n be the property of not having a factor ring isomorphic to the matrix algebra $M_{n+1}(\mathbb{k})$. Then $D_{\mathcal{P}_n}(A)$ is the singleton $\{n^2 + 2n\}$, as a subset of $D_{\mathcal{S}}(A)$. As a consequence, the \mathcal{P}_n -discriminant ideal of A is $(Q - (n^2 + 2n)) \subseteq \mathbb{k}[Q]$ and the \mathcal{P}_n -discriminant of A is the element $Q - (n^2 + 2n) \in \mathbb{k}[Q]$.

It is clear that \mathcal{S} is a Morita invariant, but \mathcal{P}_n is not for each fixed n .

Lemma 3.5 Let \mathcal{P} be a property.

(i) Suppose $\phi : A \rightarrow B$ is an isomorphism. Then ϕ preserves the following:

- (a) \mathcal{P} -locus;
- (b) \mathcal{P} -discriminant set;
- (c) \mathcal{P} -discriminant ideal;
- (d) \mathcal{P} -discriminant (if it exists).

(ii) Suppose that \mathcal{P} is a Morita invariant and that A and B are Morita equivalent. Then the algebra map ω in (E2.1.1) preserves the following:

- (a) \mathcal{P} -locus;
- (b) \mathcal{P} -discriminant set;
- (c) \mathcal{P} -discriminant ideal;
- (d) \mathcal{P} -discriminant (if it exists).

Proof (i) This is clear.

(ii) This follows from the definition, Lemma 3.1(v) and the hypothesis that \mathcal{P} is a Morita invariant. ■

In this and the next sections we study two properties that are closely related to the m-cancellative property. The retractable property was introduced in [LWZ, Definitions 2.1 and 2.5]. Next we generalize Z -retractability to the Morita setting.

Definition 3.6 Let A be an algebra.

(i) [LWZ, Definition 2.5(1)] We call A Z -retractable, if for any algebra B , an algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $\phi(Z(A)) = Z(B)$.

(ii) [LWZ, Definition 2.5(2)] We call A *strongly* Z -retractable, if for any algebra B and integer $n \geq 1$, an algebra isomorphism $\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$ implies that $\phi(Z(A)) = Z(B)$.

(iii) We call A m - Z -retractable if, for any algebra B , an equivalence of categories $M(A[t]) \cong M(B[s])$ implies that $\omega(Z(A)) = Z(B)$, where $\omega : Z(A)[t] \rightarrow Z(B)[s]$ is given as in (E2.1.1).

(iv) We call A *strongly* m - Z -retractable if, for any algebra B and $n \geq 1$, an equivalence of categories $M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$ implies that $\omega(Z(A)) = Z(B)$, where $\omega : Z(A)[t_1, \dots, t_n] \rightarrow Z(B)[s_1, \dots, s_n]$ is given as in (E2.1.1).

The following proposition is similar to [LWZ, Lemma 2.6].

Proposition 3.7 Let A be an algebra whose center $Z := Z(A)$ is an affine domain. Let \mathcal{P} be a stable Morita invariant property (resp., stable property) and assume that the \mathcal{P} -discriminant of A , denoted by d , exists.

(i) Suppose $ML_d^H(Z[t]) = Z$. Then A is m - Z -retractable (resp., Z -retractable).

(ii) Suppose that Z is strongly LND_d^H -rigid. Then A is strongly m - Z -retractable (resp., strongly Z -retractable).

Proof The proofs of (i) and (ii) are similar, so we prove only (ii). We only work on the strongly m - Z -retractable version; the strongly Z -retractable version is similar.

Suppose that $A[t_1, \dots, t_n]$ is Morita equivalent to $B[s_1, \dots, s_n]$ for some algebra B and for some $n \geq 1$. Let $\omega : Z \otimes \mathbb{k}[t] \rightarrow Z(B) \otimes \mathbb{k}[s]$ be the map given in (E2.1.1). Since \mathcal{P} is stable, $d_{\mathcal{P}}(A[t]) = d \otimes 1$, where 1 is the identity element of the polynomial ring $\mathbb{k}[t]$. In other words, the principal ideal $(d \otimes 1)$ is the \mathcal{P} -discriminant ideal of $A[t]$. Since ω preserves the discriminant ideal [Lemma 3.5(2c)] and \mathcal{P} is stable, we obtain that

$$(E2.7.1) \quad \omega((d \otimes 1)) = \omega((d) \otimes \mathbb{k}[t]) = \omega(I_{\mathcal{P}}(A[t])) = I_{\mathcal{P}}(B[s]) = I_{\mathcal{P}}(B) \otimes \mathbb{k}[s].$$

As a consequence, $I_{\mathcal{P}}(B)$ is a principal ideal, denoted by (d') , where d' is the \mathcal{P} -discriminant of B . Equation (E2.7.1) implies that

$$\omega(d \otimes 1) =_{Z(B[s])^{\times}} d' \otimes 1',$$

where $1'$ is the identity element of the polynomial ring $\mathbb{k}[s]$. Since $Z(B)$ is a domain, $Z(B[s])^{\times} = Z(B)^{\times}$. Hence ω maps d to d' up to a scalar in $Z(B)^{\times}$.

Now consider the map $\omega : Z \otimes \mathbb{k}[t] \rightarrow Z(B) \otimes \mathbb{k}[s]$ again. Since ω maps d to d' , by the strongly LND_d^H -rigidity of Z , we have

$$\omega(Z) = \omega(ML_d^H(Z \otimes \mathbb{k}[t])) = ML_{d'}^H(Z(B) \otimes \mathbb{k}[s]) \subseteq Z(B),$$

where the last \subseteq follows from the computation given in [BZ1, Example 2.4]. This means that the isomorphism ω induces an algebra map from Z to $Z(B)$. Let Z' be the subalgebra $\omega^{-1}(Z(B)) \subset Z[t]$. Then Z' contains Z , which is considered as the degree zero part of the algebra $Z[t]$, and we have

$$\begin{aligned} \text{GKdim } Z' &= \text{GKdim } Z(B) = \text{GKdim } Z(B)[s] - n = \text{GKdim } Z[t] - n \\ &= \text{GKdim } Z. \end{aligned}$$

By [BZ1, Lemma 3.2], $Z' = Z$. Therefore, $\omega(Z) = Z(B)$ as required. \blacksquare

The rest of this section follows closely [LWZ, Section 2]. By [BZ1, Section 5], effectiveness (and the dominating property) of the discriminant controls LND^H -rigidity. We now recall the definition of the effectiveness of an element. An algebra is called *PI* if it satisfies a polynomial identity.

Next we will use filtered algebras and associated graded algebras; see [YZ2, Section 1] for more details. By a filtration of a \mathbb{k} -algebra A , we mean an ascending filtration $F := \{F_i A\}_{i \geq 0}$ of vector spaces such that $1 \in F_0 A$ and $F_i A F_j A \subseteq F_{i+j} A$ for all $i, j \geq 0$. We assume that F is (separated and) exhaustive. By [YZ2, Lemma 1.1], giving a filtration on an algebra A is equivalent to giving a degree on the set of generators of A .

Definition 3.8 ([BZ1, Definition 5.1]) Let A be a domain and suppose that $Y = \bigoplus_{i=1}^n \mathbb{k} x_i$ generates A as an algebra. An element $0 \neq f \in A$ is called *effective* if the following conditions hold.

(i) There is an \mathbb{N} -filtration $\{F_i A\}_{i \geq 0}$ on A such that the associated graded ring $\text{gr } A$ is a domain (one possible filtration is the trivial filtration $F_0 A = A$). With this filtration we define the degree of elements in A , denoted by \deg_A .

(ii) For every testing \mathbb{N} -filtered PI algebra T with $\text{gr } T$ being an \mathbb{N} -graded domain and for every testing subset $\{y_1, \dots, y_n\} \subset T$ satisfying the following:

(a) it is linearly independent in the quotient \mathbb{k} -module $T/\mathbb{k}1_T$, and
 (b) $\deg_T y_i \geq \deg_A x_i$ for all i and $\deg_T y_{i_0} > \deg_A x_{i_0}$ for some i_0 ,
 then there is a presentation of f of the form $f(x_1, \dots, x_n)$ in the free algebra $\mathbb{k}\langle x_1, \dots, x_n \rangle$, such that either

$$f(y_1, \dots, y_n) = 0 \quad \text{or} \quad \deg_T f(y_1, \dots, y_n) > \deg_A f.$$

Here is an easy example.

Example 3.9 ([LWZ, Example 2.8]) Every non-invertible nonzero element in $\mathbb{k}[t]$ is effective in $\mathbb{k}[t]$.

Other examples of effective elements are given in [BZ1, Section 5]. There is another concept, called “dominating”; see [BZ1, Definition 4.5] or [CPWZ1, Definition 2.1(2)], that is similar to effectiveness. Both of these properties control LND^H -rigidity. The following result is similar to [BZ1, Theorem 5.2] and [LWZ, Theorem 2.9].

Theorem 3.10 If d is an effective (resp., dominating) element in an affine commutative domain Z , then Z is strongly LND_d^H -rigid.

Proof Since the proofs for the “effective” case and the “dominating” case are very similar, we prove only the “effective” case.

Suppose Z is generated by $\{x_j\}_{j=1}^m$. Let $\partial \in \text{LND}_d^H(Z[t_1, \dots, t_n])$ and $G := G_{\partial, t} \in \text{Aut}_{\mathbb{k}[t]}(Z[t_1, \dots, t_n][t])$ as in Definition 2.6(2). Then, for each j ,

$$G(x_j) = x_j + \sum_{i \geq 1} t^i \partial_i(x_j).$$

Since $d \in \ker \partial$, by definition,

$$(E2.10.1) \quad G(d) = d.$$

Recall from Definition 3.8 that, when d is effective, Z is a filtered algebra with \deg_Z defined as in [YZ2, Lemma 1.1]. It is clear that $Z' := Z[t_1, \dots, t_n]$ is a filtered algebra with $\deg_{Z'} z = \deg_Z z$ for all $z \in Z$ and $\deg_{Z'} t_s = 1$ for $s = 1, \dots, n$. We take the test algebra T to be $Z[t_1, \dots, t_n][t] = Z'[t]$, where the filtration on T is determined by $\deg_T(z) = \deg_Z(z)$ for all $z \in Z$, $\deg_T t_s = 1$ for $s = 1, \dots, n$, and $\deg_T t = \alpha$, where

$$\alpha > \sup\{\deg_{Z'} \partial_i(x_j) \mid j = 1, \dots, m, i \geq 0\}.$$

Now set $y_j = G(x_j) \in T$. By the choice of α , we have that

- (a) $\deg_T y_j \geq \deg_Z x_j$, and that
- (b) $\deg_T y_j = \deg_Z x_j$ if and only if $y_j = x_j$.

Let $f(x_1, \dots, x_m)$ be some polynomial presentation of d as in Definition 3.8. If $G(x_j) \neq x_j$ for some j , by the effectiveness of d as in Definition 3.8, $f(y_1, \dots, y_m) = 0$ or $\deg_T f(y_1, \dots, y_m) > \deg_Z d = \deg_T d$. So $f(y_1, \dots, y_m) \neq_{Z^\times} d$. But $f(y_1, \dots, y_m) = G(d) =_{Z^\times} d$ by (E2.10.1), a contradiction. Therefore, $G(x_j) = x_j$ for all j . As a consequence, $\partial_i(x_j) = 0$ for all $i \geq 1$, or equivalently, $x_j \in \ker \partial$. Since Z is generated by x_j 's, $Z \subseteq \ker \partial$. Thus, $Z \subseteq \text{ML}_d^H(Z[t_1, \dots, t_n])$. It is clear that $Z \supseteq \text{ML}_d^H(Z[t_1, \dots, t_n])$; see [BZ1, Example 2.4]. Therefore, $Z = \text{ML}_d^H(Z[t_1, \dots, t_n])$, as required. ■

The following corollary will be used several times.

Corollary 3.11 *Let A be an algebra such that the center of A is $\mathbb{k}[x]$. Let \mathcal{P} be a stable Morita invariant property (resp., stable property) such that the \mathcal{P} -discriminant of A , denoted by d , is a nonzero non-invertible element in $Z(A) = \mathbb{k}[x]$. Then $Z(A)$ is strongly LND_d^H -rigid and A is strongly m - Z -retractable (resp., strongly Z -retractable).*

Proof By Example 3.9, d is an effective element in $Z(A)$. By Theorem 3.10, $Z(A)$ is strongly LND_d^H -rigid. By Proposition 3.7(ii), A is strongly m - Z -retractable (resp., strongly Z -retractable). ■

4 Morita Detectability

First, we recall the detectability introduced in [LWZ]. If B is a subring of C and f_1, \dots, f_m are elements of C , then the subring generated by B and the subset $\{f_1, \dots, f_m\}$ is denoted by $B\{f_1, \dots, f_m\}$.

Definition 4.1 ([LWZ, Definition 3.1]) Let A be an algebra.

(i) We call A *detectable* if any algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $B[s] = B\{\phi(t)\}$, or equivalently, $s \in B\{\phi(t)\}$.

(ii) We call A *strongly detectable* if for each integer $n \geq 1$, any algebra isomorphism

$$\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

implies that $B[s_1, \dots, s_n] = B\{\phi(t_1), \dots, \phi(t_n)\}$, or equivalently, for each $i = 1, \dots, n$, $s_i \in B\{\phi(t_1), \dots, \phi(t_n)\}$.

In the above definition, we do not assume that $\phi(t) = s$. Every strongly detectable algebra is detectable. The polynomial ring $\mathbb{k}[x]$ is cancellative, but not detectable. By [LWZ, Lemma 3.2], if A is Z -retractable in the sense of [LWZ, Definition 2.5], then it is detectable. We first recall a definition from [LWZ, Definition 3.4].

Definition 4.2 ([LWZ, Definition 3.4]) Let A be an algebra over \mathbb{k} .

(i) We say A is *Hopfian* if every \mathbb{k} -algebra epimorphism from A to itself is an automorphism.

(ii) We say A is *strongly Hopfian* if $A[t_1, \dots, t_n]$ is Hopfian for every $n \geq 0$.

By [LWZ, Lemma 3.6], if A is detectable and strongly Hopfian, then A is cancellative. We will generalize these facts in the Morita setting. In the following definition, we use ω^{-1} instead of ω for technical reasons.

Definition 4.3 Let A be an algebra. Let ω be the map given in (E2.1.1) when in a Morita context.

(i) We call A *m -detectable* if any equivalence of categories $M(A[t]) \cong M(B[s])$ implies that

$$A[t] = A\{\omega^{-1}(s)\},$$

or equivalently, $t \in A\{\omega^{-1}(s)\}$.

(ii) We call A *strongly m-detectable* if for each $n \geq 1$, any equivalence of categories $M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$ implies that

$$A[t_1, \dots, t_n] = A\{\omega^{-1}(s_1), \dots, \omega^{-1}(s_n)\},$$

or equivalently, $t_i \in A\{\omega^{-1}(s_1), \dots, \omega^{-1}(s_n)\}$ for $i = 1, \dots, n$.

The following result is analogous to [LWZ, Lemma 3.2].

Lemma 4.4 *If A is m-Z-retractable (resp., strongly m-Z-retractable), then it is m-detectable (resp., strongly m-detectable).*

Proof We show only the “strongly” version.

Suppose that A is strongly m-Z-retractable. Let B be any algebra such that the abelian categories $M(A[t])$ and $M(B[s])$ are equivalent. Since A is strongly m-Z-retractable, the map $\omega : Z(A)[t] \rightarrow Z(B)[s]$ in (E2.1.1) restricts to an algebra isomorphism $Z(A) \rightarrow Z(B)$. Write $\phi = \omega^{-1}$ and $f_i := \phi(s_i)$ for $i = 1, \dots, n$. Then

$$\begin{aligned} Z(A)\{f_1, \dots, f_n\} &= \phi(Z(B))\{\phi(s_1), \dots, \phi(s_n)\} \\ &= \phi(Z(B)\{s_1, \dots, s_n\}) \\ &= \phi(Z(B)[s]) = Z(A)[t]. \end{aligned}$$

Then, for every i , $t_i \in Z(A)[t] = Z(A)\{f_1, \dots, f_n\} \subseteq A\{f_1, \dots, f_n\}$, as desired. ■

Next we show that m-detectability implies m-cancellative property under some mild conditions.

Example 4.5 ([LWZ, Lemma 3.5]) The following algebras are strongly Hopfian:

- (i) left or right noetherian algebras;
- (ii) finitely generated locally finite \mathbb{N} -graded algebras;
- (iii) prime affine \mathbb{k} -algebras satisfying a polynomial identity.

Lemma 4.6 *Suppose A is strongly Hopfian.*

- (i) *If A is m-detectable, then A is m-cancellative and cancellative.*
- (ii) *If A is strongly m-detectable, then A is strongly m-cancellative and strongly cancellative.*

Proof We prove only (ii).

First, we consider the Morita version. Suppose that $A[t]$ and $B[s]$ are Morita equivalent and $\omega : Z(A)[t] \rightarrow Z(B)[s]$ is the algebra isomorphism given as in (E2.1.1). Write $\phi = \omega^{-1}$ and $f_i = \phi(s_i)$ for $i = 1, \dots, n$. Then f_i are central elements in $A[t]$. Thus, $A\{f_1, \dots, f_n\}$ is a homomorphic image of $A[t_1, \dots, t_n]$ by sending $t_i \mapsto f_i$. Since A is strongly m-detectable, $A\{f_1, \dots, f_n\} = A[t]$. Then we have an algebra homomorphism

$$(E3.6.1) \quad A[t] \xrightarrow{\pi} A\{f_1, \dots, f_n\} \xrightarrow{\cong} A[t].$$

Since A is strongly Hopfian, $A[t]$ is Hopfian. Now (E3.6.1) implies that π is an isomorphism. As a consequence, $A\{f_1, \dots, f_n\} = A[f_1, \dots, f_n]$ viewing f_i as central

indeterminates in $A[f_1, \dots, f_n]$. As a consequence, $A[\underline{t}] = A[\underline{f}]$. Going back to the map

$$\omega : Z(A[\underline{t}]) = Z(A[\underline{f}]) \longrightarrow Z(B[\underline{s}]),$$

one sees that ω maps f_i to s_i for $i = 1, \dots, n$. Let J be the ideal of $Z(A[\underline{t}])$ generated by $\{f_i\}_{i=1}^n$ and J' be the ideal of $Z(B[\underline{s}])$ generated by $\{s_i\}_{i=1}^n$. Then $J' = \omega(J)$. By Lemma 3.1(v), the algebra A (which is isomorphic to $A[\underline{t}]/JA$) is Morita equivalent to B (which is isomorphic to $B[\underline{s}]/J'B$). The assertion follows.

Next we consider the “cancellative” version. Suppose that $\omega' : A[\underline{t}] \rightarrow B[\underline{s}]$ is an isomorphism that restricts to an isomorphism between the centers $\omega : Z(A)[\underline{t}] \rightarrow Z(B)[\underline{s}]$. Then ω' induces a (trivial) Morita equivalence, and ω is the map given in (E2.1.1). Re-use the notation introduced in the above proof. The above proof shows that $A[\underline{t}] = A[\underline{f}]$, where $f_i = \omega^{-1}(s_i)$ for all i . Therefore, ω' induces an isomorphism

$$A \cong A[\underline{f}]/(\{f_i\}_{i=1}^n) \xrightarrow{\omega'} B[\underline{s}]/(\{s_i\}_{i=1}^n) \cong B,$$

as desired. ■

For the rest of this section we study more properties concerning m -detectability.

Lemma 4.7 *Let A be an algebra with center Z . Suppose Z is (strongly) cancellative.*

- (i) *If Z is (strongly) detectable, then A is (strongly) m -detectable.*
- (ii) *Z is (strongly) detectable if and only if it is (strongly) m -detectable.*

Proof Following the pattern before, we prove only the “strongly” version.

(i) Suppose B is an algebra such that $A[\underline{t}]$ and $B[\underline{s}]$ are Morita equivalent. Let $\omega : Z[\underline{t}] \rightarrow Z(B)[\underline{s}]$ be the algebra isomorphism given in (E2.1.1). Since Z is strongly cancellative, one has that $Z(B) \cong Z$. Now we have an isomorphism $\omega^{-1} : Z(B)[\underline{s}] \cong Z[\underline{t}]$. Since $Z(B)$ (or Z) is strongly detectable, $t_i \in Z\{\omega^{-1}(s_1), \dots, \omega^{-1}(s_n)\}$ for all i . Thus, $t_i \in A\{\omega^{-1}(s_1), \dots, \omega^{-1}(s_n)\}$ for all i . This means that A is strongly m -detectable.

(ii) One direction is part (i). For the other direction, assume that Z is strongly m -detectable. Consider any algebra isomorphism $\phi : Z[\underline{t}] \rightarrow B[\underline{s}]$. It is clear that B is commutative and $B \cong Z$, since Z is strongly cancellative. Then ϕ induces a (trivial) Morita equivalent, and the map ω in (E2.1.1) is just ϕ . Now the strong m -detectability of Z implies that Z is strongly detectable. ■

The next result is similar to [LWZ, Proposition 3.10].

Proposition 4.8 *If the center Z of A is an affine domain of GK-dimension one that is not isomorphic to $\mathbb{k}'[x]$ for some field extension $\mathbb{k}' \supseteq \mathbb{k}$, then A is strongly m -detectable.*

Proof By [AEH, Theorem 3.3], Z is strongly retractable and cancellative. As a consequence, Z is a strongly m - Z -retractable. By Lemma 4.4, A is strongly m -detectable. ■

5 Proofs of Theorems 1.3 and 1.4

In this section we will use the results in the previous sections to show some classes of algebras are m -cancellative. We first prove Theorem 1.4.

Theorem 5.1 *If A is left (or right) noetherian, and the center of A is artinian, then A is strongly m -detectable. As a consequence, A is strongly m -cancellative.*

Proof Let Z be the center of A . Then Z is artinian by hypothesis. By [LWZ, Theorem 4.1], Z is strongly detectable and strongly cancellative. By Lemma 4.7(i), A is strongly m -detectable. By Example 4.5(i), A is strongly Hopfian. The consequence follows from Lemma 4.6(ii). ■

Theorem 1.4 is a special case of Theorem 5.1.

Theorem 5.2 *Let A be an algebra with strongly cancellative center Z . Suppose J is the prime radical of Z such that (a) J is nilpotent and (b) Z/J is a finite direct sum of fields. Then the following hold.*

- (i) A is strongly m -detectable.
- (ii) If further A is strongly Hopfian, then A is strongly m -cancellative.

Proof (i) By the proof of [LWZ, Theorem 4.2], Z is strongly detectable. By Lemma 4.7, A is strongly m -detectable.

(ii) Follows from Lemma 4.6 and part (i). ■

Next is Theorem 1.3.

Corollary 5.3 *Suppose A is strongly Hopfian and the center of A is artinian. Then A is strongly m -detectable and strongly m -cancellative.*

Proof Let Z be the center of A . By [LWZ, Theorem 4.1], Z is strongly detectable and strongly cancellative. Since Z is artinian, it satisfies conditions (a) and (b) in Theorem 5.2. The assertion follows by Theorem 5.2. ■

6 Proof of Theorem 1.6

We assume in this section that \mathbb{k} is algebraically closed. Under this hypothesis, a \mathcal{P} -discriminant ideal has the following nice property. This is one of the reasons we need the above hypothesis.

Lemma 6.1 *Let \mathcal{P} be a property. Then \mathcal{P} is stable.*

Proof Let Z be the center of A . By Definition 3.3(v), we may assume that Z is affine and write it as $\mathbb{k}[z_1, \dots, z_m]/(R)$, where $\{z_1, \dots, z_m\}$ is a generating set of Z and R is a set of relations. Every maximal ideal of Z is of the form $(z_i - \alpha_i) := (z_1 - \alpha_1, \dots, z_m - \alpha_m)$, where $\alpha_i \in \mathbb{k}$ for all i . Every maximal ideal of $Z[t]$ is of the form

$$(z_i - \alpha_i, t_j - \beta_j) := (z_1 - \alpha_1, \dots, z_m - \alpha_m, t_1 - \beta_1, \dots, t_n - \beta_n),$$

where $\alpha_i, \beta_j \in \mathbb{k}$. The natural embedding $Z \rightarrow Z[t]$ induces a projection

$$\pi : \text{MaxSpec}(Z[t]) \longrightarrow \text{MaxSpec } Z$$

by sending $\mathfrak{m} := (z_i - \alpha_i, t_j - \beta_j)$ to $\pi(\mathfrak{m}) := (z_i - \alpha_i)$.

Let $D_{\mathcal{P}}(A)$ be the \mathcal{P} -discriminant set of A . A maximal ideal \mathfrak{m} is in $D_{\mathcal{P}}(A[t])$ if and only if $A[t]/\mathfrak{m}A[t]$ does not have property \mathcal{P} . Since

$$A[t]/\mathfrak{m}A[t] \cong A/\pi(\mathfrak{m})A,$$

$\mathfrak{m} \in D_{\mathcal{P}}(A[t])$ if and only if $\pi(\mathfrak{m}) \in D_{\mathcal{P}}(A)$. This implies that $D_{\mathcal{P}}(A[t]) = D_{\mathcal{P}}(A) \times \mathbb{A}^n$. As a consequence,

$$I_{\mathcal{P}}(A[t]) = \bigcap_{\mathfrak{m} \in D_{\mathcal{P}}(A[t])} \mathfrak{m} = \left(\bigcap_{p \in D_{\mathcal{P}}(A)} p \right) \otimes \mathbb{k}[t] = I_{\mathcal{P}}(A) \otimes \mathbb{k}[t].$$

Therefore, \mathcal{P} is stable by Definition 3.3(v). ■

Let A be an algebra with the center Z being a domain. Let $\tau(A/Z)$ be the ideal of A consisting of elements in A that are annihilated by some nonzero element in Z . Define the *annihilator ideal* of Z to be

$$\kappa(A/Z) = \{z \in Z \mid z(\tau(A/Z)) = 0\}.$$

Lemma 6.2 *Retain the notation as above.*

- (i) κ is stable in the sense that $\kappa(A[t]/Z[t]) = \kappa(A/Z) \otimes \mathbb{k}[t]$.
- (ii) If A and B are Morita equivalent, then ω maps $\kappa(A/Z)$ to $\kappa(B/Z(B))$ bijectively.
- (iii) If A is left noetherian and suppose the center Z is a domain, then $\tau(A/Z) \neq 0$ if and only if $\kappa(A/Z)$ is a proper ideal, neither Z nor 0 .

Proof This is easy to check, so details are omitted. ■

Lemma 6.3 *Suppose A is a finitely generated module over its center Z and Z is a domain. If A is prime, then $\tau(A/Z) = 0$.*

Proof This is easy to check, so details are omitted. ■

Proposition 6.4 *Let A be left noetherian such that the center Z is an affine domain of GK-dimension one.*

- (i) If Z is not $\mathbb{k}[x]$, then A is strongly m - Z -retractable, m -detectable, and m -cancellative.
- (ii) If $Z = \mathbb{k}[x]$ and $\tau(A/Z) \neq 0$, then A is strongly m - Z -retractable, m -detectable, and m -cancellative.

Proof (i) By [AEH, Theorem 3.3 and Corollary 3.4], Z is strongly retractable. By Definition 3.6(iii), A is strongly m - Z -retractable. By Lemma 4.4, A is strongly m -detectable. Since A is left noetherian, by Lemma 4.6(ii), A is strongly m -cancellative.

(ii) Since A is left noetherian and $\tau(A/Z) \neq 0$, $\kappa(A/Z)$ is a nonzero proper ideal of $\mathbb{k}[x]$ by Lemma 6.2(iii). So there is a nonzero non-invertible element $f \in \mathbb{k}[x]$ such that $\kappa(A/Z) = (f)$. By Lemma 6.2(i) and (ii), κ is a stable Morita invariant property. By replacing \mathcal{P} by κ , Corollary 3.11 implies that A is strongly m - Z -retractable. The rest of the proof is similar to the proof of part (i). ■

For the rest of the section we consider the case when $Z = \mathbb{k}[x]$ and $\tau(A/Z) = 0$, or more precisely, when A is affine prime PI of GK-dimension one with $Z = \mathbb{k}[x]$. We need to recall some concepts.

Let A be an affine prime algebra of GK-dimension one. By a result of Small and Warfield [SW], A is a finitely generated module over its affine center. As a consequence, A is noetherian.

Let R be a commutative algebra, an R -algebra A is called *Azumaya* if A is a finitely generated faithful projective R -module and the canonical morphism

$$(E5.4.1) \quad A \otimes_R A^{op} \longrightarrow \text{End}_R(A)$$

is an isomorphism. By [DeI, Theorem 3.4], A is Azumaya if and only if A is a central separable algebra over R . Since we assume that \mathbb{k} is algebraically closed, we have the following equivalent definition.

Definition 6.5 ([BY, Introduction]) Let A be an affine prime \mathbb{k} -algebra which is a finitely generated module over its affine center $Z(A)$. Let n be the PI-degree of A , which is also the maximal possible \mathbb{k} -dimension of irreducible A -modules.

(i) The *Azumaya locus* of A , denoted by $\mathcal{A}(A)$, is the dense open subset of $\text{MaxSpec } Z(A)$ which parametrizes the irreducible A -modules of maximal \mathbb{k} -dimension. In other words, $\mathfrak{m} \in \mathcal{A}(A)$ if and only if $\mathfrak{m}A$ is the annihilator in A of an irreducible A -module V with $\dim V = n$, if and only if $A/\mathfrak{m}A \cong M_n(\mathbb{k})$.

(ii) If $\mathcal{A}(A) = \text{MaxSpec } Z(A)$, A is called *Azumaya*.

We can relate the Azumaya locus with the “simple”-locus. Let \mathcal{S} be the property of being simple.

Lemma 6.6 Assume that A is free over its affine center Z .

(i) $A[t]$ is free over $Z[t]$.

(ii) $\mathcal{A}(A) = \mathcal{L}_{\mathcal{S}}(A)$, where the latter is defined in Definition 3.3(i).

Proof (i) is obvious.

(ii) Since A is free over Z of rank n^2 , $A/\mathfrak{m}A$ is isomorphic to $M_n(\mathbb{k})$ if and only if $A/\mathfrak{m}A$ is simple. The assertion follows. ■

Proposition 6.7 Suppose that A is an affine prime algebra of GK-dimension one with center $\mathbb{k}[x]$.

(i) If A is not Azumaya, then A is strongly m - Z -retractable, m -detectable, and m -cancellative.

(ii) If A is Azumaya, then A is strongly m -cancellative.

Proof (i) Since the Azumaya locus is open and dense, the non-Azumaya locus of A is a proper nonzero ideal of $Z = \mathbb{k}[x]$, which is principal. Since A is prime, $\tau(A/Z) = 0$ and whence A is projective and then free over Z . By Lemma 6.6(ii), the Azumaya locus of $A[t]$ agrees with the \mathcal{S} -locus of $A[t]$. Hence, \mathcal{S} is a stable Morita invariant property such that the \mathcal{S} -discriminant is a nonzero non-invertible element in Z . By Corollary 3.11, A is strongly m - Z -retractable. The rest of the proof follows from the proof of Proposition 6.4(i).

(ii) Since A is Azumaya, by [LWZ, Lemma 4.9(3)], $A = M_n(\mathbb{k}[x])$ for some integer $n \geq 1$. If $A[t]$ is Morita equivalent to $B[s]$, then $Z(A)[t] \cong Z(B)[s]$. Since $Z(A) = \mathbb{k}[x]$ is strongly cancellative, $Z(B)$ is also isomorphic to $\mathbb{k}[x]$. If B is not Azumaya, it follows from part (i) that A and B are Morita equivalent. If B is Azumaya, then by [LWZ, Lemma 4.9(3)], B is a matrix algebra $M_{n'}(\mathbb{k}[x])$ for some $n' \geq 1$, which is also Morita equivalent to A . Therefore, A is strongly m -cancellative. ■

Now we are ready to prove Theorem 1.6.

Theorem 6.8 *Let A be an affine prime algebra of GK-dimension one.*

- (i) *A is strongly m -cancellative.*
- (ii) *If either $Z(A) \neq \mathbb{k}[x]$ or A is not Azumaya, then A is strongly m - Z -retractable and m -detectable.*

Proof Since we assume that \mathbb{k} is algebraically closed in this section, by [LWZ, Lemma 4.9], there are three cases to consider.

Case 1: $Z(A) \not\cong \mathbb{k}[x]$.

Case 2: $Z(A) \cong \mathbb{k}[x]$ and A is not Azumaya.

Case 3: $Z(A) \cong \mathbb{k}[x]$ and A is Azumaya.

Applying Proposition 6.4(i) in Case 1, Proposition 6.7(i) in Case 2 and Proposition 6.7(ii) in Case 3, the assertion follows. ■

It is clear that Theorem 1.6 is an immediate consequence of Theorem 6.8. As far as we know there are no examples of algebras with the center being an affine domain of GK-dimension one that are not m -cancellative. Therefore, we ask the following question.

Question 6.9 *Let A be a left noetherian algebra such that $Z(A)$ is an affine domain of GK-dimension one. Then is A m -cancellative?*

We finish this section with some examples of non-PI algebras that are strongly $(m-)$ cancellative.

Example 6.10 *Let Z denote the center of the given algebra A . Assume that \mathbb{k} has characteristic zero.*

- (i) *Let A be the homogenization of the first Weyl algebra that is generated by x, y, t subject to the relations*

$$[x, t] = [y, t] = 0, [x, y] = t^2.$$

It is well known that the center of A is $\mathbb{k}[t]$. Let \mathcal{S} be the property of being simple. Since $\mathfrak{m} := (t - 0)$ is the only maximal ideal of $\mathbb{k}[t]$ such that $A/\mathfrak{m}A$ is not simple, the \mathcal{S} -discriminant $d_{\mathcal{S}}(A)$ exists and equals t . By Corollary 3.11, A is strongly m - Z -retractable. By Lemma 4.4, A is strongly m -detectable. By Lemma 4.6(ii), A is both strongly cancellative and strongly m -cancellative.

- (ii) *Let A be a non-PI quadratic Sklyanin algebra of global dimension 3. It is well known that the center of A is $\mathbb{k}[g]$ where $g \in A$ has degree 3. We claim that $A/(g - \alpha)$ is simple if and only if $\alpha \neq 0$. If $\alpha = 0$, then $A/(g)$ is connected graded which is not*

simple. Now assume that $\alpha \neq 0$. It is well known that $(A[g^{-1}])_0$ is simple. Let T be the 3rd Veronese subring of A . Then $(T[g^{-1}])_0 \cong (A[g^{-1}])_0$ is simple. Now

$$T/(g - \alpha) \cong T/(\alpha^{-1}g - 1) \cong (T[(\alpha^{-1}g)^{-1}])_0 = (T[g^{-1}])_0 \cong (A[g^{-1}])_0,$$

where the second \cong is due to [RSS, Lemma 2.1]. It is clear that $A/(g - \alpha)$ contains $T/(g - \alpha)$. Since $T/(g - \alpha)$ is simple and hence has no finite dimensional modules, $A/(g - \alpha)$ does not have finite dimensional modules. Since the algebra $A/(g - \alpha)$ is affine of GK-dimension two, it must be simple. So we proved the claim.

The claim implies that the \mathcal{S} -discriminant $d_{\mathcal{S}}(A)$ exists and equals $g \in \mathbb{k}[g]$. Following the last part of the above example, A is both strongly cancellative and strongly m-cancellative.

Example 6.11 Suppose $\text{char } \mathbb{k} = 0$. Let A be the universal enveloping algebra of the simple Lie algebra sl_2 . By Example 3.4, the center of A is $\mathbb{k}[Q]$, where Q is the Casimir element. In this example, we will consider two different properties.

Let \mathcal{W} be the property of not having a factor ring isomorphic to $M_{n+1}(\mathbb{k})$ (for a fixed integer n). Then $d_{\mathcal{W}}(A) = Q - (n^2 + 2n)$, which is a nonzero non-invertible element in $\mathbb{k}[Q]$. By Corollary 3.11, A is strongly Z -retractable. By [LWZ, Lemma 3.2], A is strongly detectable, and by [LWZ, Lemma 3.6(2)], A is strongly cancellative.

Next we show that A is strongly m-cancellative by using a Morita invariant property. Let \mathcal{H} be the property that $HH_3(R) = 0$, where $HH_i(R)$ denotes the i -th Hochschild homology of an algebra R . By [We, Theorem 9.5.6], the Hochschild homology is Morita invariant. Hence \mathcal{H} is Morita invariant. We claim that the discriminant $d_{\mathcal{H}}(A)$ is $Q + \frac{1}{4}$. This claim is equivalent to the following assertions:

- (a) $HH_3(A/(Q - \lambda)) = 0$ for all $\lambda \neq -\frac{1}{4}$;
- (b) $HH_3(A/(Q + \frac{1}{4})) \neq 0$ (this is the case when $\lambda = -\frac{1}{4}$).

Let $B_{\lambda} = A/(Q - \lambda)$. Then B_{λ} agrees with the algebra B_{λ} in [FSS, Example 2.3]. By [FSS, Example 2.3], B_{λ} is a generalized Weyl algebra with $\sigma(h) = h-1$, $a = \lambda - h(h+1)$. Hence, B_{λ} satisfies the hypotheses of [FSS, Theorem 2.1]. If $\lambda \neq -\frac{1}{4}$, then $a'(h)$ and $a(h)$ are coprime. By [FSS, Theorem 2.1(1)], $HH_3(B_{\lambda}) = 0$, which is part (a). If $\lambda = -\frac{1}{4}$, then the common divisor of $a'(h)$ and $a(h)$ is $a'(h)$, which has degree 1. By [FSS, Theorem 2.1(2)], $HH_3(B_{\lambda}) = \mathbb{k}$, which is part (b). Therefore, we proved the claim. By Corollary 3.11, A is strongly m- Z -retractable. By Lemma 4.4, A is strongly m-detectable. By Lemma 4.6(ii), A is both strongly cancellative and strongly m-cancellative.

Remark 6.12 (i) The second half of Example 6.11 shows that using a Morita invariant property results a better conclusion.

(ii) Another consequence of the discussion in Example 6.11 is the following. If σ is an algebra automorphism of $A := U(sl_2)$, then $\sigma(Q) = Q$. Further, for every locally nilpotent derivation $\partial \in \text{LND}(A)$, we have $\partial(Q) = 0$. This could be a useful fact to use in calculating the automorphism group $\text{Aut}(A)$. According to [CL, Section 3.2], the full automorphism group of A is still unknown. A result of Joseph [Jo] says that $\text{Aut}(A)$ contains a wild automorphism. The automorphism of $A/(Q - \alpha)A$ was computed in [Di] when $\alpha \neq n^2 + 2n$ for all $n \in \mathbb{N}$.

7 Proof of Theorem 1.5

In this section we prove Theorem 1.5. We refer to [ASS] for basic definitions of quivers and their path algebra. Let C_n be the cyclic quiver with n vertices and n arrow connecting these vertices in one oriented direction. In representation theory of finite dimensional algebras, quiver C_n is also called *type \widetilde{A}_{n-1}* . Let $0, 1, \dots, n-1$ be the vertices of C_n , and $a_i : i \rightarrow i+1$ (in $\mathbb{Z}/(n)$) be the arrows in C_n . Then $w := \sum_{i=0}^{n-1} a_i a_{i+1} \cdots a_{i+n-1}$ is a central element in $\mathbb{k}C_n$. By [LWZ, Lemma 4.4], we have the following result concerning the center of the path algebra $\mathbb{k}Q$ when Q is connected:

$$(E6.0.1) \quad Z(\mathbb{k}Q) = \begin{cases} \mathbb{k} & \text{if } Q \text{ has no arrow,} \\ \mathbb{k}[x] & \text{if } Q = C_1 \text{ or equivalently } \mathbb{k}Q = \mathbb{k}[x], \\ \mathbb{k}[w] & \text{if } Q = C_n \text{ for } n \geq 2, \\ \mathbb{k} & \text{otherwise.} \end{cases}$$

Similar to [LWZ, Lemma 3.11], we have the following lemma, whose proof is omitted.

Lemma 7.1 *Let \mathbb{k}' be a field extension of \mathbb{k} . If $A \otimes_{\mathbb{k}} \mathbb{k}'$ is (strongly) m -detectable as an algebra over \mathbb{k}' , then A is (strongly) m -detectable as an algebra over \mathbb{k} .*

Lemma 7.2 *Let $Q = C_n$ for $n \geq 2$. Then $\mathbb{k}Q$ is strongly m -detectable and strongly m -cancellative.*

Proof By [LWZ, Lemma 4.5], $\mathbb{k}C_n$ is prime of GK-dimension one while not being Azumaya. If \mathbb{k} is algebraically closed, the assertion is a special case of Theorem 6.8(ii). If \mathbb{k} is not algebraically closed, let \mathbb{k}' be the closure of \mathbb{k} . By Theorem 6.8(ii), $\mathbb{k}'Q$ is strongly m -detectable over \mathbb{k}' . By Lemma 7.1, $\mathbb{k}Q$ is strongly m -detectable over \mathbb{k} , and then strongly m -cancellative by Lemmas 4.5(ii) and 4.6(ii). ■

We need another lemma before proving the main result of this section. The ideas of the proof are similar to the proof of [LWZ, Lemma 4.6], so the proof is omitted.

Lemma 7.3 *Let A and B be two algebras.*

- (i) *If A and B are (strongly) m -cancellative, so is $A \oplus B$.*
- (ii) *If A and B are (strongly) m -retractable, so is $A \oplus B$.*
- (iii) *If A and B are (strongly) m -detectable, so is $A \oplus B$.*

Now we are ready to prove Theorem 1.5.

Theorem 7.4 *Let Q be a finite quiver and let A be the path algebra $\mathbb{k}Q$. Then A is strongly m -cancellative. If, further, Q has no connected component being C_1 , then A is strongly m -detectable.*

Proof By Lemma 7.3, we can assume that Q is connected.

If $Q = C_1$, then $A = \mathbb{k}[x]$ and the assertion follows from Proposition 6.7(ii).

If $Q = C_n$, then this is Lemma 7.2(ii).

If $Q \neq C_n$ for any $n \geq 1$, then by (E6.0.1), the center of A is \mathbb{k} . By Theorem 5.2(i), A is strongly m -detectable. Since A is \mathbb{N} -graded and locally finite, it is strongly Hopfian by Example 4.5(ii). By Theorem 5.2(ii), A is strongly m -cancellative. This completes the proof. ■

Theorem 1.5 is clearly a consequence of the above theorem.

8 Comments, Questions, and Examples

One of the remaining questions in this project is to understand whether the cancellation property is equivalent to the m -cancellation property (as well as the d -cancellation property). We will make some comments about it in this section.

First, we will show that three cancellation properties are equivalent for commutative algebras. The next result was proved in [YZ1] using slightly different wording.

Proposition 8.1 ([YZ1, Proposition 5.1]) *Suppose that A is an Azumaya algebra over its center Z and that $\text{Spec } Z$ is connected. If $D(A)$ and $D(B)$ are triangulated equivalent for another algebra B , then A and B are Morita equivalent.*

Note that the Brauer group of a commutative algebra R , denoted by $Br(R)$, is the set of Morita-type-equivalence classes of Azumaya algebras over R ; in other words, $Br(R)$ classifies Azumaya algebras over R up to an equivalence relation [AG]. See [Sc] for some discussion about the Brauer group. One immediate consequence is the following corollary.

Corollary 8.2 *Suppose Z is a commutative algebra with $\text{Spec } Z$ connected. Then the following are equivalent.*

- (i) Z is (strongly) cancellative.
- (ii) Z is (strongly) m -cancellative.
- (iii) Z is (strongly) d -cancellative.

Proof By Proposition 8.1, it remains to show that (i) and (ii) are equivalent. By Lemma 2.4, part (i) follows from part (ii). Now we show that part (ii) is a consequence of part (i).

Suppose A is an algebra such that $Z[t]$ is Morita equivalent to $A[s]$. By the map ω in (E2.1.1), we obtain that $Z[t]$ is isomorphic to $Z(A)[s]$. Since Z is (strongly) cancellative, $Z(A) \cong Z$. Let us identify $Z(A)$ with Z . Since $Z[t]$ is Morita equivalent to $A[s]$, $A[s]$ is Morita equivalent to its center, which is $Z[s]$. Then the Brauer-class $[A[s]]$ as an element in $Br(Z[s])$ is trivial by [AG, Proposition 5.3]. Since the natural map $Br(Z) \rightarrow Br(Z[s])$ is injective, the Brauer-class $[A]$ as an element in $Br(Z)$ is trivial. By [Sc, Theorem 4] or [Ne, Proposition 4.1], A is Morita equivalent to Z , as required. ■

Corollary 8.3 *Let Z be a (strongly) detectable commutative algebra such that $\text{Spec } Z$ is connected. If A is an Azumaya algebra over Z that is strongly Hopfian, then A is both (strongly) m -cancellative and (strongly) d -cancellative.*

Proof By Proposition 8.1, we need to show only the claim that A is (strongly) m -cancellative. Since A is strongly Hopfian, the claim follows from Lemmas 4.6(ii) and 4.7(i). ■

The next example is similar to [LWZ, Example 3.3].

Example 8.4 Let $A = \mathbb{k}[x, y]/(x^2 = y^2 = xy = 0)$. By Theorem 5.1, A is strongly m -detectable. By [LWZ, Example 3.3] and Corollary 8.2, the commutative algebra A is neither retractable nor m -retractable.

For non-Azumaya (noncommutative) algebras, there is no general approach to relating the m -cancellation property with the d -cancellation property. However, most of cancellative algebras verified by using the discriminant method in [BZ1] are m -cancellative, as we will see next.

Since most of algebras that we are interested in are strongly Hopfian, to show that an algebra is m -cancellative, it suffices to show that it is m -detectable [Lemma 4.6(i)]. By Lemma 7.1, under some mild hypotheses, we can assume the base field \mathbb{k} is algebraically closed. For simplicity, we assume that \mathbb{k} is algebraically closed of characteristic zero for the rest of this section.

Let I be an ideal of a commutative algebra R . Then the *radical* of I is defined to be

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \text{Spec } R, I \subseteq \mathfrak{p}} \mathfrak{p}.$$

The *standard trace* tr_{st} defined in [BY, Sect. 2.1(2)] agrees with the *regular trace* tr_{reg} defined in [CPWZ2, p. 758]. So we take $\text{tr} = \text{tr}_{\text{st}} = \text{tr}_{\text{reg}}$ in this paper.

Proposition 8.5 Let A be a prime algebra that is finitely generated as a module over its center Z and let v be the rank of A over Z . Let $D \subseteq Z$ be either the v -discriminant ideal $D_v(B: \text{tr})$ in the sense of [CPWZ2, Definition 1.1(2)] or the modified v -discriminant ideal $MD_v(B: \text{tr})$ in the sense of [CPWZ2, Definition 1.2(2)]. Suppose that

- (i) the center Z is an affine domain and the standard trace tr maps A to Z ;
- (ii) \sqrt{D} is a principal ideal of Z generated by an element f ;
- (iii) f is an effective (resp., dominating) element in Z .

Then the following hold.

- (a) A is strongly m - Z -retractable.
- (b) A is strongly Z -retractable.
- (c) A is strongly m -detectable.
- (d) A is strongly m -cancellative.
- (e) A is strongly cancellative.

Proof Since we assume that \mathbb{k} is algebraically closed of characteristic zero, we can apply [BY, Main Theorem] by taking the standard trace. By [BY, Main Theorem], we have

$$\mathcal{V}(D) = \text{MaxSpec}(Z) \setminus \mathcal{A}(A),$$

where $\mathcal{V}(D)$ is the zero-set of D . By Lemma 6.6(ii), $\mathcal{A}(A) = L_S(A)$, where S denotes the property of being simple. Thus, the S -discriminant set of A is equal to $\mathcal{V}(D)$.

As a consequence, the \mathcal{S} -discriminant ideal of A is equal to $I(\mathcal{V}(D))$, which is \sqrt{D} . By hypothesis (ii), we obtain that the \mathcal{S} -discriminant ideal of A is a principal ideal of Z generated by an element f . Since f is effective (resp., dominating), Z is strongly LND_f^H -rigid by Theorem 3.10. Since \mathcal{S} is a stable Morita invariant property [Lemma 6.1], by Proposition 3.7(ii), A is both strongly m - Z -retractable and strongly Z -retractable. Thus, we proved parts (a) and (b). Note that part (c) follows from part (a) and Lemma 4.4. Since A is noetherian, it is strongly Hopfian [Example 4.5(1)]. Parts (d) and (e) follow from part (c) and Lemma 4.6(ii). ■

The next example is similar to [LWZ, Example 5.1].

Example 8.6 Let R be an affine commutative domain and let f be a product of a set of generating elements of R . Let

$$A = \begin{pmatrix} R & fR \\ R & R \end{pmatrix}.$$

It is easy to check that the (modified) 4-discriminant of A over its center R is the ideal generated by $-f^2$. Clearly, the radical of $(-f^2)$ is the principal ideal (f) . By the above proposition, A is strongly m - Z -retractable, m -detectable, m -cancellative, and cancellative.

Other precise examples follow, but we omit some details. See also [BZ1, Example 4.8].

Example 8.7 The following algebras are m -cancellative by verifying the hypotheses of Proposition 8.5:

- (i) skew polynomial rings $\mathbb{K}_q[x_1, \dots, x_n]$ when n is an even number and $1 \neq q$ is a root of unity;
- (ii) $\mathbb{K}\langle x, y \rangle / (x^2y - yx^2, y^2x + xy^2)$;
- (iii) quantum Weyl algebra $\mathbb{K}\langle x, y \rangle / (yx - qxy - 1)$, where $1 \neq q$ is a root of unity;
- (iv) every finite tensor product of algebras of the form (i), (ii), and (iii).

Acknowledgments The authors thank Daniel Rogalski for useful conversations on the subject and for providing the reference [RSS], Ellen Kirkman for reading an earlier version of the paper and for her useful comments, and the referees for their careful reading and valuable comments.

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