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SIMULTANEOUS MONOTONE APPROXIMATION IN LOW-ORDER MEAN

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Suppose that $f, g \in L_\infty[0,1]$ have discontinuities of the first kind only. Using the measure, $\max\{\|f-h\|_p, \|g-h\|_p\}$, of simultaneous L_p approximation, we show that the best simultaneous approximations, h_p, to f and g by nondecreasing functions converge uniformly as $p \to 1$. Part of the proof involves a discussion of discrete simultaneous approximation in a general context. We discuss the inheritance of properties of f and g by h_p, and of h_p by h_1.

1. INTRODUCTION

A context which calls for simultaneous approximation is that of fitting a multivariate function by a univariate function. For example if $f: A \times B \to \mathbb{R}$, then the problem is to approximate the set of univariate functions $\mathcal{F} := \{f(x, y_0): y_0 \in B\}$ by a single function $g: A \to \mathbb{R}$. In the present paper we shall restrict our attention to the case where $\mathcal{F}$ consists of exactly two functions. In measuring the distance from g to $\mathcal{F}$, two norms must be used; their composition is called a *vectorial* norm.

When one considers the continuum of normed linear spaces $\{L_p(\Omega, \Sigma, \mu): 1 \leqslant p \leqslant \infty\}$, three vectorial norms present themselves as being most natural for measuring simultaneous approximation as p varies. The simultaneous L_p-distance from f and g to h could be calculated by $\left(\|f-h\|_p^p + \|g-h\|_p^p\right)^{1/p}$, by $\left(\|f-h\|_p + \|g-h\|_p\right)$, or by $\max\left(\|f-h\|_p, \|g-h\|_p\right)$. In the first of these vectorial norms, the theory of simultaneous approximation is strongly related to that of single approximation on $L_p \times L_p$, and has been extensively studied [**15, 16, 17**]. The second norm has not, to our knowledge, been widely studied vis-a-vis the continuum of L_p-spaces, and is the subject of a planned future work. The third norm seems most natural for studying the uniform, as it relates to the L_p, simultaneous approximation operator, S_p; this study was begun in [**8**]. It is the norm used in the classical theory of Chebyschev centres [**18**] and provides the context in which the simultaneous approximation problem (for any compact set of approximations) is most naturally stated. In the present paper we continue the study of

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this "max norm" in L_p-simultaneous approximation, with primary focus on small values of p, and on convex discrete and monotone continuous approximation of functions on a probability space.

Most of the results to be presented here relate to the continuity of S_p; for p fixed, and as p varies. In [8], it was shown, for a large class of approximating sets in the discrete case, and for the approximating set $\mathcal{M}$ (nondecreasing functions on [0,1]) in the continuous case, that $S_p(f,g)$ converges as $p \to \infty$. In the present paper, we establish similar results for the case $p \to 1$. The existence of $\lim_{p\to 1} S_p(f,g)$ ameliorates the nonuniqueness of the 1-b.s.a. [10].

We begin with some definitions and notation. If $a,b \in \mathbb{R}$ (the set of all real numbers), let $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. If $f,g\colon \mathbb{R} \to \mathbb{R}$, define $f \vee g$ by $(f \vee g)(t) = f(t) \vee g(t)$ and $f \wedge g$ by $(f \wedge g)(t) = f(t) \wedge g(t)$. Let $(\mathcal{X},d)$ be a metric space. If $\mathcal{K} \subset \mathcal{X}$ and $f,g,h \in \mathcal{X}$, let $d(f,g;h) = d(f,h) \vee d(g,h)$, let $d(f;\mathcal{K}) = \inf_{h\in\mathcal{K}}\{d(f,h)\}$, and let $d(f,g;\mathcal{K}) = \inf_{h\in\mathcal{K}}\{d(f,g;h)\}$. We say that $h^* \in \mathcal{K}$ is a *best* (respectively, *best simultaneous*) *d-approximation* to f (respectively, to f and g) from $\mathcal{K}$ if $d(f,h^*) = d(f;\mathcal{K})$ (respectively, $d(f,g;h^*) = d(f,g;\mathcal{K})$). In this case, we say that h^* is a d-b.a. to f (respectively, d-b.s.a. to f and g). If there is a unique d-b.s.a. to f and g from $\mathcal{K}$, we denote it by $S(f,g)$. In subsequent sections of this paper we shall specialise by letting $\mathcal{X} = L_p$, but for the present we shall stay in a general context to state two theorems which we have not seen in the literature. The first relates to the continuity of S and its proof is *mutatis mutandis* the same as that of (2.5) in [14].

THEOREM 1. *If $\mathcal{K}$ is compact in $(\mathcal{X},d)$ and, for every $f,g \in \mathcal{X}$, $S(f,g)$ is uniquely defined, then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(S(f',g'),S(f,g)) < \varepsilon$ whenever $d(f,f') < \delta$ and $d(g,g') < \delta$.*

THEOREM 2. *If d is induced by a norm and if h is a d-b.s.a. to f and g from $\mathcal{K}$ but not a d-b.a. to f from $\mathcal{K}$, then $d(f,h) \leqslant d(g,h)$.*

PROOF: Suppose the theorem is false. Then $d(g,h) < d(f,h)$. Since h is not a d-b.a. to f, there exists $f^* \in \mathcal{K}$ such that $d(f,f^*) < d(f,h)$. For $\alpha \in \mathbb{R}$, let $H(\alpha) = (1-\alpha)h + \alpha f^*$, let $G(\alpha) = d(g,H(\alpha))$, and let $F(\alpha) = d(f,H(\alpha))$. Since d is induced by a norm, G and F are continuous. Thus, since $G(0) < F(0)$, there must be a $\beta > 0$ such that $G(\beta) < F(\beta)$. Since F is convex and since $F(1) < F(0)$, $F(\beta) < F(0)$. Thus $G(\beta) < F(0)$. Let $h^* = H(\beta)$. By the last two inequalities,

$$d(f,g;h^*) < d(f,h) = d(f,g;h),$$

which is a contradiction. $\square$

Let $h^* = S(f,g)$, $f^* = S(f,f)$, and $g^* = S(g,g)$. In [3], it was shown that if d is induced by an inner product, if $\mathcal{K}$ is a linear subspace, and if $f^* \neq h^* \neq g^*$, then h^*

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must be of the form

(i) $$h^* = \lambda f^* + (1-\lambda)g^*,$$

where $\lambda \in (0,1)$ is determined by the equation

(ii) $$d(f,h^*) = d(g,h^*).$$

If the requirement that $\mathcal{K}$ be a linear subspace is removed, then (i) doesn't hold in general even if we are in Hilbert space. To show this, let $f = \{3,0,5,0,7,0\}$, $g = \{-3,1,0,-2,1,-1\}$, $w = (1/15)\{1,2,1,4,1,6\}$, and let $\mathcal{K} = \mathcal{M}$, the closed convex cone of nondecreasing n-tuples in $\ell_2^n(w)$. Then $f_2 = \{1,1,1,1,1,1\}$, $g_2 = (-1/7)\{-21,6,6,6,5,5\}$ and $h_2 = \{0,1/4,1,1,1,1\}$. A simple calculation shows that $\|f-h_2\|_2^2 = \|g-h_2\|_2^2 = 569/120$, but there does not exist a $\lambda \in (0,1)$ for which $h_2 = \lambda f_2 + (1-\lambda)g_2$.

However, in the more general context of Theorem (2), (ii) does hold, and is proven in the following corollary. Geometrically speaking, the corollary says that if the relative Chebyschev centre of f and g is a nearest point to neither f nor g, then it is a relative "midpoint" of f and g.

Corollary 3. *Suppose d is induced by a norm and $\mathcal{K}$ is any convex subset of $\mathcal{X}$. If h is a d-b.s.a. to f and g from $\mathcal{K}$, but is a d-b.a. to neither f nor g, then $d(f,h) = d(g,h)$.*

Corollary (3) can be generalised to the simultaneous approximation of n functions $f^1,\dots,f^n$ as follows. If $1 \leqslant i < j \leqslant n$, if h is a d-b.s.a. of $\{f^1,\dots,f^n\}$, and if h is a d-b.a. to neither f^i nor f^j, then $d(f^i,h) = d(f^j,h)$. However, in some of the results stated below, we assume in an essential way that $n=2$.

In the remainder of this paper we shall assume that $\mathcal{X} = L_p(\Omega,\Sigma,\mu)$ (where (Ω,Σ,μ) is a probability space and $1 \leqslant p \leqslant \infty$), that $\mathcal{K}$ is an $\|\cdot\|_1$-closed convex subset of $\mathcal{X}$, and that $f,g \in L_\infty$. Let d_p be the metric induced by $\|\cdot\|_p$ and let p-b.s.a. and p-b.a. denote d_p-b.s.a. and d_p-b.a., respectively. For $1 \leqslant p \leqslant \infty$, let $\mu_p(f,g;\mathcal{K})$ consist of every p-b.s.a. to f and g from $\mathcal{K}$. If $1 < p < \infty$, then $\mu_p(f,g;\mathcal{K})$ is a singleton [**3**], which we denote by $S_p(f,g)$ or by h_p. We denote $S_p(f,f)$ by f_p.

2. Discrete Simultaneous Approximation

In this section we assume that $\Omega = \{1,2,\dots,n\}$, that $\Sigma = 2^\Omega$, that $\mu(\{i\}) = w_i > 0$ (where $\sum_{i=1}^n w_i = 1$), and that $\mathcal{K}$ is any $\|\cdot\|_1$-closed convex subset of $\mathcal{X} = \mathbb{R}^n$. The underlying norm is the weighted ℓ_p norm, defined by $\|h\|_p = \left(\sum_{i=1}^n w_i\,|h(i)|^p\right)^{1/p}$, for $1 \leqslant p < \infty$, and $\|h\|_\infty = \max_{1\leqslant i\leqslant n}(w_i\,|h(i)|)$.

We begin with a lemma that will be used in compactness arguments.

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Lemma 4. *The set $\mathcal{H} = \{h_p : 1 < p < \infty\}$ is uniformly bounded. Thus, every sequence in $\mathcal{H}$ has a convergent subsequence.*

Proof: Let $z \in \mathcal{K}$ be fixed. For any $p \in (1, \infty)$, $\|h_p\|_p - \|f\|_p \leqslant \|h_p - f\|_p \leqslant d_p(f, g; h_p) \leqslant d_p(f, g; z) \leqslant d_\infty(f, g; z)$ so $\|h_p\|_p \leqslant A := \|f\|_\infty + d_\infty(f, g; z)$ and, for $1 \leqslant i \leqslant n$, $w_i |h_p(i)|^p \leqslant A^p$. Since $w_i \leqslant 1$ and $p > 1$, $w_i |h_p(i)| \leqslant w_i^{1/p} |h_p(i)| \leqslant A$, so

$$\|h_p\|_\infty \leqslant A \max\{w_i^{-1} : 1 \leqslant i \leqslant n\}.$$

The second assertion follows from the fact that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. □

One of our primary concerns is the continuity of h_p as a function of p. The following theorem establishes this continuity on the interval $(1, \infty)$.

Theorem 5. *The function $\Pi : ((1, \infty), |\cdot|) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ defined by $\Pi(p) = h_p$ is continuous.*

Proof: If the theorem is false, then there exist $p \in (1, \infty)$ and $p_k \to p$ such that $\lim_{k\to\infty} \|h_{p_k} - h_p\|_\infty \neq 0$. By (4), $\{h_{p_k}\}$ has a subsequence $\{h_{q_k}\}$ which converges to an element $h^* \neq h_p$. We now show that, to the contrary, it must be that $h^* = h_p$.

Let $\varepsilon > 0$ be given. Since $\lim_{k\to\infty} \|z\|_{q_k} = \|z\|_p$ for every $z \in \mathbb{R}^n$, there exists N_1 such that for every $k \geqslant N_1$ and for $z = f, g$,

$$\|z - h_p\|_p - \varepsilon < \|z - h_p\|_{q_k} < \|z - h_p\|_p + \varepsilon.$$

By the definition of best simultaneous approximation, $d_{q_k}(f, g; h_{q_k}) \leqslant d_{q_k}(f, g; h_p)$ so, for every $k \geqslant N_1$,

$$\text{(i)} \qquad d_{q_k}(f, g; h_{q_k}) \leqslant d_p(f, g; h_p) + \varepsilon.$$

By our assumption, there exists N_2 such that, for every $k \geqslant N_2$, $\|h_{q_k} - h^*\|_\infty < \varepsilon$ and, for $z = f, g$,

$$\text{(ii)} \qquad \begin{aligned} \|z - h^*\|_{q_k} &\leqslant \|z - h_{q_k}\|_{q_k} + \|h_{q_k} - h^*\|_{q_k} \\ &\leqslant \|z - h_{q_k}\|_{q_k} + \eta \|h_{q_k} - h^*\|_\infty \\ &< \|z - h_{q_k}\|_{q_k} + \varepsilon. \end{aligned}$$

Let $N = N_1 \vee N_2$. By (i) and (ii), for every $k \geqslant N$,

$$d_{q_k}(f, g; h^*) < d_{q_k}(f, g; h_{q_k}) + \varepsilon \leqslant d_p(f, g; h_p) + 2\varepsilon,$$

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which implies that $d_p(f,g;h^*) < d_p(f,g;h_p) + 2\varepsilon$. Since ε is arbitrary and since h_p is the unique p-b.s.a. to f and g, it must be that $h^* = h_p$. □

The following corollary is also related to continuity, but includes the endpoints, 1 and ∞. Its proof uses the continuity of $\|z\|_p$ as a function of p and the definition of $d_p(f,g;\mathcal{K})$.

COROLLARY 6. *The function D, defined by $D(p) = d_p(f,g;\mathcal{K})$, is continuous on $[1,\infty]$.*

The following technical lemma will be used in the proof that h_p converges as $p \downarrow 1$.

LEMMA 7. *Either (i) $\|g-h\|_1 \leqslant \|f-h\|_1$ for every h in $\mu_1(f,g;\mathcal{K})$ or (ii) $\|f-h\|_1 \leqslant \|g-h\|_1$ for every h in $\mu_1(f,g;\mathcal{K})$.*

PROOF: Suppose $h', h'' \in \mu_1(f,g;\mathcal{K})$, $\|g-h'\|_1 < \|f-h'\|_1$ and $\|g-h''\|_1 > \|f-h''\|_1$. Let $h^* = (h'+h'')/2$. Then

$$\begin{aligned} d_1(f,g;h^*) &= \|f-(h'+h'')/2\|_1 \vee \|g-(h'+h'')/2\|_1 \\ &\leqslant \frac{1}{2}[(\|f-h'\|_1 + \|f-h''\|_1) \vee (\|g-h'\|_1 + \|g-h''\|_1)] \\ &< \frac{1}{2}[(\|f-h'\|_1 + \|g-h''\|_1) \vee (\|f-h'\|_1 + \|g-h''\|_1)] \\ &\leqslant \frac{1}{2}[(2\,\|f-h'\|_1) \vee (2\,\|g-h''\|_1)] \\ &= d_1(f,g;h'), \end{aligned}$$

a contradiction. □

The proof of the following theorem is modelled after the proof of [**10**, Theorem 2]. Throughout the demonstration, we shall assume without loss of generality that (7i) holds. For $1 \leqslant i \leqslant n$ define $\lambda_i : \mathbb{R}^n \to \mathbb{R}$ by $\lambda_i(h) = h(i) - f(i)$. Let $\mathcal{K}_1 = \mu_1(f,g;\mathcal{K})$. Clearly $\mathcal{K}_1$ is convex. We claim that

$$(*) \qquad \lambda_i \text{ does not change sign on } \mathcal{K}_1 .$$

Indeed, for $x, y \in \mathcal{K}_1$ and $1 \leqslant i \leqslant n$, let $s = x - f$ and $t = y - f$. If $s(i) = a > 0$ and $t(i) = -b < 0$, let $z = (bs+at)/(a+b)$. Then $z(i) = 0$ and, for $k \neq i$, $|z(k)| \leqslant (b\,|s(k)| + a\,|t(k)|)/(a+b)$, so

$$\|z\|_1 < (b\,\|s\|_1 + a\,\|t\|_1)/(a+b) = \|s\|_1 .$$

Let $x^* = (bx+ay)/(a+b)$. Since $\mathcal{K}_1$ is convex, $x^* \in \mathcal{K}$. By the last inequality, $\|x^*-f\|_1 < \|x-f\|_1$, so $d_1(f,g;x^*) = \|x^*-f\|_1 < \|x-f\|_1 = d_1(f,g;x)$. This proves $(*)$.

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Define $\gamma\colon \mathbb{R} \to \mathbb{R}$ by

$$\gamma(r) = \begin{cases} |r| \ln |r|, & r \neq 0, \\ 0, & r = 0. \end{cases}$$

For every $h \in \mathcal{K}$ and $1 \leqslant p < \infty$, let $F_h(p) = \|h - f\|_p^p$ and let

$$\Gamma(h) = F_h'(1) = \sum_{i=1}^{n} |h(i) - f(i)| \ln |h(i) - f(i)|.$$

Since γ is strictly convex on $[0, \infty)$, $(*)$ implies that Γ is strictly convex on $\mathcal{K}_1$ and so has a unique minimiser; call it h_1.

In view of (4), to show that $\lim_{p \downarrow 1} h_p$ exists, it will suffice to exhibit a vector h such that, for every sequence $\{p_k\} \downarrow 1$, $\lim_{k \to \infty} h_{p_k} = h$. The following lemma is a first step in this exhibition.

Lemma 8. *If $\{p_k : k \in \mathbb{N}\} \subset (1, \infty)$, if $p_k \downarrow 1$ and if $\|f - h_{p_k}\|_{p_k} \geqslant \|g - h_{p_k}\|_{p_k}$ for every $k \in \mathbb{N}$, then $\lim_{k \to \infty} h_{p_k} = h_1$.*

Proof: If the lemma is false, then, by (4), there exists a sequence $\{q_k\} \subset \{p_k\}$ such that $q_k \downarrow 1$ and $h_{q_k} \to z \neq h_1$. Then

$$\text{(i)} \qquad \Gamma(z) > \Gamma(h_1).$$

If $r \geqslant 0$, then the function $p \mapsto r^p$ is a convex function so the Mean Value Theorem implies that, for every $p > 1$, $r \ln r \leqslant (r^p - r)/(p - 1)$. Hence

$$\text{(ii)} \qquad \Gamma(h_{q_k}) \leqslant \frac{1}{q_k - 1} \sum_{i=1}^{n} \{|h_{q_i}(i) - f(i)|^{q_k} - |h_{q_k}(i) - f(i)|\}.$$

Since h_{q_k} is a q_k-b.s.a. to f and g from $\mathcal{K}$, we have $\|f - h_{q_k}\|_{q_k} \leqslant d_{q_k}(f, g; h_{q_k}) \leqslant d_{q_k}(f, g; h_1)$. This, along with (7i) gives

$$\text{(iii)} \qquad \|f - h_{q_k}\|_{q_k} \leqslant \|f - h_1\|_{q_k}.$$

Since $h_1 \in \mathcal{K}_1$ and (7i) holds, $\|f - h_1\|_1 = d_1(f, g; h_1) \leqslant d_1(f, g; h_{q_k})$. Since $\|z\|_p$ is a nondecreasing function of p for every z in $\mathbb{R}^n$, $\|f - h_1\|_1 \leqslant d_{q_k}(f, g; h_{q_k}) = \|f - h_{q_k}\|_{q_k}$, that is,

$$\text{(iv)} \qquad \|f - h_1\|_1 \leqslant \|f - h_{q_k}\|_{q_k}.$$

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By (ii), (iii), and (iv),

$$\text{(v)} \qquad \Gamma(h_{q_k}) \leqslant \frac{1}{q_k - 1} \sum_{i=1}^{n} \{|h_1(i) - f(i)|^{q_k} - |h_1(i) - f(i)|\}.$$

As $k \to \infty$, the right hand side of (v) approaches $\Gamma(h_1)$ so $\Gamma(z) \leqslant \Gamma(h_1)$, which contradicts (i), and proves the lemma. ∎

THEOREM 9. *The net $\{h_p : p > 1\}$ converges as $p \downarrow 1$*

PROOF: Suppose first that there is an $\alpha > 1$ such that

$$\text{(i)} \qquad \|f - h_p\|_p < \|g - h_p\|_p, \quad p \in (1, \alpha).$$

In this case, if $p_k \downarrow 1$, then, without loss of generality, $\|f - h_{p_k}\|_{p_k} < \|g - h_{p_k}\|_{p_k}$ for every $k \in \mathbb{N}$, so (2) implies that $h_{p_k} = g_{p_k}$ and, by [10], $h_{p_k} \to g_1$, the natural best ℓ_1-approximation to g from $\mathcal{K}$, and the proof is complete.

Suppose (i) does not hold. Then there exists a sequence $\{p_k\}$ which satisfies the condition in Lemma 8, namely, $p_k \downarrow 1$ and $\|f - h_{p_k}\|_{p_k} \geqslant \|g - h_{p_k}\|_{p_k}$ for every $k \in \mathbb{N}$. If $q_k \downarrow 1$ and $\|f - h_{q_k}\|_{q_k} < \|g - h_{q_k}\|_{q_k}$, let $r_k = \sup\{p < q_k : \|f - h_p\|_p \geqslant \|g - h_p\|_p\}$. We may assume without loss of generality that $\{r_k\} \subset \{p_k\}$. Then, by (5), $r_k < q_k$. By the Intermediate Value Theorem, $\|f - h_p\|_p < \|g - h_p\|_p$ for every p in (r_k, q_k), and, by (2), $h_p = g_p$ for every $p \in (r_k, q_k)$. Thus (5) implies that $\lim_{p \downarrow r_k} g_p = \lim_{p \downarrow r_k} h_p = h_{r_k}$. Since $G(p) = g_p$ is continuous on $(1, \infty)$ (the proof is similar to that of (5)), it must be that $h_{r_k} = g_{r_k}$. From the above considerations, we know that $h_{q_k} \to g_1$ and $h_{r_k} \to h_1$. But $h_{r_k} = g_{r_k} \to h_1$ so $h_{q_k} \to h_1$.

Thus, if (i) does not hold and if $q_k \downarrow 1$, then, without loss of generality, either $\{q_k\} = \{r_k\}$ or $\{q_k\} = \{r_k\} \cup \{s_k\}$, where, for every $k \in \mathbb{N}$, $\|f_{r_k} - h_{r_k}\|_{r_k} \geqslant \|g_{r_k} - h_{r_k}\|_{r_k}$ and $\|f_{s_k} - h_{s_k}\|_{s_k} < \|g_{s_k} - h_{s_k}\|_{s_k}$. Since each of $\{h_{r_k}\}$ and $\{h_{s_k}\}$ converges to h_1, so do $\{h_{q_k}\}$ and the net $\{h_p : p > 1\}$. However, $\sup(f, g; \mathcal{M}) = \overline{h} = \chi_{[0,1/2]} + 2\chi_{(1/2,1]}$ and $\inf(f, g; \mathcal{M}) = \underline{h} = \chi_{(1/2,1]}$ are not in $\mu_1(f, g; \mathcal{M})$. ∎

Combining (6) and (9), we have the following.

COROLLARY 10. *The set $\mu_1(f, g; \mathcal{K})$ is nonempty.*

3. Simultaneous Monotone L_p-Approximation, $p \in [1, \infty]$

In this section we shall assume that $\Omega = [0,1]$, that Σ consists of all Lebesgue measurable subsets of Ω, and that μ is Lebesgue measure. Let $\mathcal{K} = \mathcal{M}$, the set of all nondecreasing extended real-valued functions on Ω and let $f, g \in L_\infty$ have at most discontinuities of the first kind. Let $M = \|f\|_\infty \vee \|g\|_\infty$.

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Lemma 11. *The set $\cup_{p=1}^{\infty}\mu_p(f,g;\mathcal{M})$ is uniformly bounded by M.*

Proof: If $h \in \mu_p(f,g;\mathcal{M})$ but there is a $t \in (0,1)$ such that $h(t) > M$, then there is an $s \in (0,1)$ such that, for every $r > s$, $h(r) > M$. Let $h^* = h \wedge M$. Then $h^* \in \mathcal{M}$ and $d_p(f,g;h^*) < d_p(f,g;h)$, a contradiction. The case $\min h(t) < -M$ is treated similarly. ∎

Lemma 12. *If $1 < p < \infty$ and $\mathcal{H} \subset \mathcal{M}$ is uniformly bounded by B, then there exist $h^k \in \mathcal{H}$ and $h \in \mathcal{M}$ such that $\|h\|_\infty \leqslant B$ and $\lim_{k\to\infty} \|h - h_k\|_p = 0$.*

Proof: By Helly's Theorem [**12**], there exist $h^k \in \mathcal{H}$ and $h \in \mathcal{M}$ such that $\|h\|_\infty \leqslant B$ and $h^k \to h$ pointwise on Ω. Thus, by the Lebesgue Dominated Convergence Theorem, $\{h^i\}$ converges to h in L_p. ∎

In view of (11) we may, and will, assume that $\mathcal{M}$ consists of all nondecreasing functions h such that $\|h\|_\infty \leqslant 2M$. Thus, by (12), $\mathcal{M}$ is a compact subset of L_p for $1 < p < \infty$. By (1), S_p is a $\|\cdot\|_p$-continuous function of f and g. By a proof similar to that of (5), the following result can be obtained. If $q \in (1,\infty)$ then the function $\Pi\colon ((1,q], |\cdot|) \to L_q$ defined by $\Pi(p) = h_p$ is a continuous function of p.

We now undertake to show that $\lim_{p\downarrow 1} h_p$ exists, so that the last result can be extended to $[1,q]$.

Theorem 13. *The net $\{h_p\}$ converges uniformly as $p \downarrow 1$.*

Proof: The length of the proof, and the fact that some of its waystations are of independent interest, warrant its division into several lemmas. We begin by showing that S_p is a monotone operator.

Lemma (i) *Suppose that $f^i, g^i \in L_p$, $i = 1,2$, $1 < p < \infty$. If $f^1 \leqslant f^2$ and $g^1 \leqslant g^2$, then $S_p\, f^1g^1 \leqslant S_p\, f^2g^2$.*

Proof: Let $h^i = S_p\, f^ig^i$, $i = 1,2$, $T_1 = h^1 \wedge h^2$ and $T_2 = h^1 \vee h^2$; let $a_i = |f^i - h^i|$, $b_i = |g^i - h^i|$, $c_i = |f^i - T_i|$ and $d_i = |g^i - T_i|$, $i = 1,2$. By [**11**, Lemma 2],

$$a_2^p + a_1^p \geqslant c_2^p + c_1^p \quad \text{and} \quad b_2^p + b_1^p \geqslant d_2^p + d_1^p,$$

so

$$a_2^p \vee b_2^p \geqslant c_2^p \vee d_2^p \quad \text{or} \quad a_1^p \vee b_1^p \geqslant c_1^p \vee d_1^p.$$

If the first case holds, then upon integrating, we obtain

$$\|f^2 - h^2\|_p \vee \|g^2 - h^2\|_p \geqslant \|f^2 - T_2\|_p \vee \|g^2 - T_2\|_p.$$

Since $S_p\, f^2g^2$ is uniquely defined, $h^2 = T_2 \geqslant h^1$. By similar reasoning, if the second case holds, then $h^1 = T_1 \leqslant h^2$. This completes the proof of (i). ∎

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LEMMA (ii) *For* $1 < p < \infty$ *and* $c \in \mathbb{R}$, $S_p(f+c, g+c) = h_p + c$.

PROOF: By the definition of h_p, we have for all $h \in K$

$$\|f - h_p\|_p \vee \|g - h_p\|_p \leqslant \|f - h\|_p \vee \|g - h\|_p .$$

For any $k \in K$, there exists $h \in K$ such that $h + c = k$, so

$$\begin{aligned}\|f + c - (h_p + c)\|_p \vee \|g + c - (h_p + c)\|_p &\leqslant \|f + c - (h + c)\|_p \vee \|g + c - (h + c)\|_p \\ &= \|f + c - k\|_p \vee \|g + c - k\|_p .\end{aligned}$$

This concludes the proof of (ii). □

LEMMA (iii) If $1 < p < \infty$, if I is an open interval, and if both f and g are constant on I, then $S_p(f, g)$ is constant on I.

PROOF: Let $h = S_p(f, g)$, and let $h'|_I = -h + g + f$, and $h'|_{\Omega \setminus I} = h$. Note that $h'|_I$ is nondecreasing. For notational convenience, we let $\|k - l\| = \left(\int_I |k - l|^p\right)^{1/p}$. Then $\|f - h\| = \|g - h'\|$ and $\|g - h\| = \|f - h'\|$. If $h'' = 2^{-1}(h + h')$ and $d = 2^{-1}(\|f - h\| + \|g - h\|)$, then both $\|f - h''\| \leqslant d$ and $\|g - h''\| \leqslant d$. But this implies that

$$\|f - h'\| \vee \|g - h''\| \leqslant \|f - h\| \vee \|g - h\| .$$

Since $h'' = (g + f)/2$ is constant on I and $h = S_p(f, g)$ it must be that $h'' = h$ so $h' = h$. Thus $h|_I$ is both nondecreasing and nonincreasing, hence constant. This concludes the proof of (iii). □

Since f and g have at most discontinuities of the first kind, they can be uniformly approximated by step functions (see [**19**]). Thus, for any $n \in \mathbb{N}$ there are step functions

$$f^n = a_1 \chi_{[0,t_1]} + \sum_{i=2}^{k_n} a_i \chi_{(t_{i-1}, t_i]}, \tag{iv}$$

and

$$g^n = b_1 \chi_{[0,t_1]} + \sum_{i=2}^{k_n} b_1 \chi_{(t_{i-1}, t_i]}, \tag{v}$$

(where χ_A is the indicator function of A, that is, $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$) such that $\|f - f^n\|_\infty < n^{-1}$ and $\|g - g^n\|_\infty < n^{-1}$, where $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ is the common refinement of the partitions of [0,1] associated with the canonical representations of f^n and g^n. Let $h_p^n = S_p(f_n, g_n)$. By the last lemma, h_p^n must have the form

$$h_p^n = c_1^p \chi_{[0,t_1]} + \sum_{i=2}^{k_n} c_i^p \chi_{(t_{i-1}, t_i]}. \tag{vi}$$

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Thus, we are in the context of weighted discrete simultaneous approximation (where $f^n = \{a_i\}_{i=1}^{k_n}$, $g^n = \{b_i\}_{i=1}^{k_n}$, $h_p^n = \{c_i^p\}_{i=1}^{k_n}$ and $w_i = t_i - t_{i-1}$) so, by (9), there are numbers c_i^1, $1 \leqslant i \leqslant k_n$, such that

(vii) $$\lim_{p\downarrow 1} h_p^n = h_1^n = c_1^1\chi_{[0,t_1]} + \sum_{i=2}^{k_n} c_i^1\chi_{(t_{i-1},t_i]}.$$

LEMMA (viii) Let f^n, g^n, h_p and h_p^n be as defined above. Let h_p be the best L_p-simultaneous approximation to f and g from $\mathcal{M}$. Then for every $\varepsilon > 0$, there exists an $N = N(f,g,\varepsilon)$ such that for all $n \geqslant N$ and $p \in (1,\infty)$, $\left\|h_p^n - h_p\right\|_\infty < \varepsilon$.

PROOF: Let $\varepsilon > 0$ be given. Then there is an integer $N \geqslant 1$ such that $\|f - f^n\|_\infty < \varepsilon$ and $\|g - g^n\|_\infty < \varepsilon$ for all $n \geqslant N$. Thus, except on a set of measure zero, $n \geqslant N$ implies that

(ix) $$f^n < f + \varepsilon, \quad g^n < g + \varepsilon$$

and

(x) $$f < f^n + \varepsilon, \quad g < g^n + \varepsilon.$$

Applying (i) and (ii) to (ix) and (x) respectively, we obtain

$$h_p^n < h_p + \varepsilon, \quad \text{and} \quad h_p < h_p^n + \varepsilon,$$

which implies that $\left\|h_p^n - h_p\right\|_\infty < \varepsilon$.

We are now in a position to complete the proof of Theorem 13. Let $\varepsilon > 0$ be given. Then there exists $N \geqslant 1$ such that $\|f^n - f^m\|_\infty < \varepsilon$, and $\|g^n - g^m\|_\infty < \varepsilon$ for all $n, m \geqslant N$. An argument similar to that in the last proof shows that there exists an $N = N(f,g,\varepsilon)$ such that for every $n,m \geqslant N$ and $p \in (1,\infty)$, $h_p^n < h_p^m + \varepsilon$ and $h_p^m < h_p^n + \varepsilon$. Letting $p \downarrow 1$, we obtain

(xi) $$\|h_1^n - h_1^m\|_\infty < \varepsilon, \quad n,m \geqslant N.$$

Hence $\{h_1^n : n = 1,2,\dots\}$ converges uniformly to, say, h_1. Since the values of N in (viii) and (xi) are independent of p, (vii), (vii) and (xi) and the triangle inequality imply that h_p converges uniformly to h_1 as $p \downarrow 1$. This concludes the proof of Theorem 13. ∎

Let $h_1 = \lim_{p\downarrow 1} h_p$ and define $S_1(f,g) := h_1$. Applying a version of (6), we have that $h_1 \in \mu_1(f,g;\mathcal{M})$. This proves the following:

COROLLARY 14. *The set $\mu_1(f,g;\mathcal{M})$ is nonempty.*

We end this section with a discussion of the inheritance of the continuity of f and g by h_p. The theorem below is presented in [8], but is included here also for self-containment. We refer the reader to [1] for the definition of *approximate* continuity.

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THEOREM 15. *If f and g are approximately continuous and $p \in (1,\infty)$, then h_p is continuous on (0,1).*

PROOF: Suppose for contradiction that h_p has a jump discontinuity at $a \in (0,1)$. We may assume without loss of generality that $g(a) \leqslant f(a)$.

We may approximate the above functions by step functions. Indeed, let $\sigma = g(a)$, $\tau = f(a)$, $\lambda = h_p(a^-) = \lim_{t\uparrow a} h_p(t)$ and $\mu = h_p(a^+)$ and suppose that $\alpha > 0$. By Lemma 9, there exists an $\eta \in [\lambda,\mu]$ and $\varepsilon = \varepsilon(\alpha) > 0$ such that

$$\begin{aligned}\max\{\alpha(|\tau-\mu|^p + |\tau-\lambda|^p), \alpha(|\lambda-\sigma|^p + |\mu-\sigma|^p)\}\\ = \max\{2\alpha\,|\tau-\eta|^p\,, 2\alpha\,|\eta-\sigma|^p\} + \varepsilon.\end{aligned}$$

If α is replaced by a multiple of α in the last equality, then ε is replaced by the same multiple of ε. Thus there exists a $K > 0$ such that $\varepsilon(\alpha) = K\alpha$. Hence

$$\begin{aligned}\max\{|\tau-\mu|^p + |\tau-\lambda|^p\,, |\lambda-\sigma|^p + |\mu-\sigma|^p\}\\ = \max\{2\,|\tau-\eta|^p\,, 2\,|\eta-\sigma|^p\} + K.\end{aligned}$$

Let $h_p^r(t) = h_p(t)$ if $t > a$ and $h_p^r(t) = \mu$ if $t \leqslant a$, and define h_p^ℓ similarly, with reversed inequalities. Then each of h_p^r and h_p^ℓ is continuous at a so, by [**1**, Theorem 5.4] each of $|h_p^j - k|^p$, $j = r,l$, $k = f,g$, is approximately continuous at a. By [**1**, Theorem 8.2]

$$\lim_{\delta\to 0} \delta^{-1}\int_a^{a+\delta} |h_p^r - k|^p = |h_p^r(a) - k(a)|^p\,, \quad k = f,g,$$

and similar statements hold for h_p^ℓ, with integration from $a-\delta$ to a. Since $K > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}\max\left\{\delta^{-1}\int_{a-\delta}^{a+\delta} |h_p - f|^p\,, \delta^{-1}\int_{a-\delta}^{a+\delta} |h_p - g|^p\right\}\\ > \max\left\{\delta^{-1}\int_{a-\delta}^{a+\delta} |\eta - f|^p\,, \delta^{-1}\int_{a-\delta}^{a+\delta} |\eta - g|^p\right\}.\end{aligned}$$

If h_p^* is defined by

$$h_p^* = \begin{cases} \eta, & t \in [a-\delta, a+\delta)\,, \\ h_p(t), & \text{otherwise}, \end{cases}$$

then h_p^* is a better simultaneous L_p approximation to f and g than is h_p. ∎

If f and g are continuous, then they are quasi-continuous and approximately continuous both, so, by (13) and (15),

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Corollary 16. *If f and g are continuous, then so is h_1.*

Example (19) in Section 4 shows that not all members of $\mu_1(f,g;\mathcal{M})$ preserve the continuity of f and g. As a consequence of (3) and (13) above, we have the following.

Corollary 17. *Suppose $p \in [1,\infty)$. If $h_p \neq f_p$, then $\|f-h_p\|_p \leqslant \|g-h_p\|_p$. If $f_p \neq h_p \neq g_p$, then $\|f-h_p\|_p = \|g-h_p\|_p$.*

4. Simultaneous Monotone L_1-Approximation

The structure of the set of best simultaneous monotone L_1 approximations to an arbitrary pair of functions (f,g) is of intrinsic interest. In [**6**, **7**], assuming $f=g$, this set was completely characterised in terms of f, and in [**9**], the continuity of the multifunction $f \mapsto \mu_1(f;M)$ was studied. In this section, we present some related results in the context where f and g are not necessarily the same.

Lemma 18. *Let f and g be step functions defined over the same partition of $[0,1]$. Then there exists an element $h \in \mu_1(f,g;\mathcal{M})$ such that h is a step function of the same form as f and g.*

Proof: Let f_i and g_i be the values of f and g on the subinterval $(t_{i-1},t_i]$. Assume without loss of generality that $g_i < f_i$. Let $h \in \mu_1(f,g;\mathcal{M})$. If h is not a constant on $(t_{i-1},t_i]$, then clearly $g_i \leqslant h(x) \leqslant f_i$ for all $x \in (t_{i-1},t_i]$, otherwise both of $\|f-h\|_1$ and $\|g-h\|_1$ can be reduced simultaneously and h would not be an element of $\mu_1(f,g;\mathcal{M})$ any more. Now, we seek a constant $c \in [g_i,f_i]$ such that

$$\int_{t_{i-1}}^{t_i} (f_i - h(x))dx = \int_{t_{i-1}}^{t_i} (f_i - c)dx$$

and

$$\int_{t_{i-1}}^{t_i} (h(x) - g_i)dx = \int_{t_{i-1}}^{t_i} (c - g_i)dx.$$

But it is clear now that c is given by

$$c = (t_i - t_{i-1})^{-1} \int_{t_{i-1}}^{t_i} h(x)dx.$$

This completes the proof. ∎

Thus, for any pair of step functions f and g, there always exists a step function $h \in \mu_1(f,g;\mathcal{M})$. Clearly, such a step function is not necessarily unique. This will be shown as part of the next example.

In [5], it was shown that the set of best L_1-approximations to a bounded measurable function f by nondecreasing functions includes its supremum and infimum. However, this is not the case with $\mu_1(f,g;\mathcal{M})$.

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EXAMPLE 19. Take $f \equiv 2$ and $g \equiv 0$ on $[0,1]$. Then any function h_c of the form

$$h_c(x) = \begin{cases} c, & 0 \leqslant x \leqslant 1/2 \\ 2-c, & 1/2 < x \leqslant 1, \end{cases}$$

$c \in [0,1]$, is an element of $\mu_1(f,g;\mathcal{M})$, so $\overline{h} := \sup(f,g;\mathcal{M}) \geqslant \chi_{[0,1/2]} + 2\chi_{(1/2,1]}$ and $\underline{h} := \inf(f,g;\mathcal{M}) \leqslant \chi_{(1/2,1]}$. Thus $d_1(f,g;\overline{h}) \geqslant 3$, so $\overline{h} \notin \mu_1(f,g;\mathcal{M})$. Similarly, $\underline{h} \notin \mu_1(f,g;\mathcal{M})$. Also notice that if $h^*(x) = 2x$, then $h^* \in \mu_1(f,g;M)$.

This example shows also that the fact that both of f and g are constants doesn't imply that every element of $\mu_1(f,g;\mathcal{M})$ must be also a constant, or even a step function as is the case with $h^*(x) = 2x$. It also demonstrates the fact that continuity is not inherited from f and g by all elements of $\mu_1(f,g;\mathcal{M})$.

Next, one might ask about the relation between the set of best L_1-simultaneous approximations to a pair of functions f and g, and the set of best L_1-approximations to the mean of this pair of functions. In [**13**], it was shown that h^* is the best L_2-simultaneous approximation to two functions f and g *if and only if* h^* is the best L_2-approximation to their mean $T = (f+g)/2$, provided we define h^* as the element satisfying

$$\inf_{h\in\mathcal{M}} [\|f-h\|_2^2 + \|g-h\|_2^2]^{1/2} = \left(\|f-h^*\|_2^2 + \|g-h^*\|_2^2\right)^{1/2}.$$

This motivates us to raise a similar question for our case of best L_1-simultaneous approximation. Is $\mu_1(f,g;\mathcal{M}) \cap \mu_1(T;\mathcal{M}) \neq \emptyset$ for any pair of functions f and g; for a special pair of functions, such as continuous functions? How about if $\mu_1(T;\mathcal{M})$ is a singleton? The following example answers these questions.

EXAMPLE 20. Let $f(x) = 3-2x$ and $g(x) = 1-4x$. Then $T(x) = (1/2)(f(x)+g(x)) = 2-3x$. Clearly $T_1 \equiv 1/2$ is the unique best L_1-approximation to T by elements of $\mathcal{M}$. However $T_1 \notin \mu_1(f,g;\mathcal{M})$. Take for example $h^* \equiv 29/60 \in \mathcal{M}$. Then $d_1(f,g;h^*) < d_1(f,g;T_1)$.

However, the following lemma gives us a condition which guarantees that $\mu_1(f,g;\mathcal{M}) \subseteq \mu_1((1/2)(f+g);\mathcal{M})$.

LEMMA 21. *If* $d_1((1/2)(f+g);\mathcal{M}) \geqslant d_1(f,g;\mathcal{M})$, *then* $\mu_1(f,g;\mathcal{M}) \subseteq \mu_1((1/2)(f+g);\mathcal{M})$.

PROOF: In general, we have for any $h \in \mu_1(f,g;\mathcal{M})$

$$\begin{aligned} d^* = d_1((1/2)(f+g);M) &\leqslant (1/2)\,\|(f-h)+(g-h)\|_1 \\ &\leqslant \max(\|f-h\|_1, \|g-h\|_1) = d_1(f,g;h) = d_1. \end{aligned}$$

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So we obtain equality in the given condition of the theorem. Now, let $h_1 \in \mu_1(f,g;\mathcal{M})$, and suppose $\|f-h_1\|_1 \geqslant \|g-h_1\|_1$. Then

$$\begin{aligned} d_1 = \|f-h_1\|_1 &\geqslant (1/2)(\|f-h_1\|_1 + \|g-h_1\|_1) \\ &\geqslant \|(1/2)(f+g)-h_1\|_1 \geqslant d^* = d_1. \end{aligned}$$

Hence $h_1 \in \mu_1((1/2)(f+g);\mathcal{M})$. □

Suppose the hypothesis of (21) holds. Then $\mu_1(f,g;M) = \mu_1((1/2)(f+g);\mathcal{M})$ if $\mu_1((1/2)(f+g);\mathcal{M})$ is a singleton. This occurs when both of f and g are continuous or approximately continuous (see [**2**]). Even with the assumption of uniqueness of the best L_1-approximation to the mean $(1/2)(f+g)$, the converse of the lemma is still not true in general. The following example illustrates this fact.

EXAMPLE 22. Let $f(x) = x^2 - 1$ on $[-1,1]$ and let $g = -f$. Then

$$\mu_1(f,g;\mathcal{M}) = \mu_1((1/2)(f+g);\mathcal{M}) = (1/2)(f+g) \equiv 0.$$

However

$$d^* = d_1((1/2)(f+g);\mathcal{M}) = 0 < 2/3 = d_1 = d_1(f,g;\mathcal{M}).$$

The condition that $d^* = d_1$ is very vital. To see this, we go back to the two functions f and g given in Example (20) above. There we find that the set $\mu_1(f,g;\mathcal{M})$ consists of a single element, namely $h_1 \equiv 2\sqrt{3}-3$. However

$$\begin{aligned} d_1 = \|f-h_1\|_1 = \|g-h_1\|_1 &= 5-2\sqrt{3} \\ &> 3/4 = d^* = d_1((1/2)(f+g);h^*), \end{aligned}$$

where $h^* \equiv 1/2$ is the unique best L_1-approximation to $(1/2)(f+g)$.

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