99.26 Proof without words: cosec $2x$ = cot x − cot $2x$

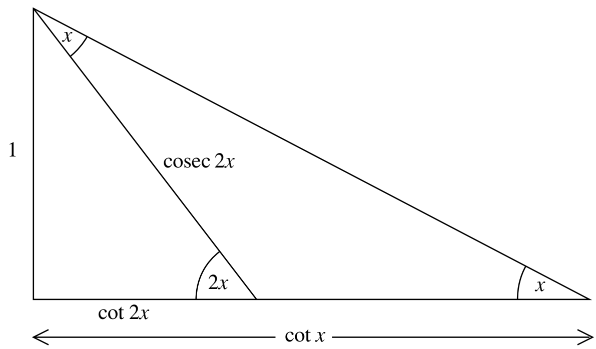

FIGURE 1

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Teaching Notes

Complex numbers and triangles

The following investigation provides excellent practice in the arithmetic of complex numbers and their representation as both points and directed edges in the Argand plane. Students should be able to conjecture a surprisingly general result about triangles which has an elementary proof.

Investigation

In the Argand plane, any three non-zero complex numbers with sum zero can be represented by the directed edges of a triangle. Examples of both a clockwise and an anticlockwise ordering of edges are shown:-

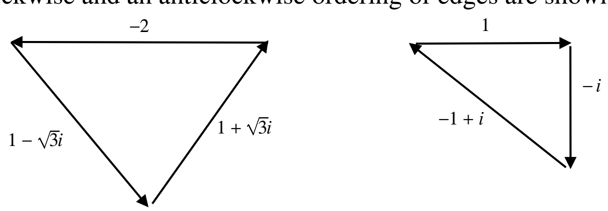

FIGURE 1

For *any* two triangles and for an *arbitrary* pairing of the edges of one triangle with those of the other, plot the points representing the three numbers $\dfrac{\text{edge of one triangle}}{\text{corresponding edge of other triangle}}$.

What do you notice?

Solution

For the two triangles used for the illustration, one particular pairing gives:

$$\frac{1+\sqrt{3}i}{-i} = -\sqrt{3}+i, \qquad \frac{-2}{-1+i} = 1+i, \qquad \frac{1-\sqrt{3}i}{1} = 1-\sqrt{3}i.$$

Plotting these points gives a triangle similar to the second triangle, whereas plotting the reciprocals of these numbers gives a triangle similar to the first triangle!

The general case

In general, the triangle whose vertices are the quotients (the *derived triangle*) is similar to the triangle whose edges are the denominators (the *denominator triangle*).

Let the edges of the numerator and denominator triangles represent the two sets of complex numbers $\{u, v, w\}$ and $\{x, y, z\}$, respectively. Define t by $txyz = wy - vz$. Then

$$u + v + w = 0 \tag{1}$$

$$x + y + z = 0. \tag{2}$$

Multiply (1) by z and (2) by w and subtract:

$$(uz - wx) + (vz - wy) = 0.$$

So, $uz - wx = txyz$ and, similarly, $vx - uy = txyz$. Then

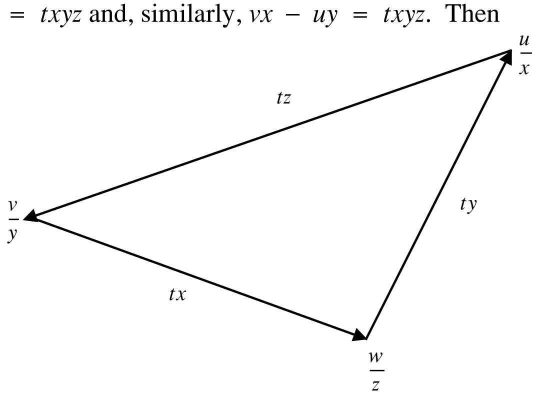

FIGURE 2

Thus, for any pairing of edges, the derived triangle is similar to the denominator triangle. Note that the degenerate case with $t = 0$ can occur. The two original triangles are then similar.

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