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ON THE CESÀRO-PERRON INTEGRAL

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In the present paper a simple proof of a theorem of Sargent on $C_\lambda P$-integral of Burkill is given.

1. INTRODUCTION

Sargent [3] has defined the $C_\lambda D$-integral (λ being a non-negative integer) and has shown that the $C_\lambda D$-integral is equivalent to the $C_\lambda P$-integral of Burkill [1]. But there is a defect in the proof of the following theorem:

THEOREM 1.1. *(Theorem VIII, Sargent [3], p.237). If f is $C_\lambda P$-integrable on $[a, b]$, then f is $C_\lambda D$-integrable on $[a, b]$ and*

$$(C_\lambda D)\int_a^b f = (C_\lambda P)\int_a^b f.$$

(For definitions of $C_\lambda D$-integrable and $C_\lambda D$-integrable, see Section 2.)

Verblunsky [5] has given a correct proof of this theorem. But his proof is very long and difficult. Here we give a simple and short proof.

We use the notation $|E|$ for the Lebesgue outer measure of a set E and f' for the derivative of the function f.

2. PRELIMINARIES

Let the real valued function F be $C_{\lambda-1}P$-integrable ($\lambda \geqslant 1$) on $[a, b]$.

DEFINITION 2.1: (Burkill [1], p.541). The λth Cesàro mean of F on $[a, b]$, $C_\lambda(F, a, b)$ is defined as follows:

$$C_\lambda(F, a, b) = \frac{\lambda}{(b-a)^\lambda}(C_{\lambda-1}P)\int_a^b (b-t)^{\lambda-1}F(t)dt.$$

If $\lambda = 0$, then $C_0(F, a, b)$ is defined to be equal to $F(b)$.

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DEFINITION 2.2: (Burkill [1], p.542). The function F is said to be C_λ-continuous at x if

$$\lim_{h\to 0} C_\lambda(F, x, x+h) = F(x).$$

DEFINITION 2.3: (Burkill [1], p.542). The upper right C_λ-derivate of F at x, $C_\lambda D^+F(x)$, is defined as follows:

$$C_\lambda D^+F(x) = \limsup_{h\to 0+} \frac{C_\lambda(F, x, x+h) - F(x)}{h/(\lambda+1)}.$$

The other derivates $C_\lambda D_+F(x)$, $C_\lambda D^-F(x)$, $C_\lambda D_-F(x)$ have the corresponding definitions. The upper and lower C_λ-derivatives $\overline{C_\lambda D}F(x)$, $\underline{C_\lambda D}F(x)$ are defined to be $\max\{C_\lambda D^+F(x), C_\lambda D^-F(x)\}$ and $\min\{C_\lambda D_+F(x), C_\lambda D_-F(x)\}$ respectively. If

$$\overline{C_\lambda D}F(x) = \underline{C_\lambda D}F(x),$$

then F is said to have a C_λ-derivative $C_\lambda DF(x)$, equal to their common value.

DEFINITION 2.4: (Sargent [3], p.221.) The function F is said to be AC^*(C_λ-sense) above over a set $E \subset [a, b]$ if to every $\varepsilon > 0$, there exists a positive number δ such that for every set of non-overlapping open intervals $\{(a_r, b_r)\}$ having end points in E with

$$\sum_r (b_r - a_r) < \delta,$$

the relations

(1) $$\sum_r \sup_{a_r<x<b_r} \{C_\lambda(F, a_r, x) - F(a_r)\} < \varepsilon$$

and

(2) $$\sum_r \sup_{a_r<x<b_r} \{F(b_r) - C_\lambda(F, b_r, x)\} < \varepsilon$$

hold.

If in the above definition the relations (1) and (2) are replaced by (1′) and (2′) as follows:

(1′) $$\sum_r \inf_{a_r<x<b_r} \{C_\lambda(F, a_r, x) - F(a_r)\} > -\varepsilon$$

(2′) $$\sum_r \inf_{a_r<x<b_r} \{F(b_r) - C_\lambda(F, b_r, x)\} > -\varepsilon$$

then F is said to be AC^*(C_λ-sense) below on $E \subset [a, b]$.

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If F is both $AC^*(C_\lambda\text{-sense})$ above and $AC^*(C_\lambda\text{-sense})$ below over $E \subset [a, b]$, then F is said to be $AC^*(C_\lambda\text{-sense})$ over E.

DEFINITION 2.5: (See Sargent [3], p.222.) The function F is said to be $ACG^*(C_\lambda\text{-sense})$ on $[a, b]$ if $[a, b]$ is expressible as the union of a countable number of sets over each of which F is $AC^*(C_\lambda\text{-sense})$.

DEFINITION 2.6: (Verblunsky [5], p.326.) The function F defined on a set E is said to be VB_λ^* on E, if there exists a constant K such that for any set of non-overlapping intervals $\{(a_i, b_i)\}$ whose end points are in E,

$$\sum_i \sup_{a_i<x<b_i} |C_\lambda(F, a_i, x) - F(a_i)| + \sum_i \sup_{a_i<x<b_i} |F(b_i) - C_\lambda(F, b_i, x)| < K.$$

DEFINITION 2.7: (Saks [2], p.224). A function G is said to fulfil the Lusin's condition (N) on a set E, if for every set $H \subset E$ of measure zero, $G(H)$ is a set of measure zero.

DEFINITION 2.8: (Burkill [1], p.548). Let f be an extended real valued function on $[a, b]$.

Then M is said to be a $C_\lambda P$-major function of f on $[a, b]$ if

(i) M is C_λ-continuous on $[a, b]$,
(ii) $M(a) = 0$,
(iii) $\underline{C_\lambda D}M(x) > -\infty$ for all $x \in [a, b]$,
(iv) $\underline{C_\lambda D}M(x) \geqslant f(x)$ for all $x \in [a, b]$.

In a similar manner, a $C_\lambda P$-minor function m of f on $[a, b]$ is defined.

The function f is said to be $C_\lambda P$-integrable on $[a, b]$ if

(i) it has at least one $C_\lambda P$-major function M and at least one $C_\lambda P$-minor function m and
(ii) $\inf\{M(b)\} = \sup\{m(b)\}$.

If f is $C_\lambda P$-integrable on $[a, b]$, the common value $\inf\{M(b)\} = \sup\{m(b)\}$ is called the $C_\lambda P$-integral of the function f on $[a, b]$ and is denoted by

$$(C_\lambda P)\int_a^b f.$$

DEFINITION 2.9: (Sargent [3], p.232). The function f is said to be $C_\lambda D$-integrable on $[a, b]$ if there exists a function F on $[a, b]$ such that

(i) F is C_λ-continuous,
(ii) F is $ACG^*(C_\lambda\text{-sense})$

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and

(iii) $C_\lambda DF(x) = f(x)$ almost everywhere.

Theorem 2.1. *(Lemma 3, Verblunsky [5], p.328). If f is $C_\lambda P$-integrable on $[a, b]$ and*

$$F(x) = (C_\lambda P)\int_a^x f,$$

then $[a, b]$ is the union of closed sets on each of which F is VB_λ^*.

Theorem 2.2. *(Theorem 2, Sargent [4], p.120). If F is C_λ-continuous on $[a, b]$, $F(a) > 0$ and $F(b) < 0$, then there is a point $c \in (a, b)$ such that $F(c) = 0$.*

Theorem 2.3. *(Theorem 6.5, Saks [2], p.227). If a function G is derivable at every point of a measurable set D, then*

$$|G(D)| \leqslant \int_D |G'|.$$

Theorem 2.4. *(Theorem II, Sargent [3], p.226). For F to be $AC^*(C_\lambda$-sense) over a bounded closed set Q with complementary intervals $\{(a_n, b_n)\}$, it is necessary and sufficient that F should be absolutely continuous over Q and $C_{\lambda-1}D$-integrable on each interval (a_n, b_n), while*

$$\sum_{n=1}^{\infty} \sup_{a_n<x<b_n} |C_\lambda(F, a_n, x) - F(a_n)| < \infty$$

$$\sum_{n=1}^{\infty} \sup_{a_n<x<b_n} |C_\lambda(F, b_n, x) - F(b_n)| < \infty.$$

3. Proof of Theorem 1.1

Let f be $C_\lambda P$-integrable on the closed interval $[a, b]$ with

$$F(x) = (C_\lambda P)\int_a^x f.$$

It is sufficient to prove that F is $ACG^*(C_\lambda$-sense) on $[a, b]$. We first show that F satisfies Lusin's condition (N).

Consider a set $E \subset [a, b]$ with $|E| = 0$. For arbitrary $\varepsilon > 0$, let M and m be a pair of $C_\lambda P$-major and minor functions of f on $[a, b]$ with $H(b) < \varepsilon$ where $H = M - m$. For every natural number n, let E_n denote the set of points x of E such that

$$\frac{\lambda+1}{h}\left[C_\lambda(M, x, x+h) - M(x)\right] > -n,$$

$$\frac{\lambda+1}{h}\left[C_\lambda(m, x, x+h) - m(x)\right] < n,$$

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whenever $0 < |h| \leqslant 1/n$. Then the sequence $\{E_n\}$ is expanding and $E = \bigcup_{n=1}^{\infty} E_n$. Again $E_n = \bigcup_{i=-\infty}^{\infty} E_n^i$, where

$$E_n^i = E_n \cap [\frac{i}{n}, \frac{i+1}{n}].$$

It is easy to show that if $\{(a_k, b_k)\}$ is a sequence of pairwise disjoint open intervals having end points in E_n^i with

$$\sum_k (b_k - a_k) < \frac{(\lambda+1)\varepsilon}{n2^{|i|}},$$

then
$$\sum_k \sup_{a_k<x<b_k} |C_\lambda(F, a_k, x) - F(a_k)| + \sum_k \sup_{a_k<x<b_k} |C_\lambda(F, b_k, x) - F(b_k)|$$
$$< 2\left[\frac{\varepsilon}{2^{|i|}} + \sum_k \{H(b_k) - H(a_k)\}\right],$$

and hence

$$\sum_k |F(b_k) - F(a_k)| < 2\left[\frac{\varepsilon}{2^{|i|}} + \sum_k \{H(b_k) - H(a_k)\}\right]. \tag{3}$$

Since $|E_n^i| = 0$, there exists a sequence $\{I_k^i\}$ of pairwise disjoint open intervals contained in $[1/n, (i+1)/n]$ such that $\bigcup_k I_k^i$ covers $E_n^i \cap (i/n, (i+1)/n)$ and

$$\sum_k |I_k^i| < \frac{(\lambda+1)\varepsilon}{n2^{|i|}}.$$

Then
$$|F(E_n^i)| \leqslant \sum_k |F(E_n^i \cap I_k^i)|.$$

We write $I_k^i = (a_k^i, b_k^i)$. Since $|F(E_n^i \cap I_k^i)|$ cannot exceed the oscillation of F on $E_n^i \cap I_k^i$ and since H is non-decreasing, from (3) it follows that

$$|F(E_n^i)| < 2\left[\frac{\varepsilon}{2^{|i|}} + \sum_k \{H(b_k^i) - H(a_k^i)\}\right].$$

Therefore
$$\begin{aligned} |F(E_n)| &\leqslant \sum_{i=-\infty}^{\infty} |F(E_n^i)| \\ &\leqslant 2\left[3\varepsilon + \sum_{i=-\infty}^{\infty} \sum_k \{H(b_k^i) - H(a_k^i)\}\right] \\ &\leqslant 6\varepsilon + 2\{H(b) - H(a)\} \\ &< 8\varepsilon. \end{aligned}$$

Since $\{E_n\}$ is expanding, it follows that

$$|F(E)| = \left|\bigcup_{n=1}^{\infty} F(E_n)\right| = \lim_n |F(E_n)| \leqslant 8\varepsilon,$$

and hence $|F(E)| = 0$. Thus F satisfies Lusin's condition (N).

Next we use Theorem 2.1, by which $[a, b]$ is the union of closed sets Q_n on each of which F is VB^*_λ. We now fix Q_n. Let $[c, d]$ be the smallest closed interval containing Q_n and let $\{(c_r, d_r)\}$ be the complementary intervals of Q_n. Since F is VB^*_λ on Q_n, it is VB on it and

$$(4) \qquad \sum_r \sup_{c_r<x<d_r} |C_\lambda(F, c_r, x) - F(c_r)| + \sum_r \sup_{c_r<x<d_r} |C_\lambda(F, d_r, x) - F(d_r)| < \infty.$$

Let $G(x) = F(x)$ on Q_n and linear on each closed interval $[c_r, d_r]$. Then G is VB on $[c, d]$ and hence G' exists finitely almost every where on $[c, d]$. Since F is C_λ-continuous on $[a, b]$, G is so on $[c, d]$. For any interval $[\alpha, \beta] \subset [c, d]$, let

$$D = \{x \in [\alpha, \beta] : G'(x) \text{ exists finitely}\}$$

and

$$H = [\alpha, \beta] - D.$$

Then $|H| = 0$. Again since F satisfies Lusin's condition (N) on $[a, b]$, G does so on $[c, d]$ and hence $|G(H)| = 0$. Now

$$\begin{aligned} |G(\beta) - G(\alpha)| &\leqslant |G[\alpha, \beta]| \qquad \text{(by Theorem 2.2)} \\ &\leqslant |G(H)| + |G(D)| \\ &= |G(D)| \\ &\leqslant \int_\alpha^\beta |G'| \qquad \text{(by Theorem 2.3).} \end{aligned}$$

This implies that G is absolutely continuous on $[c, d]$. Hence F is absolutely continuous on Q_n. Therefore by (4) and Theorem 2.4, F is $AC^*(C_\lambda\text{-sense})$ on Q_n. Thus F is $ACG^*(C_\lambda\text{-sense})$ on $[a, b]$. This completes the proof. □

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