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RESEARCH ARTICLE

The dual pair $\mathrm{Aut}(C) \times F_4$ (p-adic case)

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Abstract
We study the local theta correspondence for dual pairs of the form $\mathrm{Aut}(C) \times F_4$ over a p-adic field, where C is a composition algebra of dimension 2 or 4, by restricting the minimal representation of a group of type E. We investigate this restriction through the computation of maximal parabolic Jacquet modules and the Fourier–Jacobi functor.

As a consequence of our results, we prove a multiplicity one result for the $\mathrm{Spin}(9)$-invariant linear functionals of irreducible representations of F_4 and classify the $\mathrm{Spin}(9)$-distinguished representations.

Contents

1. Introduction

Let F be a p-adic field, that is, a nonarchimedean local field of characteristic 0 and residual characteristic $p > 0$. We study the local theta correspondence for the group of F-points of the dual pair $\mathrm{Aut}(C) \times F_4$, where C is a composition F-algebra of dimension 2 or 4, by restricting the minimal representation $(\Pi, \mathcal{V})$ of an adjoint group $\mathcal{G}_C$ of absolute type E_6 if $\dim C = 2$, and absolute type E_7 if $\dim C = 4$. For this introduction we specialize to $\dim(C) = 4$, for simplicity. In this case, the group $\mathcal{G}_C$ is split when C is split and the unique nonsplit form when C is anisotropic.

With a dual pair $\mathrm{Aut}(C) \times F_4 \subset \mathcal{G}_C$ one can lift representations from $\mathscr{G} = \mathrm{Aut}(C)(F)$ to $G = F_4(F)$ as follows. Given $\tau \in \mathrm{Irr}(\mathscr{G})$ a smooth irreducible representation of $\mathscr{G}$, the maximal τ-isotypic quotient of $\mathcal{V}$ admits an action of G and factors as $\tau \otimes \Theta(\tau)$, where $\Theta(\tau)$ is a smooth representation of G. The representation $\Theta(\tau)$ is called the big theta lift of τ. Its maximal semisimple quotient $\theta(\tau)$ (co-socle) is called the small theta lift of τ. Note that one may reverse the roles of $\mathscr{G}$ and G. The primary objective of this paper is to investigate the big and small theta lifts of the dual pair $\mathrm{Aut}(C) \times F_4 \subset \mathcal{G}_C$.

We begin by discussing the theta lift from $\mathrm{Aut}(C)$ to F_4. Our first theorem gives a qualitative behavior of the lift. It is a combination of Theorems 4.10, 6.2, and 6.3.

Theorem 1.1. *Let* $\tau \in \mathrm{Irr}(\mathrm{Aut}(C))$. *Then:*

1. $\Theta(\tau) \neq 0$ *and it is a finite-length representation of* F_4.
2. *If* τ *is tempered then* $\Theta(\tau)$ *is irreducible.*
3. *If* $\theta(\tau) \cong \theta(\tau')$, *where* $\tau' \in \mathrm{Irr}(\mathrm{Aut}(C))$, *then* $\tau \cong \tau'$.

For lifting in the opposite direction, that is, from F_4 to $\mathrm{Aut}(C)$, our main result is Theorem 7.4. It says if $\sigma \in \mathrm{Irr}(F_4)$ such that $\Theta(\sigma) \neq 0$, then $\Theta(\sigma) \in \mathrm{Irr}(\mathrm{Aut}(C))$.

For our second theorem, we specialize to the case where C is the algebra of 2×2 matrices, so $\mathrm{Aut}(C) = \mathrm{PGL}_2$. In this case, we can completely describe $\Theta(\tau)$. In order to state the results, we note that the F_4 group in this paper is not realized as a Chevalley group but as the group of automorphisms of a 27-dimensional exceptional Jordan algebra J. Thus F_4 acts on the 26-dimensional subspace J^0 of trace 0 elements in J and its maximal parabolic subgroups can be described as stabilizers of singular subspaces of J_0 [2]. In particular, F_4 has maximal parabolic subgroups Q and Q_2 stabilizing one- and two-dimensional singular spaces, respectively. (We record that Levi subgroups of Q and Q_2 have the type B_3 and $A_{2,\mathrm{long}} \times A_{1,\mathrm{short}}$, respectively.) Observe that Q and Q_2, via their actions on the stabilized one- and two-dimensional singular spaces, have quotients isomorphic to GL_1 and GL_2, respectively. In particular, a character χ of $\mathrm{GL}_1(F)$ defines a degenerate principal series representation $\mathrm{Ind}_Q^G(\chi)$, and a supercuspidal representation τ of $\mathrm{PGL}_2(F)$ defines a family of degenerate principal series representations $\mathrm{Ind}_{Q_2}^G(\tau \otimes |\det|^s)$.

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Theorem 1.2. *Let $\tau \in \mathrm{Irr}(\mathrm{PGL}_2(F))$.*

1. *If τ is a quotient of a principal series $\mathrm{Ind}_{\mathscr{B}}^{\mathscr{G}}(\chi)$ then $\Theta(\tau)$ is a quotient of $\mathrm{Ind}_Q^G(\chi)$. (For the precise statement see Theorems 6.2 and 6.3 and Proposition 6.4.)*
2. *If τ is a supercuspidal representation, then $\Theta(\tau) = \theta(\tau)$ is the unique irreducible quotient of $\mathrm{Ind}_{Q_2}^G(\tau \otimes |\det|^{3/2})$.*

In (1) $\theta(\tau)$ is always isomorphic to the co-socle of the degenerate principal series. Since the co-socle of $\mathrm{Ind}_Q^G(|\cdot|^{5/2})$ is a sum of two irreducible representations the theta correspondence is not one to one, and this is the only place where it fails.

Next we want to highlight some consequences of our results, as they relate to the relative Langlands program of Sakellaridis–Venkatesh [25]. We prove that the rank one exceptional symmetric pair (F_4, Spin_9) over a p-adic field is a Gelfand pair, a long-time open problem:

Theorem 1.3. *Let $\sigma \in \mathrm{Irr}(F_4)$. Then $\dim\mathrm{Hom}_{\mathrm{Spin}(9)}(\tilde{\sigma}, \mathbb{C}) \leq 1$. Moreover, the dimension is 1 if and only if σ is the theta lift of a generic representation of $\mathrm{PGL}_2(F)$.*

The study of symmetric spaces, and more generally spherical spaces, has a long history. In [29] van Dijk proved that real forms of the symmetric pair $(F_4, \mathrm{Spin}(9))$ are generalized Gelfand pairs, a slightly weaker statement, as it concerns unitary representations only. Recently Rubio [23] proved that $(F_4, \mathrm{Spin}(9))$ is a Gelfand pair over $\mathbb{C}$. The usual approach involves invariant distributions, see [10] or [1] for more information on this rich subject. On the other hand, Howe [11] used the dual pair $\mathrm{SL}_2 \times \mathrm{O}(n)$ to analyze the symmetric pair $(\mathrm{O}(n), \mathrm{O}(n-1))$. It was observed in [26] that Howe's strategy can be applied to all rank-one symmetric pairs. In this paper, at long last, we execute this strategy for the exceptional symmetric pair. Theorem 1.3 is a consequence of the fact that the theta correspondence relates the $\mathrm{Spin}(9)$-period on representations of F_4 to the Whittaker period on representations of $\mathrm{PGL}_2(F)$. More precisely, we have

$$\mathrm{Hom}_{\mathrm{Spin}(9)}(\tilde{\sigma}, \mathbb{C}) \cong \mathrm{Hom}_{\mathscr{U},\psi}(\Theta(\sigma), \mathbb{C})$$

where $(\mathscr{U}, \psi)$ is a Whittaker datum for $\mathrm{PGL}_2(F)$. Since we proved that $\Theta(\sigma)$ is irreducible (or zero) Theorem 1.3 follows from uniqueness of the Whittaker functional for irreducible representations of $\mathrm{PGL}_2(F)$. Moreover, since the lift from $\mathrm{PGL}_2(F)$ is completely known by Theorem 1.2, we have a classification of $\mathrm{Spin}(9)$-distinguished representations of F_4, consistent with predictions made in [25].

The primary tools in our analysis are computations of maximal parabolic Jacquet modules and the Fourier–Jacobi functor. Similar Jacquet module computations were used to study several different exceptional dual pairs in [6, 9, 19]. In particular, this type of Jacquet module computation provides an important step in establishing Howe duality and dichotomy for exceptional dual pairs containing G_2, which was recently completed in [9].

Our main new input is the use of the Fourier–Jacobi functor. This allows us to relate the $\mathrm{Aut}(C) \times F_4$ theta correspondence to a classical $\mathrm{O}(3) \times \mathrm{Sp}(6)$ theta correspondence. Using the well developed theory of this classical theta correspondence we can efficiently derive results about the $\mathrm{Aut}(C) \times F_4$ theta correspondence.

Now we outline the contents of the paper and make a few more remarks on the proofs of our main results. Section 2 introduces notation and recalls some preliminary material. Section 3 contains the computations of (twisted) Jacquet modules of the minimal representation $\mathcal{V}$ with respect to a maximal Heisenberg parabolic of F_4. These calculations are done using a filtration of $\mathcal{V}$ with respect to a maximal Heisenberg parabolic subgroup of E_7 (recalled in Theorem 3.1). This filtration was first studied in Magaard-Savin [19]. In this section we also review the Fourier–Jacobi functor.

In Section 4 we apply the results of Section 3 to study the theta lift of τ a supercuspidal representation of $\mathrm{Aut}(C)$ to F_4. The main result of this section Theorem 4.10 states that $\Theta(\tau)$ is irreducible. The proof is based on the Fourier–Jacobi functor, which is the main new input in our analysis. Its utility stems from Proposition 4.3, which says that the Fourier–Jacobi functor applied to the minimal representation of E_7 is

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isomorphic to the Weil representation as an $\mathrm{SO}(3)\times\mathrm{Sp}(6)$-representation. This is almost the setting of the classical dual pair $\mathrm{O}(3)\times\mathrm{Sp}(6)$. We use the well-developed classical theory and basic properties of the Fourier–Jacobi functor to deduce that $\Theta(\tau)$ has at most two nontrivial constituents (Corollary 4.5). Then we apply the calculations from Section 3 to show that $\Theta(\tau)$ is irreducible. Specifically, we use the twisted Jacquet module calculations to prove that $\Theta(\tau)$ has at most one nontrivial constituent (Proposition 4.8), and the untwisted Jacquet module to rule out the trivial representation (Proposition 4.9).

Next we specialize to the case when C is the algebra of 2×2 matrices, so $\mathrm{Aut}(C)=\mathrm{PGL}_2$. Section 5 is roughly analogous to Section 3. The difference is that now we use a filtration of $\mathcal{V}$ with respect to a maximal Siegel parabolic subgroup of E_7 [26] (recalled in Theorem 5.1) to compute Jacquet modules with respect to a Borel subgroup of PGL_2.

In Section 6 we describe the theta lift of representations of PGL_2 to F_4. This breaks up into two parts. First we consider constituents of principal series. For this we apply the results of Section 5 on untwisted Jacquet modules to lift the constituents of principal series of PGL_2 to F_4. Generically, the theta lift of a PGL_2 principal series is a degenerate principal series of F_4 induced from the maximal parabolic subgroup Q. The complete description of the big theta lift is contained in Theorems 6.2 and 6.3; the small theta lift is described in Proposition 6.4. The approach of this section builds upon [26].

Second, we consider supercuspidal representations in Subsection 6.4. From Theorem 4.10 we know that the theta lift of a supercuspidal representation is irreducible. Here we refine this result in Proposition 6.5 when $\mathrm{Aut}(C)=\mathrm{PGL}_2$. Specifically, we show that the theta lift is a quotient of an explicit representation of F_4 induced from the maximal parabolic subgroup Q_2. We note that this calculation uses the $G_2\times F_4\subset E_8$ dual pair studied in Magaard-Savin [19].

In Section 7 we consider the theta lift from F_4 to $\mathrm{Aut}(C)$. The main result is Theorem 7.4, which states that if $\sigma\in\mathrm{Irr}(F_4)$ and $\Theta(\sigma)\neq 0$, then $\Theta(\sigma)\in\mathrm{Irr}(\mathrm{Aut}(C))$.

In Section 8 we characterize the irreducible representations of F_4 that are $\mathrm{Spin}(9)$-distinguished, that is, possess a Spin_9-invariant linear functional. The main result, Theorem 8.1, is proved using the twisted Jacquet module calculations from Section 5. We also show in Proposition 8.4 that supercuspidal representations of PGL_2 lift to Spin_9-relatively supercuspidal representations of F_4.

Section 9 concludes the paper with analogous (but easier) results when $\dim C=2$.

2. Notation

2.1. Representation theory of p-adic groups

Let F be a nonarchimedean local field of characteristic 0 and residual characteristic $p>0$, with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{p}$. Let q be the order of the residue field. We normalize the absolute value on F so that its value is q^{-1} on any generator of $\mathfrak{p}$. We fix a nontrivial additive character $\psi: F\longrightarrow\mathbb{C}^{\times}$.

Let G be the group of F-points of a connected reductive group. Let $\mathcal{M}(G)$ be the category of smooth G-representations and let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible objects. Given $\pi\in\mathcal{M}(G)$ we write $\widetilde{\pi}$ for the smooth contragradient representation of π.

If $H\subset G$ is a closed subgroup and (σ,W) is a smooth representation of H. We write $\mathrm{Ind}_H^G(\sigma)$ for the space of right G-smooth functions $f: G\longrightarrow W$ such that for any $g\in G$ and $h\in H$ we have $f(hg)=\sigma(h)f(g)$. This is a G-representation with the action $(g\cdot f)(g')=f(g'g)$. We write $\mathrm{ind}_H^G(\sigma)\subset\mathrm{Ind}_H^G(\sigma)$ for the G-submodule of functions with compact support mod H.

Now suppose $P=MN\subset G$ is a parabolic subgroup with a Levi decomposition, and (σ,W) is a smooth representation of M inflated to P. We fix dn a Haar measure on N and let δ_P be the modular character of P defined by $d(pnp^{-1})=\delta_P(p)dn$. We write $i_M^G(\sigma)=\mathrm{Ind}_P^G(\delta_P^{1/2}\otimes\sigma)$ for normalized parabolic induction.

Let (π,V) be a smooth G-representation. If $H\subset G$ is a subgroup with a character $\chi: H\longrightarrow\mathbb{C}$, let $V_{(H,\chi)}$ denote the space of (H,χ)-coinvariants. This space can be realized as the quotient of V by the subspace $\mathrm{span}\{h\cdot v-\chi(h)v | v\in V,\ h\in H\}$ and is a representation of the subgroup of the normalizer of H that fixes χ, which we write as $\mathrm{Stab}_G(\chi)$. If χ is trivial we write $V_H=V_{(H,\chi)}$. We write $r_P(V)=\delta_P^{-1/2}\otimes V_N$ for the normalized Jacquet module of a parabolic subgroup $P=MN\subset G$.

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2.2. Composition algebras

The theta-lift examined in this paper is based on exceptional dual pairs that can be constructed using composition and Jordan algebras. We begin by collecting some information on these algebras.

Let C be a composition algebra over F with quadratic norm form n_C. We write $B_C(x, y) = n_C(x + y) - n_C(x) - n_C(y)$ for the bilinear form associated to n_C and $^{-} : C \to C$ as $x \mapsto \overline{x}$ for conjugation ([28], Section 1.3). The trace of an element of $x \in C$ is $\mathrm{Tr}_C(x) = x + \overline{x}$ and $n_C(x) = x\overline{x}$. Note that $\mathrm{Tr}_C(xy) = -B_C(x, y)$, for all $x, y \in C$.

We write C^0 for the subspace of trace 0 elements of C. The group of F-algebra automorphisms $\mathrm{Aut}(C)$ preserves the norm and acts on C^0. Thus $\mathrm{Aut}(C)$ is contained in $O(C^0, n_C)$ the orthogonal group of the norm form.

Recall that $\dim_F(C) = 1, 2, 4, 8$. Over the p-adic field F the possible composition algebras can be described explicitly. When $\dim_F(C) = 2$, then C is either a quadratic field extension of F, or C is isomorphic to the split quadratic algebra $F \oplus F$ with norm form $(x, y) \mapsto xy$. In either case, the automorphism group of $\mathrm{Aut}(C)$ is generated by the conjugate map $x \mapsto \overline{x}$ and so is isomorphic to $\mu_2 = \{\pm 1\}$.

When $\dim(C) = 4$, C is either isomorphic to the split quaternion algebra, which can be realized as the algebra of 2×2 matrices $M(2, F)$ with norm form given by the determinant; or C is isomorphic to D, the unique (up to isomorphism) quaternion division algebra over F. The group of $\mathrm{Aut}(C)$ consists of inner automorphisms (Skolem–Noether Theorem) and so is isomorphic to $PC^\times$. When $C \cong M(2, F)$, then $\mathrm{Aut}(C) \cong \mathrm{PGL}_2(F)$.

When $\dim_F(C) = 8$, then C is isomorphic to $\mathbb{O}$ the split octonion algebra over F. Its automorphism group is the F-points of an algebraic group of type G_2.

For more information on composition algebras the reader can refer to [12, 28].

2.3. Jordan algebras

Next we describe a family of Jordan algebras indexed by a composition algebra C. For more details, see Pollack [22, Chapter 2, Section 2].

Let

$$\mathcal{J} = \mathcal{J}_C = \left\{ X = \begin{pmatrix} c_1 & x_3 & \overline{x_2} \\ \overline{x_3} & c_2 & x_1 \\ x_2 & \overline{x_1} & c_3 \end{pmatrix} | c_j \in F,\ x_j \in C \right\}.$$

Given $A, B \in \mathcal{J}_C$ the Jordan multiplication is defined by

$$A * B = \frac{AB + BA}{2},$$

where AB, BA denotes usual matrix multiplication. The algebra $\mathcal{J}_C$ is equipped with a cubic norm form

$$N_{\mathcal{J}}(X) = c_1c_2c_3 - c_1n_C(x_1) - c_2n_C(x_2) - c_3n_C(x_3) + \mathrm{Tr}(x_1x_2x_3).$$

The norm form uniquely defines a symmetric trilinear form $(-, -, -)_{\mathcal{J}} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to F$ normalized so that $(X, X, X)_{\mathcal{J}} = 6N_{\mathcal{J}}(X)$.

We write $\mathrm{Tr}_{\mathcal{J}} : \mathcal{J} \to F$ for the map defined by $\mathrm{Tr}_{\mathcal{J}}(X) = c_1 + c_2 + c_3$. From this we define a nondegenerate pairing $\langle -, - \rangle_{\mathcal{J}} : \mathcal{J} \times \mathcal{J} \to F$ by $\langle X, X' \rangle_{\mathcal{J}} = \mathrm{Tr}_{\mathcal{J}}(X * X')$.

There is also a map $^{\#} : \mathcal{J} \to \mathcal{J}$ defined by

$$X^{\#} = \begin{pmatrix} c_2c_3 - n_C(x_1) & \overline{x}_2\overline{x}_1 - c_3x_3 & x_3x_1 - c_2\overline{x_2} \\ x_1x_2 - c_3\overline{x}_3 & c_1c_3 - n_C(x_2) & \overline{x}_3\overline{x}_2 - c_1x_1 \\ \overline{x}_1\overline{x}_3 - c_2x_2 & x_2x_3 - c_1\overline{x_1} & c_1c_2 - n_C(x_3) \end{pmatrix}.$$

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This map can be used to define the cross product

$$X \times Y = (X + Y)^{\#} - X^{\#} - (Y)^{\#}. \tag{2.1}$$

Alternatively $X \times Y \in \mathcal{J}$ is the unique element such that for all $Y \in \mathcal{J}$

$$\langle X \times Y, Z\rangle_{\mathcal{J}} = (X, Y, Z)_{\mathcal{J}}. \tag{2.2}$$

There is a notion of rank for elements in $\mathcal{J}$. Every element $X \in \mathcal{J}$ has rank at most 3. If $N(X) = 0$, then X has rank at most 2. If $X^{\#} = 0$, then x has rank at most 1. If $X = 0$, then X has rank 0. (Pollack [22, Chapter 3, Section 3])

We write H_C for the group of invertible linear transformations of $\mathcal{J} = \mathcal{J}_C$ that scale the norm form $N_{\mathcal{J}}$ (i.e., the group of similitudes of the cubic form), and H_C^1 for the subgroup preserving the norm form. We have a subgroup $\mathrm{Aut}(C) \times \mathrm{GL}_3(F) \to H_C$ where $g \in \mathrm{Aut}(C)$ acts naturally on entries of elements of $\mathcal{J}$, while $h \in \mathrm{GL}_3(F)$ acts on $X \in \mathcal{J}$ by

$$\det(h) \cdot (h^{-1})^{\top} X h^{-1},$$

where $h^{\top}$ denotes the transpose of h. The similitude character of this transformation of $\mathcal{J}$ is $\det(h)$. If $C = F$, then $H_F \cong \mathrm{GL}_3(F)$. In general, $\mathrm{Aut}(C) \times \mathrm{GL}_3(F)$ preserves the decomposition

$$\mathcal{J}_C = \mathcal{J}_F \oplus \mathcal{J}_{C^0}$$

where $\mathcal{J}_{C^0}$ is the subspace consisting of

$$J(x) = \begin{pmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & x_1 \\ x_2 & \overline{x}_1 & 0 \end{pmatrix},$$

where $x = (x_1, x_2, x_3) \in (C^0)^3$. The following proposition in essence restates the known fact that the dual of the standard three-dimensional representation of GL_3 is isomorphic to the exterior square of the standard representation twisted by determinant inverse. In any case it is easy to check.

Proposition 2.1. *Let V_3 be the standard representation of* $\mathrm{GL}_3(F)$. *Then $\mathcal{J}_{C^0} \cong C^0 \otimes V_3$. Explicitly, $(g, h) \in \mathrm{Aut}(C) \times \mathrm{GL}_3(F)$ acts on $J(x)$ by $J(gxh^{\top})$.*

2.4. Construction of exceptional Lie algebras

Let $\mathfrak{h}_C$ be the Lie algebra of H_C^1. We define vector spaces

$$\mathfrak{g}_{0,C} = \mathfrak{sl}(3, F) \oplus \mathfrak{h}_C,$$
$$\mathfrak{g}_{1,C} = V_3 \otimes \mathcal{J}_C,$$
$$\mathfrak{g}_{-1,C} = V_3^* \otimes \mathcal{J}_C^*,$$

where V_3 is the standard representation of $\mathfrak{sl}_3$ and V_3^* the dual of V_3. We identify $\mathcal{J}_C^*$ with $\mathcal{J}_C$ using the trace form. Consider the vector space

$$\mathfrak{g}_C = \mathfrak{g}_{0,C} \oplus \mathfrak{g}_{1,C} \oplus \mathfrak{g}_{-1,C}.$$

The space $\mathfrak{g}_C$ can be given the structure of a Lie algebra that extends the Lie algebra structure on $\mathfrak{g}_{0,C}$ and the natural action of $\mathfrak{g}_{0,C}$ on $\mathfrak{g}_{1,C} \oplus \mathfrak{g}_{-1,C}$. (See [24], Section 1.3.) We write $\langle -, - \rangle_C$ for the Killing form of $\mathfrak{g}_C$. The Lie algebra $\mathfrak{g}_F$ is the split simple Lie algebra of type F_4. The Lie algebra $\mathfrak{g}_C$ is a simple Lie algebra of type E_n, where $n = 6, 7, 8$ when $\dim_F(C) = 2, 4, 8$, respectively.

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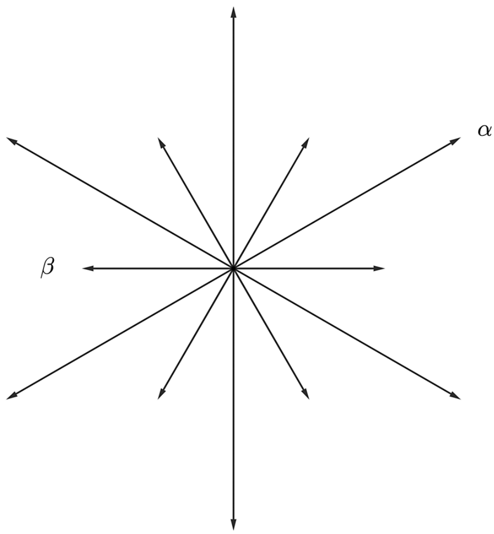
Let $Y \in \mathfrak{sl}(3,F)$ and $X \in \mathcal{J}_C$, then $X \mapsto YX + XY^\top$ defines a Lie algebra action of $\mathfrak{sl}(3,F)$ on $\mathcal{J}_C$ extending the analogous action of $\mathfrak{sl}(3,F)$ on $\mathcal{J}_F$. This action induces an inclusion of Lie algebras $\mathfrak{h}_F \cong \mathfrak{sl}(3,F) \hookrightarrow \mathfrak{h}_C$, which induces an inclusion of Lie algebras $\mathfrak{g}_F \hookrightarrow \mathfrak{g}_C$.

Next we describe a Heisenberg parabolic subalgebra in $\mathfrak{g}_C$. Let $\mathfrak{t}$ be the subalgebra of diagonal matrices in $\mathfrak{sl}(3,F) \subset \mathfrak{g}_{0,C}$. The adjoint action of $\mathfrak{t}$ on $\mathfrak{g}_C$ provides a decomposition

$$\mathfrak{g}_C = \bigoplus_{\gamma \in \mathfrak{t}^*} \mathfrak{g}_\gamma,$$

where $\mathfrak{g}_\gamma = \{X \in \mathfrak{g}_C | [h,X] = \gamma(h)X \text{ for all } h \in \mathfrak{t}\}$. The weights $\gamma \neq 0$ such that $\mathfrak{g}_\gamma \neq 0$ form a relative root system $\underline{\Phi}$ of type G_2 ([7, Sections 9.2, 10.8]). Note that the long relative root spaces are all isomorphic to F and sit in $\mathfrak{sl}(3,F)$ while the short relative root spaces are isomorphic to $\mathcal{J}_C$ or $\mathcal{J}_C^*$. Finally, observe that $\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{h}_C$.

We let $\{\alpha, \beta\}$ be a set of simple roots in the G_2 relative root system so that α is long, β is short. Then the maximal root $\alpha_{\max} = 2\alpha + 3\beta$ is a long root. Thus, without loss of generality, we can assume that $h_{\alpha_{\max}} = \mathrm{diag}(1,0,-1) \in \mathfrak{sl}(3,F)$.



The element $h_{\alpha_{\max}}$ defines a $\mathbb{Z}$-grading on $\mathfrak{g}_C$ supported on $\{0, \pm 1, \pm 2\}$. For $j \in \mathbb{Z}$, let

$$\mathfrak{g}_C(j) = \{x \in \mathfrak{g}_C \mid [h_{\alpha_{\max}}, x] = jx\}.$$

Let $\mathfrak{p} = \oplus_{j \geq 0} \mathfrak{g}_C(j)$. Then $\mathfrak{p}$ is a Heisenberg parabolic subalgebra with Levi subalgebra

$$\mathfrak{m} = \mathfrak{g}_C(0) = \mathfrak{t} \oplus \mathfrak{h}_C \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$$

and nilpotent radical $\mathfrak{n} = \oplus_{j>0} \mathfrak{g}_C(j)$ with one-dimensional center

$$\mathfrak{z} = \mathfrak{g}_C(2) = \mathfrak{g}_{\alpha_{\max}}.$$

Let $\mathcal{G}_C = \mathrm{Aut}(\mathfrak{g}_C)$. If $C \neq F$ then the connected component of $\mathcal{G}_C$ is an adjoint group of type E_n. The group $\mathcal{G}_F$ is F_4. We omit the subscript C when no confusion can arise. Then the maximal parabolic subalgebra $\mathfrak{p}$ corresponds to a maximal parabolic subgroup $\mathcal{P} = \mathcal{MN}$ in $\mathcal{G}$. Let $\mathcal{Z}$ be the center of $\mathcal{N}$. Then $\mathcal{M}$ acts on $\mathcal{N}/\mathcal{Z} \cong \mathfrak{n}/\mathfrak{z}$. The space $\mathfrak{n}/\mathfrak{z}$ admits a symplectic and a quartic form, and $\mathcal{M}$ acts as a group of similitudes of these two forms.

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2.5. A symplectic space

Using Pollack [22, Chapter 3] we give an explicit construction of the reductive group $\mathcal{M}$ and its representation on the symplectic space $\mathfrak{n}/\mathfrak{z}$. In terms of the restricted root system, we have

$$\mathfrak{n}/\mathfrak{z} \cong \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+2\beta} \oplus \mathfrak{g}_{\alpha+3\beta}.$$

We identify $\mathcal{J}$ and $\mathcal{J}^*$ using the trace form, so $\mathfrak{g}_\gamma \cong \mathcal{J}$ for any short root γ. Thus we can identify $\mathfrak{n}/\mathfrak{z}$ with

$$\mathbb{W} = \mathbb{W}_C = F \oplus \mathcal{J} \oplus \mathcal{J} \oplus F.$$

So any element $w \in \mathbb{W}$ is a quadruple $w = (a, b, c, d)$, where $a, d \in F$ and $b, c \in \mathcal{J}$. The space $\mathbb{W}$ comes with a symplectic form

$$\langle (a, b, c, d), (a', b', c', d') \rangle_{\mathbb{W}} = ad' - \mathrm{Tr}(b * c') + \mathrm{Tr}(c * b') - da',$$

and a quartic form

$$q(a, b, c, d) = (ad - \mathrm{Tr}(b * c))^2 + 4aN(c) + 4dN(b) - 4\mathrm{Tr}(b^\# * c^\#).$$

Let $(-, -, -, -)_{\mathbb{W}}$ be the unique symmetric 4-linear form on $\mathbb{W}$ such that $(v, v, v, v) = 2q(v)$. Then $\mathcal{M}$ is isomorphic to the group of similitudes

$$M_C = \{(g, \nu) \in \mathrm{GL}(\mathbb{W}) \times \mathrm{GL}_1(F) | \langle gv, gv' \rangle = \nu \langle v, v' \rangle,\ q(gv) = \nu^2 q(v) \text{ for all } v, v' \in \mathbb{W}\}.$$

We write M_C^1 for the subgroup of elements where the similitude factor ν is equal to 1.

We highlight a few subgroups of M_C^1. If $h \in H_C$ with similitude factor λ, then the map $(a, b, c, d) \mapsto (\lambda a, hb, \tilde{h}c, \lambda^{-1}d)$, where the action of $\tilde{h}$ on $\mathcal{J}$ is defined through the identification of $\mathcal{J}$ with $\mathcal{J}^*$ via the trace pairing, defines an element of M_C^1. We abuse notation and let H_C denote this subgroup.

For $x \in \mathcal{J}$ let $n(x)$ be the map defined by

$$n(x)(a, b, c, d) = (a, b + ax, c + b \times x + ax^\#, d + \mathrm{Tr}(c * x) + \mathrm{Tr}(b * x^\#) + aN(x)). \tag{2.3}$$

The map $n(x) \in M_{\mathcal{J}}$ and has similitude factor equal to 1. The group generated by these elements is isomorphic to the unipotent group $\exp(\mathfrak{g}_\beta) \subset \mathcal{M}$.

Similarly, for $x \in \mathcal{J}$ let $\overline{n}(x)$ be the map defined by

$$\overline{n}(x)(a, b, c, d) = (a + \mathrm{Tr}(b * x) + \mathrm{Tr}(c * x^\#) + dN(x), b + c \times x + dx^\#, c + dx, d).$$

The map $\overline{n}(x) \in M_C$ and has similitude factor equal to 1. The group generated by these elements is isomorphic to the unipotent group $\exp(\mathfrak{g}_{-\beta}) \subset \mathcal{M}$.

The two abelian groups generated by $n(x)$ and $\overline{n}(x)$, respectively, are unipotent radicals of two opposite maximal parabolic subgroups in M_C^1 with the Levi factor H_C. These two parabolic groups are conjugate by

$$(a, b, c, d) \mapsto (-d, c, -b, a). \tag{2.4}$$

If $\lambda \in \mathrm{GL}_1(F)$, then

$$s_\lambda : (a, b, c, d) \mapsto (\lambda^2, \lambda b, c, \lambda^{-1}d)$$

and

$$s_\lambda^* : (a, b, c, d) \mapsto (\lambda^{-1}, b, \lambda c, \lambda^2 d)$$

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are two elements of M_C with the similitude factor $\nu = \lambda$. Thus M_C is generated by M_C^1 and any of the two one-parameter groups s or s^*. We have written down both of these two groups for the sake of symmetry but also because they generate a two-dimensional torus whose Lie algebra is $\mathfrak{t} \subset \mathfrak{m}$.

Observe that $\mathrm{Aut}(C) \subset M_C$ where $\mathrm{Aut}(C)$ acts on the coordinates of $\mathbb{W}_C$. The centralizer of $\mathrm{Aut}(C)$ in M_C is M_F, the Levi of the Heisenberg maximal parabolic of F_4. This group is isomorphic to $\mathrm{GSp}_6(F)$, as one can see from root data, for example. We shall fix an isomorphism $M_F \cong \mathrm{GSp}_6(F)$ as follows. Recall that $\mathbb{W}_C$ is a symplectic space. Under the action of $\mathrm{Aut}(C)$ it decomposes as

$$\mathbb{W}_C = \mathbb{W}_F \oplus (\mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0}).$$

If V_6 is a six-dimensional symplectic space then $C^0 \otimes V_6$ is a symplectic space obtaining by tensoring the quadratic space C^0 and the symplectic space V_6. We pick V_6 so that

$$\mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0} \cong C^0 \otimes V_6, \tag{2.5}$$

given by $(J(x), J(y)) \mapsto (x_1, x_2, x_3, y_1, y_2, y_3)$, is an isomorphism of symplectic spaces. Since M_F commutes with $\mathrm{Aut}(C)$, and $\mathrm{Aut}(C)$ acts on C^0 irreducibly, M_F must act on V_6, giving an identification with $\mathrm{GSp}_6(F)$. Let sim denote the usual similitude character of $\mathrm{GSp}_6(F)$. Observe that the similitude character of M_C restricts to sim under the identification.

2.6. *Orbits*

We now describe orbits of $M_{\mathcal{J}}$ acting on $\mathbb{W} = \mathbb{W}_C$. Given $v = (a, b, c, d) \in \mathbb{W}_{\mathcal{J}}$ define $v^\flat = (a^\flat, b^\flat, c^\flat, d^\flat)$ ([22, Proposition 1.0.3]), where

$$\begin{aligned}
a^\flat &= -a(ad - \mathrm{Tr}(b * c)) - 2N(b);\\
b^\flat &= -2c \times b^\# + 2ac^\# - (ad - \mathrm{Tr}(b * c))b;\\
c^\flat &= 2b \times c^\# - 2bd^\# + (ad - \mathrm{Tr}(b * c))c;\\
d^\flat &= d(ad - \mathrm{Tr}(b * c)) + 2N(c).
\end{aligned}$$

Over the algebraic closure the orbits are classified by the rank for elements in $\mathbb{W}$, defined as follows. Let $v \in \mathbb{W}$. The element v has rank at most 4. If $q(v) = 0$, then v has rank at most 3. If $v^\flat = 0$, then v has rank at most 2. If $(v, v, w, w') = 0$ for all $w, w' \in (v)^\perp$ (the orthogonal complement with respect to $\langle -, - \rangle_{\mathbb{W}}$), then v has rank at most 1. If $v = 0$, then v has rank 0.

We need the following proposition [8, Proposition 8.1] and a simple corollary.

Proposition 2.2. *A nonzero element $(a, b, c, d) \in \mathbb{W}$ has rank 1 if and only if*

1. $b^\# - ac = 0$,
2. $c^\# - db = 0$,
3. $ad = h(b) * \tilde{h}(c)$ *for all $h \in H_C$, where $\tilde{h}$ is the dual action of h on $\mathcal{J}^*$ identified with $\mathcal{J}$ using the trace form.*

Corollary 2.3. $\Xi = (1, b, c, d) \in \mathbb{W}$ *has rank 1 if and only if $c = b^\#$ and $d = N(b)$.*

Proof. If Ξ has rank 1, then Proposition 2.2 implies $c = b^\#$ and $d = b * b^\# = N(b)$ (use $h = 1$ in (3)). In the opposite direction use $(b^\#)^\# = N(b) \cdot b$ and $\tilde{h}(b^\#) = h(b)^\#/\nu$ where ν is the similitude factor of h. □

2.7. *The dual pair* $\mathrm{Aut}(C) \times \mathcal{G}_F$ *in* $\mathcal{G}_C$

Observe that $\mathrm{Aut}(C)$ naturally acts on $\mathcal{J}_C$ preserving the norm $N_{\mathcal{J}}$. Thus $\mathrm{Der}(C)$, the Lie algebra of $\mathrm{Aut}(C)$, is a subalgebra of $\mathfrak{h}_C$. From the construction of $\mathfrak{g}_C$ in Subsection 2.4 it is evident that the centralizer of $\mathrm{Der}(C)$ in $\mathfrak{g}_C$ contains $\mathfrak{g}_F$, the Lie algebra of $\mathcal{G}_F$, the group of type F_4. This group is simply connected with trivial center. Thus the inclusion of Lie algebras lifts to an inclusion $\mathcal{G}_F \subseteq \mathcal{G}_C$.

Lemma 2.4. *$\mathrm{Aut}(C) \times \mathcal{G}_F$ is a maximal subgroup in $\mathcal{G}_C$.*

Proof. Observe that, under the adjoint action restricted to $\mathrm{Der}(C) \oplus \mathfrak{g}_F$,

$$\mathfrak{g}_C = \mathrm{Der}(C) \oplus \mathfrak{g}_F \oplus C^0 \otimes V_{26}$$

where C^0 and V_{26} are irreducible representations of $\mathrm{Der}(C)$ and $\mathfrak{g}_F$ respectively. The irreducibility of $C^0 \otimes V_{26}$ implies that $\mathrm{Der}(C) \oplus \mathfrak{g}_F$ is a maximal subalgebra of $\mathfrak{g}$.

First assume $\dim(C) \neq 2$. Then $\mathrm{Aut}(C) \times \mathcal{G}_F$ is connected. Since $\mathrm{Der}(C) \oplus \mathfrak{g}_F$ is maximal, any proper subgroup H of $\mathcal{G}_C$ containing $\mathrm{Aut}(C) \times \mathcal{G}_F$ must have $\mathrm{Aut}(C) \times \mathcal{G}_F$ as its connected component of identity. Now $\mathrm{Aut}(C) \times \mathcal{G}_F$ has no outer automorphisms, so if H is disconnected, then there exists a semisimple finite-order element $z \in \mathcal{G}_C$ centralizing $\mathrm{Aut}(C) \times \mathcal{G}_F$. Since $C^0 \otimes V_{26}$ is irreducible, z would have to act on it as -1. But the centralizer of the semisimple element z must contain a maximal torus. Since $\mathrm{Aut}(C) \times \mathcal{G}_F$ has rank 5 and $\mathcal{G}_C$ has rank 7, this is a contradiction.

Second, assume $\dim C = 2$. Note that in this case $\mathcal{G}_C$ is disconnected and the component group is generated by the outerautomorphism of the E_6 Dynkin diagram, which has order 2. As above we can see that $\mathcal{G}_F$ is maximal in $\mathcal{G}_C^\circ$, the connected component of the identity. Since the generator of $\mathrm{Aut}(C) = \mu_2$ is the outerautomorphism in $\mathcal{G}_C$, the conclusion of the lemma holds in this case, too. □

We are now in a position to understand rational $\mathcal{G}_C$-conjugacy classes of $\mathrm{Aut}(C) \times \mathcal{G}_F \subset \mathcal{G}_C$ over F. Over $\bar{F}$, there is one conjugacy class, that is, any subgroup of $\mathcal{G}_C$ isomorphic to $\mathrm{Aut}(C) \times \mathcal{G}_F$ is conjugate to $\mathrm{Aut}(C) \times \mathcal{G}_F$, this is due to Dynkin. By Lemma 2.4, the normalizer of $\mathrm{Aut}(C) \times \mathcal{G}_F$ in $\mathcal{G}_C$ is $\mathrm{Aut}(C) \times \mathcal{G}_F$, hence conjugacy classes over F correspond to the kernel of the map of pointed sets

$$H^1(F, \mathrm{Aut}(C)) \times H^1(F, \mathcal{G}_F) \to H^1(F, \mathcal{G}_C).$$

If F is p-adic, then $H^1(F, \mathcal{G}_F)$ is trivial, hence we are reduced to the map $H^1(F, \mathrm{Aut}(C)) \to H^1(F, \mathcal{G}_C)$. This maps sends a rational form C' of C (i.e., an element of $H^1(F, \mathrm{Aut}(C))$) to $\mathcal{G}_{C'}$. This map is clearly injective, hence we have only one conjugacy class. Furthermore, injectivity implies that $\mathrm{Aut}(C) \times \mathcal{G}_F$ can be a subgroup of $\mathcal{G}_{C'}$ only when $C \cong C'$.

In this paper we use two different constructions of the dual pair $\mathrm{Aut}(C) \times \mathcal{G}_F \subseteq \mathcal{G}_C$ to investigate the theta correspondence. From the above these two subgroups are conjugate. Thus the results obtained using each construction are compatible.

2.8. Degenerate principal series on F_4

In this section, we collect the results of Choi–Jantzen [4] that describe the structure of degenerate principal series on F_4. Henceforth we write $G = \mathcal{G}_F$ for the unique group of type F_4 over F.

We note that in this paper we use the Bourbaki labeling of simple roots of F_4 [3, Plate VIII], but Choi–Jantzen [4] use the reverse order, that is, the two labelings are related by the permutation $(14)(23)$, written in disjoint cycle notation.

Proposition 2.5 (Theorems 3.1 and 6.1 [4]). *Let χ be a character of $F^\times$ such that $\chi = |-|^s \chi_0$, where χ_0 is a unitary character and $s \in \mathbb{C}$. Let Q be a maximal parabolic subgroup associated with the simple root α_4 and let ϖ_4 be the fundamental weight that pairs nontrivially with $\alpha_4^\vee$.*

1. *$i_Q^G(\chi \circ \varpi_4)$ is reducible if and only if $s = \pm\frac{11}{2}, \pm\frac{5}{2}, \pm\frac{1}{2}$ with χ_0 trivial, or $s = \pm\frac{1}{2}$ with χ_0 of order 2.*
2. *If $s = \pm\frac{11}{2}, -\frac{5}{2}, \pm\frac{1}{2}$ and χ_0 has order dividing 2, then $i_Q^G(\chi \circ \varpi_4)$ has a unique irreducible quotient.*
3. *If $s = \frac{5}{2}$ and χ_0 is trivial, then $i_Q^G(\chi \circ \varpi_4)$ has a maximal semisimple quotient of the form $\sigma^+ \oplus \sigma^-$, where σ^+ and σ^- are distinct irreducible representations of G.*

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2.9. $\mathrm{PGL}_2(F)$

In this subsection, we recall basic facts about $\mathscr{G} = \mathrm{PGL}_2(F)$.

Lemma 2.6. *The characters of $\mathscr{G}$ are in bijection with the quadratic characters of $F^\times$.*

Let $\mathscr{B} = \mathscr{T}\mathscr{U}$ be a Borel subgroup of $\mathscr{G}$ and let $\overline{\mathscr{B}} = \mathscr{T}\overline{\mathscr{U}}$ be its opposite. We fix an identification $\mathscr{T} \cong F^\times$ so that $\mathscr{T}$ acts on $\overline{\mathscr{U}}$ by multiplication. In particular, the modular character is $\delta_{\overline{\mathscr{B}}}(t) = |t|$. Let χ be a character of $\mathscr{T}$ and define $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi) = \mathrm{Ind}_{\overline{\mathscr{B}}}^{\mathscr{G}}(\delta_{\overline{\mathscr{B}}}^{1/2} \cdot \chi)$. Let St denote the Steinberg representation of $\mathscr{G}$. We have the following well-known result:

Lemma 2.7. *Let χ be a character of $F^\times$ such that $\chi = |-|^s\chi_0$, where χ_0 is a unitary character and $s \in \mathbb{C}$. The representation $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ is irreducible unless $s = \pm 1/2$ and χ_0 is a quadratic character.*

1. *If $s = 1/2$ and χ_0 is a quadratic character. Then $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ has length 2. The unique irreducible quotient is the one-dimensional representation obtained by inflating χ_0 to $\mathscr{G}$. The unique irreducible submodule is* $\mathrm{St} \otimes \chi_0$.
2. *If $s = -1/2$ and χ_0 is a quadratic character. Then $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ has length 2. The unique irreducible quotient is* $\mathrm{St} \otimes \chi_0$. *The unique irreducible submodule the one-dimensional representation obtained by inflating χ_0 to $\mathscr{G}$.*

2.10. Theta lifting preliminaries

In this subsection we establish preliminaries to discuss a theta lift for $\mathrm{Aut}(C) \times F_4 \subset E_7$.

Recall that associated to a quaternion algebra C we have the following groups: $\mathcal{G}$ is the F-points of a connected semisimple adjoint group of type E_7; G is the F-points of a connected semisimple group of type F_4; $\mathscr{G}$ is the F-points of the automorphism group of C. Let $(\Pi, \mathcal{V}) = (\Pi_{\min}, \mathcal{V}_{\min})$ be the minimal representation of $\mathcal{G}$. (See [14] for split groups; [7] for nonsplit.)

To begin we define the big theta lift. Let τ be an irreducible smooth $\mathscr{G}$-representation. The maximal τ-isotypic quotient of $\mathcal{V}$ is naturally a $\mathscr{G} \times G$-representation $\mathcal{V}_\tau$. Furthermore, $\mathcal{V}_\tau$ admits a factorization $\mathcal{V}_\tau \cong \tau \otimes \Theta(\tau)$, where $\Theta(\tau)$ is a G-representation. We call $\Theta(\tau)$ the big theta lift of τ with respect to the restriction of $\mathcal{V}$ to $\mathscr{G} \times G$. (For details, see [20, Chapter 2, Section 3].) Let $\theta(\tau)$ be the maximal semisimple quotient (cosocle) of $\Theta(\tau)$.

The main objective of this work is to investigate $\Theta(\tau)$ and $\theta(\tau)$. The following simple lemma is important for our analysis. If V is a vector space, we write V^* for its linear dual.

Lemma 2.8. *Let $\tau \in \mathrm{Irr}(\mathscr{G})$. Let $U \subset G$ be a unipotent subgroup with a character Ψ and let $H = \mathrm{Stab}_G(U, \Psi)$. There is an H-module isomorphism*

$$(\Theta(\tau)_{(U,\Psi)})^* \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, \tau),$$

where the H acts on $\mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, \tau)$ by $(h \cdot f)(v) = f(h^{-1} \cdot v)$.

In particular, there is an isomorphism of G-modules

$$\Theta(\tau)^* \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, \tau).$$

Proof. Because τ is irreducible, it follows that $\Theta(\tau) \cong (\mathcal{V} \otimes \tilde{\tau})_{\mathscr{G}}$ as G-modules. Taking (U, Ψ)-coinvariants gives $\Theta(\tau)_{(U,\Psi)} \cong [(\mathcal{V} \otimes \tilde{\tau})_{\mathscr{G}}]_{(U,\Psi)}$ as H-modules. Since H and $\mathscr{G}$ commute we can commute the coinvariants to get an H-module isomorphism $\Theta(\tau)_{(U,\Psi)} \cong (\mathcal{V}_{(U,\Psi)} \otimes \tilde{\tau})_{\mathscr{G}}$. Next we take the linear dual and apply the $\otimes$-Hom adjunction to get an isomorphism of H-modules

$$(\Theta(\tau)_{(U,\Psi)})^* \cong \mathrm{Hom}((\mathcal{V}_{(U,\Psi)} \otimes \tilde{\tau})_{\mathscr{G}}, \mathbb{C}) \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, (\tilde{\tau})^*).$$

Since $\mathcal{V}_{(U,\Psi)}$ is a smooth $\mathscr{G}$-module we have $\mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, (\tilde{\tau})^*) \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, \tilde{\tilde{\tau}})$. Since τ is irreducible we have $\tilde{\tilde{\tau}} \cong \tau$ as $\mathscr{G}$-modules. Thus as H-modules $(\Theta(\tau)_{(U,\Psi)})^* \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}_{(U,\Psi)}, \tau)$. □

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3. Jacquet modules I

Let $\mathcal{G} = \mathcal{G}_C$ and $G = \mathcal{G}_F$. Let $\mathrm{Aut}(C) \times G$ be the dual pair described in subsection 2.7 and let $\mathcal{P} = \mathcal{M}\mathcal{N}$ be the Heisenberg maximal parabolic subgroup in $\mathcal{G}$. Since $\mathrm{Aut}(C) \subset \mathcal{M}$, the centralizer of $\mathrm{Aut}(C)$ in $\mathcal{P}$ is the Heisenberg maximal parabolic $P = MN$ in G. We write $\overline{\mathcal{P}} = \mathcal{M}\overline{\mathcal{N}}$ and $\overline{P} = M\overline{N}$ for the parabolic subgroups opposite to $\mathcal{P}$ and P, respectively.

Our study of the theta lifting is based on two tools. The first are functors of twisted co-invariants of the minimal representation $\mathcal{V}$ of $\mathcal{G}$. The second is the Fourier–Jacobi functor. To compute these functors, we use a $\overline{\mathcal{P}}$-filtration of $\mathcal{V}$ due to Magaard-Savin [19, Theorem 6.1], which we now recall. In the following, Ω is the minimal nontrivial $\mathcal{M}$-orbit in $\mathcal{N}/Z$. Under the isomorphism $\mathcal{N}/Z \cong \mathbb{W}_C$, Ω is the set of rank 1 elements in $\mathbb{W}_C$.

Theorem 3.1. *Let $(\Pi, \mathcal{V})$ be the minimal representation of $\mathcal{G}$ and let $\overline{\mathcal{P}} = \mathcal{M}\overline{\mathcal{N}}$ be the Heisenberg parabolic subgroup of $\mathcal{G}$ opposite to $\mathcal{P}$. Let $\overline{Z}$ be the center of $\overline{\mathcal{N}}$. Then $\mathcal{V}$ has a $\overline{\mathcal{P}}$-filtration given by the exact sequence*

$$0 \to C_c^\infty(\Omega) \to \mathcal{V}_{\overline{Z}} \to \mathcal{V}_{\overline{\mathcal{N}}} \to 0. \tag{3.1}$$

Furthermore, the action of $\overline{\mathcal{P}}$ is described as follows:

1. *Let $m\overline{n} \in \mathcal{M}\overline{\mathcal{N}}$ and $f \in C_c^\infty(\Omega)$. Then*

$$[\Pi(\overline{n})f](x) = \psi(\langle x, \overline{n}\rangle) f(x);$$
$$[\Pi(m)f](x) = \chi_C(m)|\det(m)|^{s/d} f(m^{-1} \cdot x).$$

2. $\mathcal{V}_{\overline{\mathcal{N}}} \cong [\mathcal{V}(\mathcal{M}) \otimes |\det|^{t/d}] \oplus \chi_C|\det|^{s/d}$*, where $\mathcal{V}(\mathcal{M})$ is the minimal representation of $\mathcal{M}$ (center acting trivially).*

Here det *is the determinant of the representation of $\mathcal{M}$ acting on $\overline{\mathcal{N}}/\overline{Z}$; χ_C is a quadratic character, trivial unless C is a quadratic field, and then corresponding to C by the local class field theory; d is the dimension of $\mathcal{N}/Z$. The values of s, t, and d are given in the following table.*

$\mathcal{G}$	s	t	d
E_6	4	3	20
E_7	6	4	32
E_8	10	6	56

Remark: In Magaard-Savin [19], the groups are split. Nevertheless, their proof still applies to $\mathcal{G} = \mathcal{G}_C$, where C is a split or nonsplit composition algebra over F. The quadratic twist by χ_C was observed in [8].

Given $\tau \in \mathrm{Irr}(\mathscr{G})$ there is a surjective map $\mathcal{V} \twoheadrightarrow \tau \otimes \Theta(\tau)$. To study $\Theta(\tau)$ we apply the functor of $(\overline{N}, \Psi)$-coinvariants to sequence (3.1), where Ψ is a character of $\overline{N}$.

Let $M_\Psi = \mathrm{Stab}_M(\Psi)$. Let $\mathcal{N}(\Psi) = \{n \in \mathcal{N} | \psi(\langle n, \overline{n}\rangle) = \Psi(\overline{n}) \text{ for all } \overline{n} \in \overline{N}\}$. Let $\Omega_\Psi = \Omega \cap \mathcal{N}(\Psi)$.

Lemma 3.2. *The restriction map $C_c^\infty(\Omega) \to C_c^\infty(\Omega_\Psi)$ induces a $\mathscr{G} \times M_\Psi$-module isomorphism*

$$C_c^\infty(\Omega)_{(\overline{N},\Psi)} \cong C_c^\infty(\Omega_\Psi).$$

Proof. The proof is the same as Magaard-Savin [19], Lemma 2.2. □

To describe $C_c^\infty(\Omega_\Psi)$ as a $\mathscr{G} \times M_\Psi$-module we need an explicit description of Ω_Ψ, which we take up in the next subsection. When Ψ is nontrivial, this gives a complete description of $\mathcal{V}_{(\overline{N},\Psi)}$, as we see in the next lemma.

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Lemma 3.3. *Suppose that Ψ is nontrivial. By applying $(\overline{N}, \Psi)$ coinvariants to the exact sequence (3.1) we get a $\mathcal{G} \times M_\Psi$-module isomorphism*

$$C_c^\infty(\Omega)_{(\overline{N},\Psi)} \cong \mathcal{V}_{(\overline{N},\Psi)}.$$

Proof. Since the functor $(-)_{(\overline{N},\Psi)}$ is exact and $(\mathcal{V}_{\overline{\mathcal{N}}})_{(\overline{N},\Psi)} = 0$ the result follows. □

3.1. Fiber calculation

The main objective of this section is to compute $C_c^\infty(\Omega)_{(\overline{N},\Psi)}$, where Ψ is of rank 3 or rank 0. This is accomplished in Proposition 3.7 for rank 3, and Proposition 3.11 for rank 0. The rank of Ψ will be defined in terms of the rank of elements of $\mathbb{W}_F$ (Subsection 2.5). We explain this after setting up some notation.

The map $\mathcal{N}/Z \to \mathrm{Hom}(\overline{\mathcal{N}}/\overline{Z}, \mathbb{C}^\times)$ defined by $n \mapsto \psi(\langle n, -\rangle)$ defines an isomorphism of $\mathcal{N}/Z$ with the Pontryagin dual of $\overline{\mathcal{N}}/\overline{Z}$. Similarly, by restriction this map defines an isomorphism between N/Z and the Pontryagin dual of $\overline{N}/\overline{Z}$.

We identify $\mathcal{N}/Z$ with $\mathbb{W}_C$ and $\mathcal{M}$ with M_C so that the adjoint action of $\mathcal{M}$ on $\mathcal{N}/Z$ corresponds to the action of M_C on $\mathbb{W}_C$. This also fixes an identification of N/Z with $\mathbb{W}_F = \mathbb{W}_C^{\mathrm{Aut}(C)}$ and of $M \subset \mathcal{M}$ with $M_F \subset M_C$. Thus we can view the character Ψ as an element of $\mathbb{W}_F$. We define the rank of Ψ to be the rank of its associated element in $\mathbb{W}_F$.

Now we reinterpret the set $\mathcal{N}(\Psi)$ as the fiber of a map $\mathbf{F}$, defined below.

Let $\mathbf{f} : \mathcal{J}_C \to \mathcal{J}_F$ be the map defined by

$$\begin{pmatrix} a & x & \overline{z} \\ \overline{x} & b & y \\ z & \overline{y} & c \end{pmatrix} \mapsto \begin{pmatrix} a & \frac{\mathrm{Tr}(x)}{2} & \frac{\mathrm{Tr}(z)}{2} \\ \frac{\mathrm{Tr}(x)}{2} & b & \frac{\mathrm{Tr}(y)}{2} \\ \frac{\mathrm{Tr}(z)}{2} & \frac{\mathrm{Tr}(y)}{2} & c \end{pmatrix}.$$

Let $\mathbf{F} : \mathbb{W}_C \to \mathbb{W}_F$ be defined by

$$(a, b, c, d) \mapsto (a, \mathbf{f}(b), \mathbf{f}(c), d).$$

With the identifications above, the natural restriction map from the Pontryagin dual of $\overline{\mathcal{N}}/\overline{Z}$ to the Pontryagin dual of $\overline{N}/\overline{Z}$ is realized as the map $\mathbf{F} : \mathbb{W}_C \to \mathbb{W}_F$. Viewing Ψ as an element of $\mathbb{W}_F$, we have $\mathcal{N}(\Psi) = \mathbf{F}^{-1}(\Psi)$. Thus our next objective is to describe the intersection of the fibers of $\mathbf{F}$ with Ω.

Proposition 3.4. *Let C be any composition algebra. Let $\xi = (1, 0, c, d) \in \mathbb{W}_F$. The set $\mathbf{F}^{-1}(\xi) \cap \Omega$ consists of*

$$(1, J(x), J(x)^\#, N_{\mathcal{J}}(J(x)))$$

for all $x = (x_1, x_2, x_3) \in (C^0)^3$ such that

1. $c = \frac{1}{2}(\mathrm{Tr}(x_i x_j))$
2. $d = \mathrm{Tr}(x_1 x_2 x_3)$.

Proof. Let $\Xi = (a', b', c', d') \in \mathbf{F}^{-1}(\xi) \cap \Omega$.

Since $\Xi \in \mathbf{F}^{-1}(\xi)$, it follows that $\Xi = (1, b', c', d)$, where

$$\begin{aligned} b' &= J(x_1, x_2, x_3), \text{ with } x_j \in C^0; \\ \mathbf{f}(c') &= c. \end{aligned}$$

Since $\Xi \in \Omega$, by Proposition 2.2, $d = N(b') = \mathrm{Tr}(x_1 x_2 x_3)$ and

$$c' = (b')^\# = \begin{pmatrix} -N(x_1) & \overline{x_2 x_1} & x_3 x_1 \\ x_1 x_2 & -N(x_2) & \overline{x_3 x_2} \\ \overline{x_1 x_3} & x_2 x_3 & -N(x_3) \end{pmatrix} = \begin{pmatrix} x_1^2 & x_2 x_1 & x_3 x_1 \\ x_1 x_2 & x_2^2 & x_3 x_2 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix}.$$

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Thus

$$c = \mathbf{f}(c') = \frac{1}{2}\begin{pmatrix} \mathrm{Tr}(x_1^2) & \mathrm{Tr}(x_1x_2) & \mathrm{Tr}(x_1x_3) \\ \mathrm{Tr}(x_1x_2) & \mathrm{Tr}(x_2^2) & \mathrm{Tr}(x_2x_3) \\ \mathrm{Tr}(x_1x_3) & \mathrm{Tr}(x_2x_3) & \mathrm{Tr}(x_3^2) \end{pmatrix}. \qquad \square$$

Now we describe $\mathbf{F}^{-1}((1,0,c,d)) \cap \Omega$, where $c \in \mathcal{J}_F$ has rank 3 and $\dim C = 4$.

Proposition 3.5. *Assume* $\dim C = 4$. *Let* $\xi = (1,0,c,d) \in \mathbb{W}_F$ *such that* $c \in \mathcal{J}_F$ *has rank 3. Then* $\Omega_\xi = \mathbf{F}^{-1}(\xi) \cap \Omega$ *is nonempty if and only if*

1. $\mathrm{Aut}(C) \cong \mathrm{SO}(3,c)$,
2. $d^2 = -4\det(c)$*, so that* ξ *has rank 3.*

If that is the case, then Ω_ξ *is a principal homogeneous space for* $\mathrm{Aut}(C)$.

Proof. We start with a lemma.

Lemma 3.6. *Let* $x = (x_1, x_2, x_3) \in (C^0)^3$. *Then we have the following identity of sextic polynomials*

$$-4\det(\frac{1}{2}\mathrm{Tr}(x_ix_j)) = [\mathrm{Tr}(x_1x_2x_3)]^2.$$

Proof. Let $g \in \mathrm{GL}_3(F)$. Let $y = (y_1, y_2, y_3) \in (C^0)^3$ defined by $y = xg$. It is clear that

$$\det(\mathrm{Tr}(y_iy_j)) = \det(g)^2 \cdot \det(\mathrm{Tr}(x_ix_j)).$$

On the other hand, $\mathrm{Tr}(x_1x_2x_3)$ is a nontrivial trilinear, skew-symmetric form. Since C^0 has dimension 3, the form induces an isomorphism of $\wedge^3 C^0$ and F. Hence

$$\mathrm{Tr}(y_1y_2y_3) = \det(g) \cdot \mathrm{Tr}(x_1x_2x_3).$$

Since $\mathrm{GL}_3(F)$ acts transitively on the open set of all bases (x_1, x_2, x_3) of C^0, it suffices now to check the identity on one basis of C^0. So let us take usual i, j, k such that $i^2 = a$, $j^2 = b$, $ij = k$ and $k^2 = -ab$. Then both sides of the proposed identity are equal to $(2ab)^2$. $\square$

Now, the if and only if statement is a simple combination of the lemma and Proposition 3.4. For the last statement, on the structure of the fiber, observe that the set of x such that $c = \frac{1}{2}(\mathrm{Tr}(x_ix_j))$ is a principal homogeneous space for $\mathrm{O}(C^0)$. Since $\mathrm{O}(C^0) \cong \mathrm{SO}(C^0) \times \{\pm 1\}$ and $\mathrm{Tr}((-x_1)(-x_2)(-x_3)) = -\mathrm{Tr}(x_1x_2x_3)$, the additional equation $d = \mathrm{Tr}(x_1x_2x_3)$ assures that the fiber Ω_ξ is a principal homogenous space for $\mathrm{SO}(C^0) = \mathrm{Aut}(C)$. $\square$

We shed some light on $\mathrm{Stab}_M(\Psi)$ for rank 3 characters. Recall that we have $\mathrm{GL}_3(F) \subset M$ such that $g \in \mathrm{GL}_3(F)$ acts on $\xi = \xi_\Psi = (1,0,c,d) \in \mathbb{W}_F$ by

$$(\det(g), 0, \det(g)^{-1} gcg^\top, \det(g)^{-1} d).$$

Hence $g \in \mathrm{Stab}_M(\Psi)$ if and only if $\det(g) = 1$ and $gcg^\top = c$. In other words the stabilizer of ξ in $\mathrm{GL}_3(F)$ is the group $\mathrm{SO}(3,c)$. Thus we have an action of $\mathrm{Aut}(C) \times \mathrm{SO}(3,c)$ on Ω_ξ. Explicitly, $(g,h) \in \mathrm{Aut}(C) \times \mathrm{SO}(3,c)$ acts on $x = (x_1, x_2, x_3) \in \Omega_\xi$ by

$$x \mapsto (gx_1, gx_2, gx_3)h^\top.$$

We have the following corollary to Proposition 3.5.

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Corollary 3.7. *Assume* $\dim C = 4$. *Let* Ψ *be a rank* 3 *character of* $\overline{N}$ *corresponding to* $\xi = (1, 0, c, d) \in \mathbb{W}_F$ *such that* $\mathrm{SO}(3, c) \cong \mathrm{Aut}(C)$*, where* $\mathrm{SO}(3, c) \subseteq \mathrm{Stab}_M(\Psi)$ *described above. Then there are isomorphisms of* $\mathrm{Aut}(C) \times \mathrm{SO}(3, c)$*-modules*

$$\mathcal{V}_{(\overline{N},\Psi)} \cong C_c^\infty(\Omega)_{(\overline{N},\Psi)} \cong C_c^\infty(\Omega_\xi) \cong C_c^\infty(\mathrm{Aut}(C)) \cong C_c^\infty(\mathrm{SO}(3, c))$$

where the last two isomorphisms depend on a choice of a point in Ω_ξ*, giving identifications of* Ω_ξ *with* $\mathrm{Aut}(C)$ *and* $\mathrm{SO}(3, c)$*, and an isomorphism* $\mathrm{SO}(3, c) \cong \mathrm{Aut}(C)$.

Next we give the analog of Proposition 3.5, where C is a quadratic composition algebra.

Proposition 3.8. *Assume* $\dim C = 2$. *Let* $\xi = (1, 0, c, d) \in \mathbb{W}_F$. *If the set* $\Omega_\xi = \mathbf{F}^{-1}(\xi) \cap \Omega$ *is nonempty then* $d = 0$ *and* c *has rank at most one. If* c *has rank one, then* Ω_ξ *is a principal homogeneous* $\mathrm{Aut}(C) \cong \mathrm{O}(2)$*-space, possibly with no rational points.*

Proof. This follows from Proposition 3.4 using that C^0 is one-dimensional. □

Now we discuss the rank 0 case, that is, Ψ is trivial. We begin by computing $\mathbf{F}^{-1}(0) \cap \Omega$. Observe that $\mathbf{F}^{-1}(0) = \mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0}$. Recall that we have identified $M \cong \mathrm{GSp}(V_6)$ such that

$$\mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0} \cong C^0 \otimes V_6$$

via the map $(J(x), J(y)) \mapsto (x_1, x_2, x_3, y_1, y_2, y_3)$ (Subsection 2.5).

Proposition 3.9. *Let* C *be a composition algebra.*

1. *If* C^0 *is anisotropic then* $\mathbf{F}^{-1}(0) \cap \Omega = \emptyset$.
2. *Suppose* C *is a split quaternion algebra. Then* $\Omega_0 = \mathbf{F}^{-1}(0) \cap \Omega$ *consists of nonzero pure tensors*

$$x \otimes v \in C^0 \otimes V_6$$

where $x^2 = 0$.

Proof. Let $\Xi \in \mathbf{F}^{-1}(0) = \mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0}$, then $\Xi = (0, b, c, 0)$, where

$$b = J(\beta_1, \beta_2, \beta_3) \in \mathcal{J}_{C^0},$$
$$c = J(\gamma_1, \gamma_2, \gamma_3) \in \mathcal{J}_{C^0}.$$

If $\Xi \in \Omega$, then by Lemma 2.2 we know that b or c is not equal to 0 and

$$b^\# = 0,$$
$$c^\# = 0,$$
$$b * c = 0.$$

The equation $b^\# = 0$ implies that $\beta_i^2 = \beta_j^2 = \beta_i\beta_j = 0$. Similarly, the equation $c^\# = 0$ implies that $\gamma_i^2 = \gamma_j^2 = \gamma_i\gamma_j = 0$. If C^0 is anisotropic then there are no nonzero nilpotent elements. This proves the first claim. Now assume $C = \mathrm{M}(2, F)$, the algebra of 2×2 matrices. The equation $b * c = 0$ implies $\beta_i\gamma_j + \gamma_j\beta_i = 0$ for all i and j. We need the following lemma.

Lemma 3.10. *Let* $\beta, \gamma \in \mathrm{M}(2, F)$. *If* $\beta^2 = 0$, $\gamma^2 = 0$ *and* $\beta\gamma + \gamma\beta = 0$. *Then* β *and* γ *are proportional.*

Proof. This is trivial if β or γ is 0, so suppose not. Then $\ker\beta = \mathrm{Im}\beta$ and $\ker\gamma = \mathrm{Im}\gamma$ are one dimensional. The equation $\beta\gamma = -\gamma\beta$ implies that γ acts on $\ker\beta$. Thus $\ker\beta = \mathrm{Im}\beta = \ker\gamma = \mathrm{Im}\gamma$ and so β and γ are proportional. □

It follows that all β_i and γ_j are linearly dependent, proving the proposition. □

Now we can describe $\mathcal{V}_{\overline{N}}$.

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Proposition 3.11. *Let C be a composition algebra over F.*

1. *If C^0 is anisotropic, then $\mathcal{V}_{\overline{N}} \cong \mathcal{V}_{\overline{\mathcal{N}}}$.*
2. *If $C = \mathrm{M}(2,F)$, write $\mathscr{G}$ for $\mathrm{PGL}_2(F) = \mathrm{Aut}(C)$, then $\mathcal{V}_{\overline{N}}$ has a composition series with a quotient $\mathcal{V}_{\overline{\mathcal{N}}}$ and a submodule*

$$\mathrm{Ind}_{\mathscr{B}\times Q}^{\mathscr{G}\times \mathrm{GSp}_6}(C_c^\infty(F^\times)) \otimes |\mathrm{sim}|^3$$

where $\mathscr{B}$ is a Borel subgroup of $\mathscr{G}$, and Q is a maximal parabolic in $\mathrm{GSp}_6(F)$ stabilizing a line in V_6, and sim *is the similitude character of* GSp_6. *The induction is not normalized.*

Proof. By Theorem 3.1, $\mathcal{V}_{\overline{N}}$ has a filtration with quotient $\mathcal{V}_{\overline{\mathcal{N}}}$ and submodule

$$C_c^\infty(\Omega)_{\overline{N}} \cong C_c^\infty(\Omega_0).$$

Now we apply by Proposition 3.9. If C^0 is anisotropic then Ω_0 is empty and we are done. So suppose $C = \mathrm{M}(2,F)$. Then $\Omega_0 \subset C^0 \otimes V_6$ consists of nonzero pure tensors $x \otimes v$ such that $x^2 = 0$. Fix $\omega = x \otimes v$. The stabilizer in $\mathscr{G}$ of the line through x is a Borel subgroup $\mathscr{B}$, and the stabilizer in GSp_6 of the line through v is a maximal parabolic subgroup Q. The stabilizer of $x \otimes v$ is a subgroup of $\mathscr{B} \times Q$ such that the quotient is $F^\times$. Observe that $C_c^\infty(\Omega_0)$, as a $\mathscr{G} \times \mathrm{GSp}_6(F)$-module, is obtained by compact induction of the trivial representation of the stabilizer of $x \otimes v$. Hence, using induction in stages,

$$C_c^\infty(\Omega_0) \cong \mathrm{Ind}_{\mathscr{B}\times Q}^{\mathscr{G}\times \mathrm{GSp}_6}(C_c^\infty(F^\times))$$

where the induction is not normalized. This completes the proof, after taking into account additional twisting by the character of $\mathcal{M}$ in Theorem 3.1. □

We remark that the variant of the previous proposition, when C is an octonion algebra, was obtained in [27].

3.2. Fourier–Jacobi functor

Now we recall the definition of the Fourier–Jacobi functor. (For more details, see Weissman [30].) By the Stone-Von-Neumann theorem, the group N has a unique irreducible smooth representation with central character ψ, denoted by (ρ_ψ^N, W_ψ). By [30, Proposition 2.5], there is a unique extension of (ρ_ψ^N, W_ψ) to a projective representation of M_1N, where M_1 is the commutator subgroup of M. Furthermore, $\widetilde{\mathrm{Sp}}(14,F)$, the two-fold cover of the symplectic group $\mathrm{Sp}(14,F)$ where $14 = \dim(N/Z)$, also acts on W_ψ via the Weil representation. So, $\widetilde{M}_1 \cong \widetilde{\mathrm{Sp}}(6,F)$, the metaplectic double cover of $M_1 \cong \mathrm{Sp}(6,F)$, acts on W_ψ through the Weil representation of $\widetilde{\mathrm{Sp}}(14,F)$.

We now make a brief comment on why the embedding $M_1 \hookrightarrow \mathrm{Sp}(14,F)$ induced by the action of M_1 on N/Z induces an embedding $\widetilde{M}_1 \hookrightarrow \widetilde{Sp}(14,F)$. The representation of M_1 on N/Z is irreducible and corresponds to the third fundamental weight of $M_1 \cong \mathrm{Sp}(6,F)$ (Bourbaki labeling). Let $H \subset M_1$ be a long-root $\mathrm{SL}(2)$ subgroup. Then the restriction of the M_1-representation N/Z to H decomposes into four copies of the trivial representation and five copies of the unique two-dimensional representation of $\mathrm{SL}(2)$. Thus the image of H in $\mathrm{Sp}(14,F)$ sits diagonally inside five commuting long-root $\mathrm{SL}(2)$ subgroups of $\mathrm{Sp}(14)$. Since five is odd and each root is long, the preimage of H in $\widetilde{Sp}(14,F)$ does not split. Therefore the preimage of M_1 in $\widetilde{Sp}(14,F)$ does not split, giving the embedding $\widetilde{M}_1 \hookrightarrow \widetilde{Sp}(14,F)$.

If (π, V) is a smooth representation of G, then the Fourier–Jacobi functor with respect to the Heisenberg parabolic P sends π to

$$\mathrm{FJ}(\pi) = \mathrm{Hom}_N(W_\psi, V_{(Z,\psi)}).$$

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The space $\mathrm{FJ}(\pi)$ is an $\widetilde{M_1}$-module with the action defined by $[m \cdot f](w) = \pi(m)f(m^{-1}w)$, where the action of $\widetilde{M}_1$ on $V_{(Z,\psi)}$ factors through M_1. The Fourier–Jacobi functor does not depend on ψ ([30, Proposition 3.1]).

Remark. The work of Weissman [30] assumes that the groups involved are simply laced. However, the results that we require also hold for the non-simply laced group F_4. In particular we use [30, Corollary 6.1.4], which states that if the Fourier–Jacobi functor kills an irreducible representation then that representation is the trivial representation. In fact, this statement holds outside of type C_n.

We use the sequence (3.1) and the Fourier–Jacobi functor to investigate the constituents of $\Theta(\tau)$ via the surjection $\mathcal{V} \twoheadrightarrow \tau\otimes\Theta(\tau)$. This is done in two steps. First, we use the Fourier–Jacobi functor in conjunction with a classical theta correspondence to show that $\Theta(\tau)$ has at most two nontrivial constituents. Second, by applying twisted coinvariants to the sequence (3.1) along with another application of the Fourier–Jacobi functor we show that $\Theta(\tau)$ has a single constituent, which is nontrivial, that is, $\Theta(\tau)$ is nontrivial and irreducible.

4. Lifting supercuspidal representations from $\mathrm{Aut}(C)$ to F_4

Our objective in this section is to investigate $\Theta(\tau)$, where τ is a supercuspidal representation of $\mathscr{G}$, using the tools of Section 3. The main result is Theorem 4.10, where we show that $\Theta(\tau)$ is irreducible, and $\Theta(\tau_1) \cong \Theta(\tau_2)$ implies that $\tau_1 \cong \tau_2$, where τ_1, and τ_2 are supercuspidal.

4.1. At most two nontrivial constituents

We begin by using the Fourier–Jacobi functor and a classical theta correspondence to show that $\Theta(\tau)$ has at most two nontrivial constituents. This is accomplished in Corollary 4.5.

Let $\mathcal{M}_1 \subset \mathcal{M}$ be the commutator subgroup. Let $\mathcal{P}_1 = \mathcal{M}_1\mathcal{N}$.

Lemma 4.1. *Let $\rho_\psi^{\mathcal{N}}$ be the unique irreducible smooth representation of $\mathcal{N}$ with central character ψ. As $\mathcal{P}_1$-modules $\mathcal{V}_{(Z,\psi)} \cong \rho_\psi^{\mathcal{N}}$.*

Proof. The canonical $\mathcal{P}_1$-module map $\mathrm{Hom}_{\mathcal{N}}(\rho_\psi^{\mathcal{N}}, \mathcal{V}_{(Z,\psi)}) \otimes \rho_\psi^{\mathcal{N}} \to \mathcal{V}_{(Z,\psi)}$ is an isomorphism by [30, Proposition 3.2], and $\mathrm{Hom}_{\mathcal{N}}(\rho_\psi^{\mathcal{N}}, \mathcal{V}_{(Z,\psi)}) \cong \mathbb{C}$ with trivial $\mathcal{P}_1$-action by [7, Definition 3.6]. □

Lemma 4.2. *Let $P_1 = M_1N$. As an $\mathrm{Aut}(C) \times P_1$-module,*

$$\mathcal{V}_{(Z,\psi)} \cong \rho_\psi^N \otimes \omega_\psi,$$

where ω_ψ is the Weil representation of $O(C^0) \times \widetilde{\mathrm{Sp}}(V_6)$ as a dual pair in $\widetilde{\mathrm{Sp}}(C^0 \otimes V_6)$. Under this isomorphism, $\mathscr{G}$ acts on the second factor, while the action of M_1 is on both factors. (We note that M_1 does not act on either factor individually. Rather $\widetilde{M}_1$ acts genuinely on both factors, thus the diagonal action factors through M_1.)

Proof. By Lemma 4.1, $\mathcal{V}_{(Z,\psi)} \cong \rho_\psi^{\mathcal{N}}$ as $\mathcal{P}_1$-modules. We must describe the restriction to $\mathscr{G} \times M_1$.

Let $N^\perp \subseteq \mathcal{N}$ be the subgroup containing Z such that $N^\perp/Z$ is the orthogonal complement of the symplectic subspace $N/Z \subseteq \mathcal{N}/Z$. By Moeglin-Vignéras-Waldspurger [20, Chapitre 2, I.6 (2) and II.1 (6)], it follows that $\rho_\psi^{\mathcal{N}} \cong \rho_\psi^N \otimes \rho_\psi^{N^\perp}$ as $\widetilde{\mathrm{Sp}}(14, F)N \times \widetilde{\mathrm{Sp}}(18, F)$-modules. Note that $\mathscr{G}$ acts trivially on N and so it acts trivially on ρ_ψ^N.

Recall from Subsection 2.5 the identification of $\mathcal{N}/Z$ with $\mathbb{W}_C$. This then identifies N/Z with $\mathbb{W}_F$, and $N^\perp/Z$ with $\mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0}$. From the isomorphism $\mathcal{J}_{C^0} \oplus \mathcal{J}_{C^0} \cong C^0 \otimes V_6$ of symplectic spaces (line (2.5)), we see that the action of $\mathscr{G} \times M_1$ on $\rho_\psi^{N^\perp}$ is through the action of the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(C^0 \otimes V_6)$ restricted to $O(C^\circ) \times \widetilde{\mathrm{Sp}}(V_6)$. □

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Using Lemma 4.2 we show in the next proposition that the Fourier–Jacobi functor with respect to P applied to $\mathcal{V}$ is isomorphic to the Weil representation. This allows us to study $\Theta(\tau)$ using a classical $O(3) \times \mathrm{Sp}(6)$ theta correspondence.

Proposition 4.3. *The $\mathscr{G} \times \widetilde{M}_1$-module $\mathrm{FJ}(\mathcal{V})$ is isomorphic to the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(C^0 \otimes V_6)$ restricted to $\mathscr{G} \times \widetilde{M_1}$. (Recall that $\mathscr{G} \cong SO(C^0)$ and $M_1 \cong \mathrm{Sp}(6, F)$.)*

Proof. By definition $\mathrm{FJ}(\mathcal{V}) = \mathrm{Hom}_N(\rho_\psi^N, \mathcal{V}_{(Z,\psi)})$. By Lemma 4.2, $\mathcal{V}_{Z,\psi} \cong \omega_\psi \otimes \rho_\psi^N$ as $\mathscr{G} \times M_1 N$-modules. Thus as $\mathscr{G} \times \widetilde{M}_1$-modules

$$\mathrm{FJ}(\mathcal{V}) \cong \mathrm{Hom}_N(\rho_\psi^N, \omega_\psi \otimes \rho_\psi^N).$$

Since ρ_ψ^N is a finitely generated N-module, $\mathrm{Hom}_N(\rho_\psi^N, \omega_\psi \otimes \rho_\psi^N) \cong \omega_\psi \otimes \mathrm{Hom}_N(\rho_\psi^N, \rho_\psi^N)$. By Schur's lemma $\mathrm{Hom}_N(\rho_\psi^N, \rho_\psi^N) \cong \mathbb{C}$. Thus $\mathrm{FJ}(\mathcal{V}) \cong \omega_\psi$ as $\mathscr{G} \times \widetilde{M}_1$-modules. □

Now we introduce some notation to discuss the $O(C^0) \times \mathrm{Sp}(6, F)$ theta correspondence. Since $\mathscr{G} \times M_1 \cong \mathrm{SO}(C^0) \times \mathrm{Sp}(6, F)$, we almost have a classical dual pair. The representation τ admits two extensions to the group $O(C^0) \cong \mathrm{SO}(C^0) \times \{\pm id_{C^0}\}$ determined by whether $-id_{C^0}$ acts by ± 1. We write $\tau^\pm$ for the two extensions and $\Theta^\dagger(\tau^\pm)$ for the big theta lift of $\tau^\pm$ with respect to the action of $O(C^0) \times \widetilde{\mathrm{Sp}}(6, F)$ on the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(18, F)$.

Proposition 4.4. *Let $\tau \in \mathrm{Irr}(\mathscr{G})$. There is a surjective $\widetilde{M}_1$-module homomorphism*

$$\Theta^\dagger(\tau^+) \oplus \Theta^\dagger(\tau^-) \twoheadrightarrow \mathrm{FJ}(\Theta(\tau)).$$

Proof. We apply the Fourier–Jacobi functor, which is exact, to the surjective map $\mathcal{V} \twoheadrightarrow \tau \otimes \Theta(\tau)$ to get a map of $\mathscr{G} \times \widetilde{M}$-modules

$$\mathrm{FJ}(\mathcal{V}) \twoheadrightarrow \tau \otimes \mathrm{FJ}(\Theta(\tau)). \tag{4.1}$$

By Proposition 4.3, we know that $\mathrm{FJ}(\mathcal{V}) \cong \omega_\psi$ as $\mathscr{G} \times \widetilde{M}_1$-modules. Therefore, we have a surjective $O(C^0) \times \widetilde{\mathrm{Sp}}(6, F)$-module map

$$\omega_\psi \twoheadrightarrow (\tau^+ \otimes \Theta^\dagger(\tau^+)) \oplus (\tau^- \otimes \Theta^\dagger(\tau^-)).$$

Upon restricting to $SO(C^0) \times \widetilde{\mathrm{Sp}}(6, F) \cong \mathscr{G} \times \widetilde{M}_1$ we get a surjective homomorphism

$$\mathrm{FJ}(\mathcal{V}) \cong \omega_\psi \twoheadrightarrow \tau \otimes (\Theta^\dagger(\tau^+) \oplus \Theta^\dagger(\tau^-)).$$

Moreover, this is the surjection onto the maximal τ-isotypic quotient of $\mathrm{FJ}(\mathcal{V})$. Thus the map from line (4.1) factors through the maximal τ-isotypic quotient to give a surjection

$$\tau \otimes (\Theta^\dagger(\tau^+) \oplus \Theta^\dagger(\tau^-)) \twoheadrightarrow \tau \otimes \mathrm{FJ}(\Theta(\tau)).$$

By construction, this map factors over the tensor product and the result follows. □

Corollary 4.5. *Let $\tau \in \mathrm{Irr}(\mathscr{G})$ be a supercuspidal. The G-module $\Theta(\tau)$ has at most two nontrivial irreducible subquotients, each with multiplicity at most 1.*

Proof. This follows from Proposition 4.4 and the following two results. First, when $\tau^\pm$ is supercuspidal, the $\widetilde{M}_1$-modules $\Theta^\dagger(\tau^+)$ and $\Theta^\dagger(\tau^-)$ are irreducible and distinct (Kudla [17]; Moeglin-Vignéras-Waldspurger [20, Chapitre 3, IV, 4.]). Second, the Fourier–Jacobi functor is exact and the only irreducible representation that it kills is the trivial representation. (See [30, Proposition 3.1; Corollary 6.1.4] and our remark in Subsection 3.2.) □

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4.2. Unique nontrivial constituent

In this subsection, we show that $\Theta(\tau)$ has exactly one nontrivial constituent.

Using Propositions 3.7 and 3.11 we can compute twisted coinvariants of $\Theta(\tau)$.

Proposition 4.6. *Let $(1, 0, c, d) \in \mathbb{W}_F$ be an element of rank 3 such that $\mathrm{SO}(c, 3) \cong \mathrm{Aut}(C)$. Let $\Psi^{\pm}$ be the character of $\overline{N}/\overline{Z}$ corresponding to the element $(1, 0, c, \pm d) \in \mathbb{W}_F \cong N/Z$. Let $\tau \in \mathrm{Irr}(\mathscr{G})$ (not necessarily supercuspidal). Then as $\mathrm{SO}(c, 3) \subset \mathrm{Stab}_M(\Psi^{\pm})$-modules*

$$\Theta(\tau)_{(\overline{N},\Psi^{\pm})} \cong \widetilde{\tau}.$$

and $\Theta(\tau)$ must have a nontrivial constituent.

Furthermore, if ρ_1 and ρ_2 are distinct irreducible subquotients of $\Theta(\tau)$, then

1. $(\rho_j)_{(N,\Psi^+)} \cong (\rho_j)_{(N,\Psi^-)}$ *as $\mathrm{SO}(c, 3)$-modules, $j = 1, 2$;*
2. *for $\epsilon \in \{\pm\}$, $(\rho_1)_{(N,\Psi^{\epsilon})}$ and $(\rho_2)_{(N,\Psi^{\epsilon})}$ cannot both be nonzero.*

Proof. The first part is a simple consequence of Corollary 3.7, and Lemma 2.8.

Suppose that ρ_1, ρ_2 are two distinct irreducible subquotients of $\Theta(\tau)$. Note that the characters $\Psi^{\pm}$ are M-conjugate, because $s_{-1}(1, 0, c, d) = (1, 0, c, -d)$. Thus

$$(\rho_j)_{(\overline{N},\Psi^+)} \cong (\rho_j)_{(\overline{N},\Psi^-)}.$$

Finally $(\rho_1)_{(\overline{N},\Psi^+)}$ and $(\rho_2)_{(\overline{N},\Psi^+)}$ cannot both be nonzero because this would imply that the irreducible $\mathrm{Aut}(C)$-module $\Theta(\tau)_{(\overline{N},\Psi^+)} \cong \widetilde{\tau}$ has length greater than or equal to 2. □

The next lemma employs two Heisenberg parabolic subgroups in G. Let $\overline{P} = M\overline{N}$ and $\overline{P}' = M'\overline{N}'$ be two Heisenberg parabolic subgroups. Let $\overline{Z} \subset \overline{N}$ and $\overline{Z}' \subset \overline{N}'$ be the centers of the Heisenberg groups. Furthermore, suppose that $\overline{Z}$ ($\overline{Z}'$) is the root subgroup associated to the G_2 relative root $2\alpha + 3\beta$ ($\alpha + 3\beta$).

We also use the following notation. Let $\overline{N}^{\alpha}$ be the subgroup of $\overline{N}$ generated by the root subgroups of the roots $\{2\alpha + 3\beta, \alpha + 3\beta, \alpha + 2\beta, \alpha + \beta\}$ in the G_2 relative root system. Let $\overline{N}_{\alpha+\beta} = M' \cap \overline{N}$, which is the root subgroup of $\alpha + \beta$. Let $L \subset M$ be the subgroup generated by elements $hs^*_{\det(h)}$, where $h \in H_F$. For a character Ψ of $\overline{N}$, we abuse notation and continue to write Ψ for its restriction to $\overline{N}^{\alpha}$ and $\overline{N}_{\alpha+\beta}$.

Lemma 4.7. *Let σ be a smooth representation of G. Let Ψ be the character of $\overline{N}/\overline{Z}$ corresponding to the element $(1, 0, c, d) \in W_F$, where $\mathrm{SO}(3, c) \cong \mathscr{G}$. Then as $\mathrm{Stab}_L((\overline{N}^{\alpha}, \Psi)) = \mathrm{Stab}_L((\overline{N}_{\alpha+\beta}, \Psi)) \cong O(3, c)$-modules*

$$\sigma_{(\overline{N}^{\alpha},\Psi)} \cong \mathrm{FJ}'(\sigma)_{(\overline{N}_{\alpha+\beta},\Psi)}.$$

Proof. We begin with some preliminaries. Let W^+ be the subgroup of $\overline{N}'$ generated by the root subgroups of the relative roots $\{\alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$ in the G_2 relative root system. We extend the character ψ to W^+ so that it is trivial on the $\alpha + 2\beta$ and $2\alpha + 3\beta$ root spaces, and continue to call this extended character ψ. Note that this is the restriction of Ψ to W^+. Now we prove the lemma.

From Weissman [30, Proposition 3.2], we have $\sigma_{\overline{Z}',\psi} \cong \mathrm{Hom}_{\overline{N}'}(\rho_{\psi}^{\overline{N}'}, \sigma_{(\overline{Z}',\psi)}) \otimes \rho_{\psi}^{\overline{N}'}$ as $M_1' \ltimes \overline{N}'$-modules. (Remember, $\widetilde{M}_1'$ acts genuinely on each factor.) It suffices for us to restrict the action of M_1' to the subgroup $L \cong \mathrm{GL}(3, F)$.

Since W^+/Z' is a maximal isotropic subspace of N'/Z', the (W^+, ψ)-coinvariants of $\rho_{\psi}^{\overline{N}'}$ is a one-dimensional space. Thus as $\mathrm{Stab}_L((W^+, \psi)) = L$-modules

$$(\sigma_{(\overline{Z}',\psi)})_{(W^+,\psi)} \cong \mathrm{Hom}_{\overline{N}'}(\rho_{\psi}^{\overline{N}'}, \sigma_{(\overline{Z}',\psi)}) = FJ'(\sigma).$$

Applying $(\overline{N}_{\alpha+\beta},\Psi)$-coinvariants and using transitivity of coinvariants we get an isomorphism of $\mathrm{Stab}_L((\overline{N}^{\alpha},\Psi))=\mathrm{Stab}_L((\overline{N}_{\alpha+\beta},\Psi))\cong O(3,c)$-modules

$$\sigma_{(\overline{N}^{\alpha},\Psi)}\cong[(\sigma_{(\overline{Z}',\psi)})_{(W^+,\psi)}]_{(\overline{N}_{\alpha+\beta},\Psi)}\cong FJ'(\sigma)_{(\overline{N}_{\alpha+\beta},\Psi)}.\qquad\square$$

Proposition 4.8. *Let $\tau\in\mathrm{Irr}(\mathscr{G})$ be supercuspidal. Then $\Theta(\tau)$ has a unique nontrivial irreducible subquotient.*

Proof. By Proposition 4.6, $\Theta(\tau)$ has at least one nontrivial irreducible subquotient.

By Proposition 4.4 (applied using P') we know that there is an $\widetilde{M}_1'$-module surjection

$$\Theta^{\dagger}(\tau^+)\oplus\Theta^{\dagger}(\tau^-)\twoheadrightarrow FJ'(\Theta(\tau)).$$

Since we are assuming that τ is supercuspidal it follows that $\Theta^{\dagger}(\tau^{\pm})$ is an irreducible $\widetilde{M}_1'$-module. Thus $\mathrm{FJ}'(\Theta(\tau))$ is completely reducible of length at most 2.

Suppose that $\Theta(\tau)$ has two distinct irreducible subquotients σ^+,σ^- different from the trivial representation. Since $\sigma^{\pm}$ is not trivial $\mathrm{FJ}'(\sigma^{\pm})\neq 0$. Then without loss of generality we may assume that we have $\widetilde{M}_1'$-module isomorphisms $\Theta^{\dagger}(\tau^{\pm})\cong\mathrm{FJ}'(\sigma^{\pm})$.

We take $(\overline{N}_{\alpha+\beta},\Psi)$-coinvariants and apply Lemma 4.7 to get $O(3,c)$-module isomorphisms $\Theta^{\dagger}(\tau^{\pm})_{(\overline{N}_{\alpha+\beta},\Psi)}\cong(\sigma^{\pm})_{(\overline{N}^{\alpha},\Psi)}$.

Now by an analog of Proposition 4.6 in the classical case, we have $\Theta^{\dagger}(\tau^{\pm})_{(\overline{N}_{\alpha+\beta},\Psi)}\cong\widetilde{\tau}^{\pm}$ as $\mathrm{O}(3,c)$-modules.

The natural $\mathrm{SO}(3,c)$-module quotient maps $(\sigma^{\pm})_{(\overline{N}^{\alpha},\Psi)}\longrightarrow(\sigma^{\pm})_{(\overline{N},\Psi^{\pm})}$ define an isomorphism

$$\widetilde{\tau}\cong(\sigma^{\pm})_{(\overline{N}^{\alpha},\Psi)}\longrightarrow(\sigma^{\pm})_{(\overline{N},\Psi^{+})}\oplus(\sigma^{\pm})_{(\overline{N},\Psi^{-})}$$

of $\mathrm{SO}(3,c)$-modules. But by Proposition 4.6, $(\sigma^{\epsilon})_{(\overline{N},\Psi^{+})}\oplus(\sigma^{\epsilon})_{(\overline{N},\Psi^{-})}=0$ for at least one $\epsilon\in\{\pm\}$. Thus $\widetilde{\tau}=0$, a contradiction. Therefore, $\Theta(\tau)$ must have at most one nontrivial irreducible subquotient. $\square$

Finally, we rule out the existence of trivial irreducible subquotients of $\Theta(\tau)$.

Proposition 4.9. *Let $\tau\in\mathrm{Irr}(\mathscr{G})$ be supercuspidal. Then $\Theta(\tau)$ does not contain an irreducible subquotient that is trivial.*

Proof. Suppose that the trivial representation 1 is a subquotient of $\Theta(\tau)$, then $\tau\otimes 1_{\overline{N}}$ is a subquotient of $\mathcal{V}_{\overline{N}}$. Observe that $1_{\overline{N}}$ is the trivial representation of M. By Proposition 3.11, $\mathcal{V}_{\overline{N}}$ has a $\mathscr{G}\times M$-module filtration with $\mathcal{V}_{\overline{\mathcal{N}}}$ as a quotient. From Theorem 3.1,

$$\mathcal{V}_{\overline{\mathcal{N}}}\cong(\mathcal{V}(\mathcal{M})\otimes|\det|^{3/32})\oplus|\det|^{6/32},$$

where the center of M acts trivially on $\mathcal{V}(\mathcal{M})$. Thus the center of M acts by two nontrivial characters on the two summands of $\mathcal{V}_{\overline{\mathcal{N}}}$ hence the trivial representation of M cannot be a subquotient of $\mathcal{V}_{\overline{\mathcal{N}}}$. The bottom part of the filtration, which appears if C is split, is a principal series representation of $\mathscr{G}$, and hence τ cannot be a subquotient there. $\square$

Now we prove our main theorem on the theta lift of super cuspidal representations.

Theorem 4.10. *Let $\tau_1,\tau_2\in\mathrm{Irr}(\mathscr{G})$ where τ_1 is supercuspidal.*

1. *The theta lift $\Theta(\tau_1)$ is an irreducible representation of G.*
2. *If $\theta(\tau_1)\cong\theta(\tau_2)$, then $\tau_1\cong\tau_2$.*

Proof. (1) By Propositions 4.8 and 4.9 the representation $\Theta(\tau)$ is irreducible. (2) By the assumption, and using (1), we have a surjection $\Theta(\tau_2)\twoheadrightarrow\Theta(\tau_1)$. Then, by Proposition 4.6,

$$\widetilde{\tau}_2\cong\Theta(\tau_2)_{(N,\Psi^{\pm})}\twoheadrightarrow\Theta(\tau_1)_{(N,\Psi^{\pm})}\cong\widetilde{\tau}_1.\qquad\square$$

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5. Jacquet modules II

In this section, we take C to be the split quaternion algebra $M(2, F)$ of 2×2 matrices with entries in F. In particular, $\mathscr{G} = \mathrm{Aut}(C) = \mathrm{PGL}_2(F)$. The main objective of this section is to compute the Jacquet module of the minimal representation of E_7 with respect to a Borel subgroup of $\mathscr{G}$. These calculations are applied in Section 6 to compute the big theta lift of constituents of principal series of $\mathscr{G}$ to G.

Our approach follows the argument of Savin [26] and Magaard-Savin [19] utilizing the exact sequence from [26, Theorem 6.5], which we recall after introducing some notation.

Fix a Borel subgroup $\mathcal{B} \subset \mathcal{G}$, let $\mathcal{P} \supset \mathcal{B}$ be the unique maximal parabolic subgroup corresponding to the E_6 subdiagram inside the E_7 diagram. Fix a Levi decomposition $\mathcal{P} = \mathcal{M}\mathcal{N}$ and note that $\mathcal{N}$ can be given the structure of the exceptional cubic Jordan algebra $\mathcal{J} = \mathcal{J}_{\mathbb{O}}$ [15]. We identify $\mathcal{N}$ with $\mathcal{J}$ as F-vector spaces. Under this identification $\mathcal{M}$ is the group of linear transformations of $\mathcal{J}$ that preserve the cubic norm form of $\mathcal{J}$ up to scaling. The semisimple part of $\mathcal{M}$ is a group of type E_6. (For details see [16].)

Let ω be the set of singular points in $\mathcal{J} \cong \mathcal{N}$, that is, the highest weight vectors for a Borel subgroup in $\mathcal{M}$. Equivalently, ω is the set of rank 1 elements in $\mathcal{J}$.

Theorem 5.1 (Magaard-Savin [19], Theorem 1.1; Savin [26], Theorem 6.5). *Let $\mathcal{P} = \mathcal{M}\mathcal{N}$ be the maximal parabolic subgroup defined above. Let $\overline{\mathcal{P}} = \mathcal{M}\overline{\mathcal{N}}$ be its opposite. The minimal representation $(\Pi, \mathcal{V})$ of $\mathcal{G}$ has a $\overline{\mathcal{P}}$-invariant filtration*

$$0 \to C_c^\infty(\omega) \to \mathcal{V} \to \mathcal{V}_{\overline{\mathcal{N}}} \to 0. \tag{5.1}$$

Here $C_c^\infty(\omega)$ denotes the space of locally constant, compactly supported functions on ω, and $\mathcal{V}_{\overline{\mathcal{N}}}$ is the space of $\overline{\mathcal{N}}$-coinvariants of $\mathcal{V}$. Furthermore, the $\overline{\mathcal{P}}$-module structure is given by:

1. *Let $f \in C_c^\infty(\omega)$ and let $m\overline{n} \in \overline{\mathcal{P}} = \mathcal{M}\overline{\mathcal{N}}$. Then*

$$[\Pi(\overline{n})f](x) = \psi(\langle x, \overline{n}\rangle) f(x) \tag{5.2}$$

and

$$[\Pi(m)f](x) = |\det(m)|^{s/d} f(m^{-1}x). \tag{5.3}$$

2.

$$\mathcal{V}_{\overline{\mathcal{N}}} \cong \mathcal{V}(\mathcal{M}) \otimes |\det|^{t/d} + |\det|^{s/d}, \tag{5.4}$$

where $\mathcal{V}(\mathcal{M})$ is the minimal representation of $\mathcal{M}$ (center acting trivially).

Above $\langle -, - \rangle : \mathcal{N} \times \overline{\mathcal{N}} \to F$ is the F-valued pairing induced by the Killing form on $\mathrm{Lie}(\mathcal{G})$, *and* det *is the determinant of the representation of $\mathcal{M}$ on $\overline{\mathcal{N}}$, and d is the dimension of $\mathcal{N}$. The values of s and t are given in the following table.*

$\mathcal{G}$	s	t
E_6	4	2
E_7	6	3

It will be convenient to describe the dual pair $\mathscr{G} \times G$ in terms of the parabolic subgroup $\mathcal{P}$. If we identify $\mathcal{N} \cong \mathcal{J}$ and $\mathcal{M}$ with the group of similitudes of the norm form, then $G \cong \mathrm{Aut}(\mathcal{J})$ sits in $\mathcal{M}$. Let $\mathscr{G}$ be the centralizer of G in $\mathcal{G}$. Let $\mathscr{B} = \mathscr{T}\mathscr{U} \subset \mathscr{G}$ be Borel subgroup defined by

$$\mathscr{T} = \mathscr{G} \cap \mathcal{M} \text{ and } \mathscr{U} = \mathscr{G} \cap \mathcal{N}.$$

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Then $\mathcal{T}$ is the center of $\mathcal{M}$ and $\mathcal{U}$ the set of scalar matrices in $\mathcal{J}$ under the identification $\mathcal{N} \cong \mathcal{J}$. Recall that, for the purpose of describing representations of $\mathcal{G}$, we identified $\mathcal{T}$ with $F^\times$ such that $\mathcal{T}$ acts on $\overline{\mathcal{U}}$ by multiplication, it follows that $\mathcal{T} \cong F^\times$ acts on $\mathcal{N} \cong \mathcal{J}$ by inverse scalar multiplication. Thus the character $|\det(m)|^{s/d}$ restricted to $\mathcal{T}$ is $|t|^6$.

5.1. Untwisted Jacquet modules

Our objective in this section is to describe the $\mathcal{T} \times G$-module $r_{\overline{\mathcal{B}}}(C_c^\infty(\omega))$. This is accomplished in Proposition 5.5.

For $S \subset \overline{\mathcal{N}}$ we write

$$S^\perp = \{x \in \mathcal{N} | \psi(\langle x, s\rangle) = 1, \text{ for all } s \in S\}.$$

Let $\mathcal{J}^0$ be the set of trace 0 elements in $\mathcal{J}$. Under the identification $\mathcal{N} \cong \mathcal{J}$, we have $\overline{\mathcal{U}}^\perp \cong \mathcal{J}^0$. Let $\omega_0 = \omega \cap \mathcal{J}^0$.

Lemma 5.2. *The restriction map $C_c^\infty(\omega) \to C_c^\infty(\omega_0)$ induces an isomorphism $r_{\overline{\mathcal{B}}}(C_c^\infty(\omega)) \cong C_c^\infty(\omega_0)$ of $(\mathcal{T} \times G)$-modules, where the action on $C_c^\infty(\omega_0)$ is given by*

$$((t, g) \cdot f)(x) = |t|^{6-\frac{1}{2}} f(g^{-1} \cdot xt), \qquad (t, g) \in \mathcal{T} \times G.$$

Proof. The proof is the same as Magaard-Savin [19], Lemma 2.2. □

Lemma 5.3 (Aschbacher [2], section (8.6)). *The action of $\mathcal{T} \times G$ on ω_0 is transitive.*

Proof. This follows from translating Aschbacher's terminology into ours. □

If $x_0 \in \omega_0$, then, as $\mathcal{T} \times G$-modules

$$C_c^\infty(\omega_0) \cong |-|^{\frac{11}{2}} \cdot \mathrm{ind}_{\mathrm{Stab}_{\mathcal{T}\times G}(x_0)}^{\mathcal{T}\times G}(1)$$

Next, we want to describe the stabilizer of a point in ω_0. The highest weight of the action of G on $\mathcal{J}^0 \subset \mathcal{J}$ is the fundamental weight ϖ_4 taking value 1 on the simple coroot $\alpha_4^\vee$ and 0 on the other simple coroots. (We are using the Bourbaki labeling for simple roots [3, Plate VIII]. In particular, α_4 is the short simple root of degree 1 in the Dynkin diagram.) The next lemma follows directly from definitions.

Lemma 5.4. *Let $x_0 \in \omega_0$ be a highest weight vector with respect to the Borel subgroup B. Let $Q \supseteq B$ be the maximal parabolic subgroup of G obtained by removing α_4 from the F_4 Dynkin diagram. This yields a parabolic subgroup of type B_3. Then*

$$\mathrm{Stab}_{\mathcal{T}\times G}(x_0) = \{(t, q) \in \mathcal{T} \times Q \mid t = \varpi_4(q)\}.$$

By transitivity of induction, we have the $\mathcal{T} \times G$-module isomorphism

$$\mathrm{ind}_{\mathrm{Stab}_{\mathcal{T}\times G}(x_0)}^{\mathcal{T}\times G}(1) \cong \mathrm{Ind}_{\mathcal{T}\times Q}^{\mathcal{T}\times G}(\mathrm{ind}_{\mathrm{Stab}_{\mathcal{T}\times G}(x_0)}^{\mathcal{T}\times Q}(1)) \cong \mathrm{Ind}_{\mathcal{T}\times Q}^{\mathcal{T}\times G}(C_c^\infty(F^\times)).$$

In order to write the last module in terms of normalized parabolic induction, we need to replace $C_c^\infty(F^\times)$ by $\delta_Q^{-1/2} \cdot C_c^\infty(F^\times)$. Recall that $\delta_Q^{1/2}(q) = |\varpi_4(q)|^{-11/2}$. We also need to bring back the twist by $|t|^{11/2}$. Observe that the two exponents are inverses of each other. Hence

$$r_{\overline{\mathcal{B}}}(C_c^\infty(\omega)) \cong i_{\mathcal{T}\times Q}^{\mathcal{T}\times G}(C_c^\infty(F^\times)), \tag{5.5}$$

where the action of $\mathcal{T} \times Q$ on $C_c^\infty(F^\times)$ is given by

$$((t, q) \cdot f)(x) = f(\varpi_4(q^{-1})xt). \tag{5.6}$$

Putting everything together we have the following description of $r_{\overline{\mathcal{B}}}(\mathcal{V})$.

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Proposition 5.5. *As a representation of $\mathscr{T} \times G$, the module $r_{\overline{\mathscr{B}}}(\mathcal{V})$ has a filtration with successive quotients*

$$|-|^{5/2} \cdot \mathcal{V}(\mathcal{M}) \oplus |-|^{11/2},$$

$$i_{\mathscr{T}\times Q}^{\mathscr{T}\times G}(C_c^\infty(F^\times)),$$

where the action of $\mathscr{T} \times Q$ on $C_c^\infty(F^\times)$ is given by equation (5.6).

5.2. Twisted Jacquet modules

In this subsection we compute the twisted Jacquet modules of $\mathcal{V}$ with respect to $(\overline{\mathscr{U}}, \psi)$. (Recall that $\overline{\mathscr{U}} \cong F$, so ψ defines a character of $\overline{\mathscr{U}}$.) Note that $G = \mathrm{Stab}_{\mathcal{M}}((\overline{\mathscr{U}}, \psi))$.

Lemma 5.6. *The inclusion $C_c^\infty(\omega) \hookrightarrow \mathcal{V}$ (from line (5.1)) induces an isomorphism of G-modules.*

$$C_c^\infty(\omega)_{(\overline{\mathscr{U}},\psi)} \cong \mathcal{V}_{(\overline{\mathscr{U}},\psi)}. \tag{5.7}$$

Proof. Apply $(\overline{\mathscr{U}}, \psi)$-coinvariants, which is exact, to line (5.1). Note that $(\mathcal{V}_{\overline{\mathcal{N}}})_{(\overline{\mathscr{U}},\psi)} = 0$. □

To study $C_c^\infty(\omega)_{(\overline{\mathscr{U}},\psi)}$ we need to consider the set of rank 1 trace 1 elements in $\mathcal{J}$. Viewing ω as the set of rank 1 elements in $\mathcal{J}$, we define $\omega_1 = \{x \in \omega | \mathrm{Tr}(x) = 1\}$.

Lemma 5.7. *The restriction map $C_c^\infty(\omega) \to C_c^\infty(\omega_1)$ induces a G-module isomorphism $C_c^\infty(\omega)_{(\overline{\mathscr{U}},\psi)} \cong C_c^\infty(\omega_1)$. (The action of G on $C_c^\infty(\omega_1)$ is the same as on line (5.3).)*

Proof. This is proved as in Magaard-Savin [19], Lemma 2.2. □

Lemma 5.8. *The action of G on ω_1 is transitive. Let $v_0 = \mathrm{diag}(1, 0, 0) \in \mathcal{J}$, then $\mathrm{Stab}_G(v_0) \cong \mathrm{Spin}(9, F)$ (split spin group). Thus the map $G/\mathrm{Stab}_G(v_0) \to \omega_1$ defined by $g\mathrm{Stab}_G(v_0) \mapsto g \cdot v_0$ is a bijection.*

Proof. This is Corollary 5.8.2 and Theorem 7.1.3 in Spinger-Veldkamp [28]. We make a few remarks and match our notation with Springer-Veldkamp.

Our v_0 is the u in Springer-Veldkamp. The space E_0 in loc. cit. is then the elements of $\mathcal{J}$ of the form $\mathrm{diag}(0, b, -b) + J(x, 0, 0)$, where $b \in F$ and $x \in \mathbb{O}$. This is a nine-dimensional orthogonal space where the quadratic form is the restriction of the trace form of $\mathcal{J}$ to E_0. This form on E_0 is nondegenerate. So by Springer-Veldkamp Theorem 7.1.3, we see that $\mathrm{Stab}_G(v_0) \cong \mathrm{Spin}(Q, E_0)$. □

Remark: We note that the quadratic space in the previous lemma (Q, E_0) decomposes as an orthogonal sum of a one-dimensional quadratic space $(Q_0', F \cdot v)$, where $Q_0'(v) = 2$, and the eight-dimensional quadratic space associated to the split octonion algebra $\mathbb{O}$.

Lemma 5.9. *Let $v_0 = \mathrm{diag}(1, 0, 0) \in \mathcal{J}$.*

There is an isomorphism of G-modules $\mathrm{ind}_{\mathrm{Stab}_G(v_0)}^G(1) \to C_c^\infty(\omega_1)$ defined by

$$f \mapsto (g^{-1} \cdot v_0 \mapsto f(g)).$$

Proof. A direct calculation shows this map is a G-module homomorphism. Here we are using the fact that any character of G is trivial.

One can directly check that the inverse map is given by $F \mapsto (g \mapsto F(g^{-1} \cdot v_0))$. □

Proposition 5.10. *By combining the isomorphisms of Lemmas 5.6, 5.7, and 5.9 we see that as G-modules*

$$\mathcal{V}_{(\overline{\mathscr{U}},\psi)} \cong \mathrm{ind}_{\mathrm{Stab}(v_0)}^G(1).$$

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6. Lifting from PGL(2) to F_4

We continue to use the notation from Section 5. In particular, $C = M(2, F)$. In this section, we explicitly describe the theta lift from $\mathscr{G}$ to G.

In Subsections 6.1 and 6.2 we compute the big theta lift of constituents of principal series; the small theta lift is computed in Subsection 6.3. In subsection 6.4, we revisit the theta lift of supercuspidal representations.

6.1. Principal series

Now we begin the calculation of $\mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, \tau)$, where $\tau = i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ is a principal series (not necessarily irreducible) of $\mathscr{G}$ induced from a character $\chi : \mathscr{T} \to \mathbb{C}^\times$.

By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, \tau) \cong \mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(\mathcal{V}), \chi). \tag{6.1}$$

Note that $\mathrm{Stab}_{\overline{\mathcal{P}}}(\overline{\mathscr{U}}) = (\mathscr{T} \times G)\overline{\mathcal{N}}$. Thus we apply the exact functor $r_{\overline{\mathscr{B}}}$ to sequence (5.1) to get a sequence of $\mathscr{T} \times G$-modules

$$0 \to r_{\overline{\mathscr{B}}}(C_c^\infty(\omega)) \to r_{\overline{\mathscr{B}}}(\mathcal{V}) \to \delta_{\overline{\mathscr{B}}}^{-1/2} \otimes \mathcal{V}_{\overline{\mathcal{N}}} \to 0. \tag{6.2}$$

We apply the functor $\mathrm{Hom}_{\mathscr{T}}(-, \chi)$ to (6.2) to get a long exact sequence. Let X, Y, Z be the nonzero $\mathscr{T} \times G$-modules in sequence (6.2) from left to right, respectively. Then the long exact sequence is:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Z, \chi) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y, \chi) \xrightarrow{\iota} \mathrm{Hom}_{\mathscr{T}}(X, \chi) \\ \longrightarrow \mathrm{Ext}^1_{\mathscr{T}}(Z, \chi) \longrightarrow \mathrm{Ext}^1_{\mathscr{T}}(Y, \chi) \longrightarrow \mathrm{Ext}^1_{\mathscr{T}}(X, \chi) \longrightarrow \ldots \end{aligned} \tag{6.3}$$

The following lemma shows that ι is an isomorphism if χ avoids a finite set of characters. It is a simple consequence of Theorem 5.1.

Lemma 6.1. *Assume that $\chi \neq |\cdot|^{5/2}$ and $|\cdot|^{11/2}$. Then*

$$\mathrm{Hom}_{\mathscr{T}}(\delta_{\overline{\mathscr{B}}}^{-1/2} \otimes \mathcal{V}_{\overline{\mathcal{N}}}, \chi) = 0 \text{ and}$$
$$\mathrm{Ext}^1_{\mathscr{T}}(\delta_{\overline{\mathscr{B}}}^{-1/2} \otimes \mathcal{V}_{\overline{\mathcal{N}}}, \chi) = 0,$$

and the map ι in the long exact sequence (6.3) induces an isomorphism

$$\mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(\mathcal{V}), \chi) \cong \mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(C_c^\infty(\omega)), \chi).$$

Now we can prove the main result of this subsection.

Theorem 6.2. *Let χ be a character of $\mathscr{T}$ such that $\chi \neq |\cdot|^{-5/2}$ and $|\cdot|^{-11/2}$. Let π be an irreducible quotient of $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$. Then $\Theta(\pi)$ is a quotient of $i_Q^G(\chi \circ \varpi_4)$. Moreover, if $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ is irreducible, then $\Theta(i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)) = i_Q^G(\chi \circ \varpi_4)$.*

Proof. Observe that π is a submodule of $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi^{-1})$, since π is self-dual. By Lemma 2.8,

$$\Theta(\pi)^* \cong \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, \pi) \subseteq \mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi^{-1}))$$

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and we are going to compute the latter space. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathscr{G}}(\mathcal{V}, i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi^{-1})) \cong \mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(\mathcal{V}), \chi^{-1}).$$

Since $\chi^{-1} \neq |\cdot|^{5/2}$ and $|\cdot|^{11/2}$ we can apply Lemma 6.1 to get

$$\mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(\mathcal{V}), \chi^{-1}) \cong \mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(C_c^\infty(\omega)), \chi^{-1}).$$

By equation (5.5) we have

$$\mathrm{Hom}_{\mathscr{T}}(r_{\overline{\mathscr{B}}}(C_c^\infty(\omega)), \chi^{-1}) \cong \mathrm{Hom}_{\mathscr{T}}(i_{\mathscr{T}\times Q}^{\mathscr{T}\times G}(C_c^\infty(F^\times)), \chi^{-1}).$$

The maximal χ^{-1}-isotypic quotient of the $\mathscr{T} \times Q$-module $C_c^\infty(F^\times)$ is $\chi^{-1} \otimes \chi \circ \varpi_4$. Thus by [5, Lemma 9.4],

$$\mathrm{Hom}_{\mathscr{T}}(i_{\mathscr{T}\times Q}^{\mathscr{T}\times G}(C_c^\infty(F^\times)), \chi^{-1}) \cong \mathrm{Hom}_{\mathbb{C}}(i_Q^G(\chi \circ \varpi_4), \mathbb{C}) = i_Q^G(\chi \circ \varpi_4)^*. \qquad \square$$

Any irreducible non-supercuspidal representation of $\mathscr{G}$ is either an irreducible quotient of $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$, where $|\chi| = |-|^s$ with $s \geq 0$, or it is a quadratic twist of Steinberg. But these representations are quotients of $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(\chi)$ such that $|\chi| = |\cdot|^{-1/2}$, so Theorem 6.2 applies to all irreducible non-supercuspidal representations. However, it does not provide a full understanding of the big theta lift of constituents of reducible principal series. We resolve this point in the next subsection.

6.2. *Trivial and Steinberg*

In this subsection we study the theta lifts of the trivial and Steinberg representations of $\mathscr{G}$, along with their twists.

Theorem 6.3. *Let χ be a character of $\mathscr{T}$ such that $\chi = \chi_0|-|^{1/2}$, where χ_0 is a quadratic character. Then*

1. $\Theta(\chi_0)$ *is the unique irreducible quotient of* $i_Q^G(\chi \circ \varpi_4)$.
2. $\Theta(\mathrm{St} \otimes \chi_0)$ *is the unique irreducible submodule of* $i_Q^G(\chi \circ \varpi_4)$.

Proof. We already know that $\Theta(\chi_0)$ is a quotient of $i_Q^G(\chi \circ \varpi_4)$ and $\Theta(\mathrm{St} \otimes \chi_0)$ is a quotient of $i_Q^G(\chi^{-1} \circ \varpi_4)$, which is the same as a submodule of $i_Q^G(\chi \circ \varpi_4)$. (The representations $i_Q^G(\chi \circ \varpi_4)$ and $i_Q^G(\chi^{-1} \circ \varpi_4)$ each have length 2. Moreover, the irreducible sub of one is the quotient of the other [4, Theorem 6.1].)

We work with both cases simultaneously. By Proposition 4.6,

$$\Theta(\chi_0)_{(\overline{N},\Psi)} \cong \chi_0^{-1} \text{ and } \Theta(\mathrm{St} \otimes \chi_0)_{(\overline{N},\Psi)} \cong \mathrm{St} \otimes \chi_0^{-1}.$$

Thus χ_0^{-1} is a quotient of $i_Q^G(\chi \circ \varpi_4)_{(\overline{N},\Psi)}$ while $\mathrm{St} \otimes \chi_0^{-1}$ is a submodule. This implies that neither $\Theta(\chi_0)$ nor $\Theta(\mathrm{St} \otimes \chi_0)$ could be isomorphic to $i_Q^G(\chi \circ \varpi_4)$. $\square$

6.3. *Small theta*

In this section we describe $\theta(\pi)$, where π is a constituent of a principal series of $\mathscr{G}$. The next proposition follows from Propositions 2.5, 6.3, and Theorem 6.2.

Proposition 6.4. *Let χ be a character of $\mathscr{T}$ so that $\chi = |-|^s \cdot \chi_0$, where $s \geq 0$ and χ_0 is a unitary character of $\mathscr{T}$.*

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1. *If $s \neq \frac{1}{2}$ or χ_0 is not of order dividing 2, then $i_{\mathscr{B}}^{\mathscr{G}}(\chi)$ is irreducible, and*
   (a) *if $s \neq \frac{5}{2}, \pm\frac{11}{2}$ or χ_0 is not trivial, then $i_Q^G(\chi \circ \varpi_4)$ is irreducible, so $\theta(i_{\mathscr{B}}^{\mathscr{G}}(\chi)) = \Theta(i_{\mathscr{B}}^{\mathscr{G}}(\chi)) \cong i_Q^G(\chi \circ \varpi_4)$;*
   (b) *if $s = \frac{11}{2}$ and χ_0 is trivial, then $\theta(i_{\mathscr{B}}^{\mathscr{G}}(\chi))$ is the unique irreducible quotient of $i_Q^G(|-|^{\frac{11}{2}} \circ \varpi_4)$, which is the trivial representation of G.*
   (c) *if $s = \frac{5}{2}$ and χ_0 is trivial, then $\theta(i_{\mathscr{B}}^{\mathscr{G}}(\chi))$ is the unique semisimple quotient of $i_Q^G(|-|^{\frac{5}{2}} \circ \varpi_4)$, which has the form $\sigma^+ \oplus \sigma^-$ where σ^+ and σ^- are distinct irreducible representations of G.*
2. *If $s = \frac{1}{2}$ and χ_0 has order dividing 2, then:*
   (a) *$\theta(\chi_0) = \Theta(\chi_0)$ is the unique irreducible quotient of $i_Q^G(\chi \circ \varpi_4)$;*
   (b) *$\theta(\mathrm{St} \otimes \chi_0) = \Theta(\mathrm{St} \otimes \chi_0)$ is the unique irreducible submodule of $i_Q^G(\chi \circ \varpi_4)$.*

6.4. Supercuspidal representations

In this subsection we revisit the theta lift of supercuspidal representations of $\mathrm{PGL}_2(F)$. This calculation involves the $F_4 \times G_2$ dual pair inside of E_8.

For this subsection we maintain our previous notation with the following exceptions. We redefine P, M, and N below. We write $(\Pi_n, \mathcal{V}_n)$ for the the minimal representation of E_n.

Let τ be a supercuspidal representation of $\mathrm{PGL}_2(F)$. Then $\sigma := \Theta(\tau)$ is irreducible by Theorem 4.10. Let $Q_2 \subset G$ be the maximal parabolic that stabilizes a two-dimensional singular (also called amber) subspace in the 26-dimensional representation. The standard Q_2 (corresponding to a fixed choice of positive roots) is the stabilizer of the amber space spanned by the weights ϖ_4 and $\varpi_4 - \alpha_4$. We note that the Levi of Q_2 has type $A_{2,\mathrm{long}} \times A_{1,\mathrm{short}}$. Observe that Q_2 has a quotient isomorphic to GL_2 given by the action of Q_2 on the stabilized amber space. With this identification, det can be naturally viewed as a character of Q_2, and τ can be inflated to Q_2. The modular character is $\rho_{Q_2}(g) = |\det(g)|^{7/2}$. We have the following:

Proposition 6.5. *σ is the unique irreducible quotient of $\mathrm{Ind}_{Q_2}^G(\tau \otimes |\det|^{3/2})$.*

Proof. Let $P = MN \subset G_2$ be the Heisenberg parabolic. Then $M \cong \mathrm{GL}_2$ and $G \times \mathrm{GL}_2$ is a subgroup of the Levi factor E_7 in E_8 such that the quotient by the center of the Levi gives the dual pair $G \times \mathrm{PGL}_2$ in the adjoint E_7. In Magaard-Savin [19, Theorem 7.6], $r_P(\mathcal{V}_8)$ was shown to have a $G \times \mathrm{GL}_2$-module filtration with three pieces. The top (quotient) is

$$\mathcal{V}_7 \otimes |\det|^{3/2} \oplus 1 \otimes |\det|^{7/2}.$$

Since $\sigma \otimes \tau$ is a quotient of $\mathcal{V}_7$, by Frobenius reciprocity, $\sigma \otimes \mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^{3/2})$ is a quotient of $\mathcal{V}_8$. Here we are using that $\mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^s)$ reduces only for $s = \pm 1/2$, in particular, $\mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^{3/2})$ is irreducible. Hence σ is a quotient of $\Theta(\mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^{3/2}))$ and this is what we shall compute. To that end, since $\mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^{3/2}) \cong \mathrm{Ind}_P^{G_2}(\tilde{\tau} \otimes |\det|^{-3/2})$ (using an intertwining operator), we are computing

$$\mathrm{Hom}_{G_2}(\mathcal{V}_8, \mathrm{Ind}_P^{G_2}(\tilde{\tau} \otimes |\det|^{-3/2})) \cong \mathrm{Hom}_{\mathrm{GL}_2}(r_P(\mathcal{V}_8), \tilde{\tau} \otimes |\det|^{-3/2}).$$

Since $-3/2 \neq 3/2, 7/2$, we see that the top quotient of the filtration of $r_P(\mathcal{V}_8)$ can be ignored. Since τ is supercuspidal, the intermediate subquotient can be ignored as well, so the computation reduces to the bottom of the filtration of $r_P(\mathcal{V}_8)$ where it follows at once that

$$\Theta(\mathrm{Ind}_P^{G_2}(\tau \otimes |\det|^{3/2})) \cong \mathrm{Ind}_{Q_2}^G(\tau \otimes |\det|^{3/2}).$$

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Thus σ is a quotient of $\mathrm{Ind}_{Q_2}^{G}(\tau \otimes |\det|^{3/2})$. This induced representation is a quotient of a standard module for the parabolic subgroup contained in Q_2 with the Levi $A_{1,\mathrm{short}}$. Thus $\mathrm{Ind}_{Q_2}^{G}(\tau \otimes |\det|^{3/2})$ has a unique irreducible quotient. $\square$

7. Lifting from F_4 to $\mathrm{Aut}(C)$

In this section, we study the theta lift from F_4 to $\mathrm{Aut}(C)$. Specifically, let $\sigma \in \mathrm{Irr}(G)$. We study $\Theta(\sigma)$ with respect to the minimal representation $(\Pi, \mathcal{V})$ on $\mathcal{G}$, utilizing our results on the lifting from $\mathscr{G}$ to G. To begin, we show that the lifting from G to $\mathscr{G}$ has finite length.

Proposition 7.1. *Let $\sigma \in \mathrm{Irr}(G)$. Then $\Theta(\sigma)$ has finite length.*

Proof. We prove this assuming that $\mathscr{G} = \mathrm{PGL}_2(F)$; the nonsplit case (i.e., when C is anisotropic) is easier. If $\Theta(\sigma) = 0$ we are done, so suppose that $\Theta(\sigma) \neq 0$.

Since supercuspidal representations can be split off, the $\mathscr{G}$-representation decomposes as

$$\Theta(\sigma) = \Theta(\sigma)_{ps} \oplus \Theta(\sigma)_{sc},$$

where $\Theta(\sigma)_{sc}$ is the submodule generated by all of the supercuspidal submodules and $\Theta(\sigma)_{ps}$ is the complement all of whose constituents are constituents of principal series. (When C is anisotropic, $\Theta(\sigma)_{ps} = 0$.) To prove the proposition it suffices to show that $\Theta(\sigma)_{ps}$ and $\Theta(\sigma)_{sc}$ have finite length.

We begin with $\Theta(\sigma)_{sc}$. Recall that $\Theta(\sigma)_{sc}$ is completely reducible. Thus if $\Theta(\sigma)_{sc} \neq 0$, then there is a supercuspidal representation $\pi \in \mathrm{Irr}(\mathscr{G})$ such that there is a surjective $\mathscr{G} \times G$ map $\Pi \twoheadrightarrow \pi \otimes \sigma$. By Theorem 4.10 part (1), $\Theta(\pi)$ is irreducible, so $\sigma \cong \Theta(\pi)$. Moreover, by Theorem 4.10 part (2) it follows that $\Theta(\sigma)_{sc} \cong \pi$. In particular, $\Theta(\sigma)_{sc}$ has finite length. (This proves the result when C is anisotropic.)

Next we consider $\Theta(\sigma)_{ps}$. Note that if ρ is any smooth representation of $\mathscr{G}$, then ρ_{ps} is of finite length if and only if $\rho_{\overline{\mathscr{U}}}$ is finite dimensional. By Lemma 2.8 $(\Theta(\sigma)_{\overline{\mathscr{U}}})^* \cong \mathrm{Hom}_G(\Pi_{\overline{\mathscr{U}}}, \sigma)$. So we show that $\mathrm{Hom}_G(\Pi_{\overline{\mathscr{U}}}, \sigma)$ is finite dimensional.

It suffices to analyze the hom-space for each piece of the filtration of $\Pi_{\overline{\mathscr{U}}}$ from Proposition 5.5. The quotient $\Pi_{\overline{\mathcal{N}}}$ is finite length as a G-module by Theorem 5.1 and Corollary 9.2 (which does not depend on this result). So, $\mathrm{Hom}_G(\Pi_{\overline{\mathcal{N}}}, \sigma)$ is finite dimensional.

Now we consider the submodule $i_{\mathscr{T}\times Q}^{\mathscr{T}\times G}(C_c^\infty(F^\times)) \cong i_Q^G(C_c^\infty(F^\times))$. Let L be a Levi subgroup of Q. By Bernstein's second adjointness, we have

$$\mathrm{Hom}_G(i_Q^G(C_c^\infty(F^\times)), \sigma) \cong \mathrm{Hom}_L(C_c^\infty(F^\times), r_{\overline{Q}}(\sigma)).$$

The action of L on $C_c^\infty(F^\times)$ factors through the fundamental weight $\varpi_4 : L \twoheadrightarrow F^\times$. Thus L acts on $C_c^\infty(F^\times)$ through the geometric action of $F^\times$. Since $\dim(\mathrm{Hom}_{F^\times}(C_c^\infty(F^\times), \chi)) = 1$ for any character χ, it follows that $\dim(\mathrm{Hom}_L(C_c^\infty(F^\times), r_{\overline{Q}}(\sigma)))$ is no larger than the number of one-dimensional constituents of $r_{\overline{Q}}(\sigma)$. $\square$

In a moment we shall make the computation of $\mathrm{Hom}_G(\Pi_{\overline{\mathscr{U}}}, \sigma)$ more precise, but first note the following corollary:

Corollary 7.2. *If $\Theta(\sigma) \neq 0$, then σ is a quotient of $\Theta(\pi)$ for some $\pi \in \mathrm{Irr}(\mathscr{G})$. Moreover $\theta(\sigma)$ is irreducible and $\theta(\sigma_1) \cong \theta(\sigma_2) \neq 0$ implies $\sigma_1 \cong \sigma_2$ except in one case when $\sigma_1 \oplus \sigma_2$ is the co-socle of $i_Q^G(|-|^{\frac{5}{2}} \circ \varpi_4)$.*

Proof. Since $\Theta(\sigma)$ has finite length, it has an irreducible quotient π. Then clearly σ is a quotient of $\Theta(\pi)$. The other statements are now trivial consequences of what we know about the lift from $\mathscr{G}$. $\square$

Lemma 7.3. *Let $\pi \in \mathrm{Irr}(\mathscr{G})$ such that $\sigma \stackrel{\mathrm{def}}{=} \Theta(\pi) \in \mathrm{Irr}(G)$. Then $\Theta(\sigma) \cong \pi$.*

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Proof. Let Ψ be a rank 3 character of $\overline{N}$ as in Corollary 3.7. We apply $(\overline{N}, \Psi)$-coinvariants to the natural surjective map $\mathcal{V} \twoheadrightarrow \Theta(\sigma) \otimes \sigma$ to get a surjective map $\mathcal{V}_{(\overline{N},\Psi)} \twoheadrightarrow \Theta(\sigma) \otimes \sigma_{(\overline{N},\Psi)}$. By Corollary 3.7 we have $\mathcal{V}_{(\overline{N},\Psi)} \cong C_c^\infty(\mathscr{G})$ and by Proposition 4.6 we have $\sigma_{(\overline{N},\Psi)} \cong \widetilde{\pi}$. Thus $\Theta(\sigma) \cong \pi$. □

Remark: In the previous lemma, the assumption that σ is irreducible is required for the definition of $\Theta(\sigma)$.

Theorem 7.4. *Let* $\sigma \in \mathrm{Irr}(G)$ *such that* $\Theta(\sigma) \neq 0$. *Then* $\Theta(\sigma) \in \mathrm{Irr}(\mathscr{G})$.

Proof. By Proposition 7.1, $\Theta(\sigma)$ is finite length. So, there exists $\pi \in \mathrm{Irr}(\mathscr{G})$ such that $\Theta(\sigma) \twoheadrightarrow \pi$. If $\Theta(\pi)$ is irreducible, then $\sigma \cong \Theta(\pi)$, and thus Lemma 7.3 implies that $\pi \cong \Theta(\sigma)$.

It remains to consider the case where $\Theta(\pi)$ is reducible. By Theorem 4.10, part (1), this can occur only if $\mathscr{G} \cong \mathrm{PGL}_2(F)$. Moreover, by Proposition 6.4, we see that π must be isomorphic to $i_{\mathscr{B}}^{\mathscr{G}}(|-|^s)$, where $s \in \{\frac{5}{2}, \frac{11}{2}\}$. Thus σ is a quotient of $\Theta(\pi) \cong i_Q^G(\chi \circ \varpi_4)$, which by Proposition 2.5 implies that σ is one of three possible representations. When $s = \frac{11}{2}$, then σ is the trivial representation; when $s = \frac{5}{2}$, then σ is one of the two irreducible representations of the co-socle of $i_Q^G(|-|^{\frac{5}{2}} \circ \varpi_4)$, which we call σ^+ and σ^-. Moreover, $\Theta(\sigma)$ has the irreducible principal series $i_{\mathscr{B}}^{\mathscr{G}}(|-|^s)$ as a quotient, so $\dim(\Theta(\sigma)_{\overline{\mathscr{U}}}) \geq 2$.

From the proof of Proposition 7.1 $\dim(\Theta(\sigma))_{\overline{\mathscr{U}}}$ is less than or equal to $a + b$, where $a = \dim(\mathrm{Hom}_G(\mathcal{V}_{\overline{N}}, \sigma))$ and b is the number of constituents of $r_Q(\sigma)$ of dimension 1. In either case, $a = 1$ by Theorem 5.1 and Corollary 9.2 (which does not depend on this result).

Suppose that σ is trivial (so $s = \frac{11}{2}$). Then $b = 1$ and the result follows in this case.

Suppose that $\sigma \cong \sigma^\pm$ (so $s = \frac{5}{2}$). We claim that $b = 1$ in this case too, from which the result follows. The representations σ^+ and σ^- have Iwahori-fixed vectors, and the corresponding Hecke algebra H_G-modules are $E_{\mathcal{G}}$ and $E_{\mathcal{G}''}$ in [18, page 640]. On the level of H_G-modules, the functor $r_{\overline{Q}}$ correspond to restricting to the Hecke algebra $H_L \subset H_G$ of the Levi subgroup L. Now it is easy to check that $E_{\mathcal{G}}$ and $E_{\mathcal{G}''}$ embed into $i_Q^G(|-|^{-\frac{5}{2}} \circ \varpi_4)$, giving us the claimed identification with σ^+ and σ^-, and that $r_{\overline{Q}}(\sigma^\pm)$ are of length two, with only one one-dimensional summand, each, as desired. □

8. Spin (9) distinguished representations of F_4

The objective of this section is to prove a multiplicity one result for Spin(9)-invariant linear functionals and characterize the Spin(9)-distinguished representations of F_4 as those arising from the theta lift of generic representations on $\mathrm{PGL}_2(F)$. We continue to use the notation of Section 5. In particular, $\mathscr{G} = \mathrm{PGL}_2(F)$. Let $H = \mathrm{Stab}_G(v_0) \cong \mathrm{Spin}(9, F)$. (Recall Lemma 5.8.)

Theorem 8.1. *Let σ be an irreducible representation of G. Then the dimension of* $\mathrm{Hom}_H(\tilde{\sigma}, \mathbb{C})$ *is at most* 1. *Moreover, $\tilde{\sigma}$ is H-distinguished if and only if* $\Theta(\sigma)$ *is generic.*

Proof. By Lemma 2.8 there is an isomorphism

$$(\Theta(\sigma)_{(\overline{\mathscr{U}},\psi)})^* \cong \mathrm{Hom}_G(\mathcal{V}_{(\overline{\mathscr{U}},\psi)}, \sigma).$$

By Proposition 5.10,

$$\mathrm{Hom}_G(\mathcal{V}_{(\overline{\mathscr{U}},\psi)}, \sigma) \cong \mathrm{Hom}_G(\mathrm{ind}_H^G(1), \sigma).$$

By taking duals and applying Frobenius reciprocity we have

$$\mathrm{Hom}_G(\mathrm{ind}_H^G(1), \sigma) \cong \mathrm{Hom}_H(\tilde{\sigma}, \mathbb{C}).$$

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By Theorem 7.4, $\Theta(\sigma)$ is irreducible, if nonzero. Thus by the multiplicity one theorem for Whittaker functionals, $\dim((\Theta(\sigma)_{(\overline{\mathscr{U}},\psi)})^*) \leq 1$. Thus $\dim(\mathrm{Hom}_H(\tilde{\sigma},\mathbb{C})) \leq 1$. □

In fact, we can remove reference to the smooth dual in Theorem 8.1, because the F_4 representations that arise as lifts from PGL_2 are self-dual.

Proposition 8.2. *If* $\Theta(\sigma) \neq 0$*, then* $\sigma \cong \widetilde{\sigma}$.

Proof. Since $\Theta(\sigma) \neq 0$ there exists $\pi \in \mathrm{Irr}(\mathscr{G})$ such that σ is a constituent of $\Theta(\pi)$. We prove the result by considering two cases.

First suppose that π is a supercuspidal representation of $\mathscr{G}$. Since supercuspidal representation split off, there is a $\mathscr{G}$-module decomposition such that $\mathcal{V} = \mathcal{V}^\pi \oplus \mathcal{V}^{\pi,\perp}$, where $\mathcal{V}^\pi$ is the maximal π-isotypic subspace of $\mathcal{V}$ and $\mathcal{V}^{\pi,\perp}$ is the canonical complementary $\mathscr{G}$-submodule. Since the actions of $\mathscr{G}$ and G commute we see that G acts on both $\mathcal{V}^\pi$ and $\mathcal{V}^{\pi,\perp}$. Since all of the constituents of $\mathcal{V}^{\pi,\perp}$ are not isomorphic to π it follows that the $\mathscr{G} \times G$-module surjection

$$\mathcal{V} \twoheadrightarrow \pi \otimes \Theta(\pi)$$

is trivial on $\mathcal{V}^{\pi,\perp}$ and so we have a $\mathscr{G} \times G$-module surjection

$$\mathcal{V}^\pi \twoheadrightarrow \pi \otimes \Theta(\pi). \tag{8.1}$$

Since by definition $\pi \otimes \Theta(\pi)$ is the maximal π-isotypic quotient of $\mathcal{V}$ it follows that (8.1) is an isomorphism $\mathcal{V}^\pi \cong \pi \otimes \Theta(\pi)$. Since $\mathcal{V}$ is a unitary $\mathcal{G}$-representation and $\mathcal{V}^\pi \subseteq \mathcal{V}$ it follows that $\mathcal{V}^\pi$ is a unitary $\mathscr{G} \times G$-representation. Thus we have $\mathscr{G} \times G$-module isomorphisms

$$\widetilde{\pi} \otimes \widetilde{\Theta(\pi)} \cong \widetilde{\mathcal{V}^\pi} \cong \overline{\mathcal{V}^\pi} \cong \mathcal{V}^{\bar{\pi}} \cong \mathcal{V}^{\widetilde{\pi}}.$$

All irreducible representations of $\mathscr{G}$ are self-dual, so we have $\widetilde{\pi} \cong \pi$. From the above chain of isomorphisms it follows that $\Theta(\pi)$ is self-dual.

Since π is supercuspidal, Theorem 4.10 implies $\Theta(\pi) = \sigma$. Thus σ is self-dual.

Now suppose that π is a constituent of the principal series $i_{\mathscr{B}}^{\mathscr{G}}(\chi)$. Then by Theorem 6.2, σ is a constituent of $i_Q^G(\chi \circ \varpi_4)$.

We claim that all of the constituents of $i_Q^G(\chi \circ \varpi_4)$ are self-dual. By Choi–Jantzen [4], Theorem 6.1, the length of $i_Q^G(\chi \circ \varpi_4)$ is less than 3. In each of the following three cases we use that there is a nonzero intertwining operator $i_Q^G(\chi^{\pm 1} \circ \varpi_4) \longrightarrow i_Q^G(\chi^{\mp 1} \circ \varpi_4) \cong \widetilde{i_Q^G}(\chi^{\pm 1} \circ \varpi_4)$.

When $i_Q^G(\chi \circ \varpi_4)$ is irreducible we are done. When $i_Q^G(\chi \circ \varpi_4)$ has length 2, then [4, Theorem 6.1,1.] implies that $i_Q^G(\chi^{\pm 1} \circ \varpi_4)$ has a unique irreducible sub and a unique irreducible quotient, which are distinct. Thus the nonzero intertwining operators imply the self-duality of the irreducible constituents of $i_Q^G(\chi^{\pm 1} \circ \varpi_4)$.

When $i_Q^G(\chi^{\pm 1} \circ \varpi_4)$ has length 3, then $\chi = |-|^{\pm\frac{5}{2}}$. In this case, $i_Q^G(|-|^{-\frac{5}{2}} \circ \varpi_4)$ has a unique irreducible quotient, and the intertwining operator shows that it is self dual. There is also a decomposible submodule with two distinct constituents, call them σ^+ and σ^-. Using the $\mu_2 \times G$ dual pair considered in Section 9 we can show that σ^+ and σ^- are self-dual. Specifically, $\mathcal{V}_6$ the minimal representation of E_6 decomposes under the action of $\mu_2 \times G$ as $\mathcal{V}_6 \cong \sigma^+ \oplus \sigma^-$ (Theorem 9.3). Now if I is an Iwahori subgroup of G, then $\dim((\sigma^+)^I) = 5$ and $\dim((\sigma^-)^I) = 2$. Thus σ^+ and σ^- are self-dual. □

Combining Theorem 8.1 and Proposition 8.2 directly gives the following corollary, which simply restates Theorem 8.1 with $\tilde{\sigma}$ replaced by σ.

Corollary 8.3. *Let σ be an irreducible representation of G. Then the dimension of* $\mathrm{Hom}_H(\sigma,\mathbb{C})$ *is at most* 1. *Moreover, σ is H-distinguished if and only if* $\Theta(\sigma)$ *is generic.*

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Let σ be an irreducible H-distinguished representation of G. Pick a nonzero $\lambda \in \mathrm{Hom}_H(\sigma, \mathbb{C})$. Recall, from [21], that σ is an H-relatively supercuspidal representation if one (or equivalently every) generalized matrix coefficient $g \mapsto \lambda(gv)$ is a compactly supported function on $H\backslash G$. As a consequence of our results, we have the following:

Proposition 8.4. *Let σ be an irreducible representation of G. If σ is a theta lift of a supercuspidal representation of* $\mathrm{PGL}_2(F)$*, then σ is H-relatively supercuspidal.*

Proof. Observe that σ is H-relatively supercuspidal if and only if σ is a submodule of $C_c^\infty(H\backslash G)$. Now write $\sigma = \Theta(\pi)$ where $\pi \in \mathrm{Irr}(\mathscr{G})$ is supercuspidal. Since π is supercuspidal, as in Proposition 8.2, we have an embedding of $\mathscr{G} \times G$-modules $\pi \otimes \Theta(\pi) \hookrightarrow \mathcal{V}$. By taking $(\overline{\mathscr{U}}, \Psi)$-coinvariants and applying Proposition 5.10 we get an embedding of G-modules $\Theta(\pi) \hookrightarrow C_c^\infty(H\backslash G)$, where $H \cong \mathrm{Spin}(9)$. Thus $\sigma = \Theta(\pi)$ is H-relatively supercuspidal.

We give a second proof of this proposition. If σ is not H-relatively supercuspidal then, by a result of Kato and Takano [13, Theorem 7.1], σ must be a subquotient of $i_Q^G(\chi \circ \varpi_4)$. This contradicts that σ is a theta lift of a supercuspidal representation of $\mathrm{PGL}_2(F)$. Here we used that any θ-split parabolic subgroup for the rank one symmetric space $H\backslash G$ is in the G-conjugacy class of Q, see [25, A3.6]. □

9. Dual pair $\mu_2 \times F_4 \subset E_6$

In this section, we study the theta lift associated to the dual pair $\mu_2 \times F_4 \subset E_6$, where E_6 is of adjoint form with two connected components where the action of the nontrivial component is through the outer automorphism of E_6. This situation arises from the construction of Subsection 2.4 by taking C to be a quadratic composition algebra.

The analysis of this case is similar to and simpler than the case of E_7 considered in Section 4, so we will be brief. We note that the results of Subsection 9.1 could have been proved after Section 3, but our proof of the result of Subsection 9.2 utilizes Theorem 6.2.

Let $\mathcal{G}$ be the F-points of the adjoint form of E_6 constructed using the quadratic composition algebra C with two connected components where the nontrivial component acts through the outer automorphism associated with a choice of simple roots Δ. Let $(\Pi, \mathcal{V})$ be the minimal representation of $\mathcal{G}$. Let G be the fixed points in the identity component of $\mathcal{G}$ under the action of the outer automorphism. Then if we identify μ_2 with the subgroup of $\mathcal{G}$ generated by the outer automorphism, then $\mu_2 \times G \subset \mathcal{G}$ is a dual pair.

Let τ^+ and τ^- be the trivial and nontrivial characters of μ_2, respectively. There is a surjective map $\mathcal{V} \twoheadrightarrow \tau^\pm \otimes \Theta(\tau^\pm)$. The goal of this section is to compute $\Theta^\pm = \Theta(\tau^\pm)$.

9.1. Lifting from μ_2 to F_4

Theorem 9.1. *The G-module $\Theta^\pm$ is irreducible and $\Theta^+ \ncong \Theta^-$.*

Proof. First, we show that $\mathrm{FJ}(\Theta^\pm) \cong \omega_\psi^\pm$, where ω_ψ^+ and ω_ψ^- are the even and odd Weil representation of $\mathrm{Sp}(6, F)$, respectively. This follows from the analogs of Lemmas 4.1 and 4.2 and Propositions 4.3 and 4.4. The main difference in this case is that the dimension of $N^\perp/Z$ is 6, so ω_ψ is the Weil representation of $\mathrm{Sp}(6, F)$, and $\mathrm{Aut}(C) \cong \mu_2$ is the full orthogonal group $\mathrm{O}(C^0)$ (as opposed to $\mathrm{SO}(C^0)$). Since $\omega_\psi^+ \ncong \omega_\psi^-$ it follows that $\Theta^+ \ncong \Theta^-$.

From this we also see that $\Theta^\pm$ has exactly one nontrivial constituent, since the Fourier–Jacobi functor is exact and only kills the trivial representation.

Second, we show that $\Theta^\pm$ does not contain the trivial representation as a constituent. If it does, then $\mathcal{V}_{\overline{N}}$ contains the trivial representation of M as a constituent. However, by Proposition 3.11 part (1) and Theorem 3.1,

$$\mathcal{V}_{\overline{N}} \cong \mathcal{V}(\mathcal{M}) \otimes |\det|^{2/20} \oplus \chi_C |\det|^{4/20}.$$

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We see that the center of M acts by nontrivial characters on the two summands of $\mathcal{V}_{\overline{N}}$, thus $\mathcal{V}_{\overline{N}}$ cannot contain the trivial representation of M as a constituent. Therefore $\mathcal{V}$ cannot contain the trivial representation of G as a constituent.

From this it follows that $\Theta^{\pm}$ is irreducible. □

Corollary 9.2. *As a $\mu_2 \times G$-module, $\mathcal{V} \cong \Theta^+ \oplus \Theta^-$.*

9.2. $C = F \oplus F$

In this section we use our results on the $\mathrm{PGL}(2) \times F_4 \subset E_7$ dual pair to make the lift of μ_2 to F_4 induced from the split form of E_6 explicit.

Throughout we use the notation of Section 5 and we write $\mathcal{V}_n$ for the minimal representation of E_n. We apply $\overline{\mathscr{U}}$-coinvariants to sequence (5.1) to get the surjective $\mathscr{T} \times G$-module map

$$(\mathcal{V}_7)_{\overline{\mathscr{U}}} \twoheadrightarrow (\mathcal{V}_7)_{\overline{\mathcal{N}}} \cong \mathcal{V}_6 \otimes |-|^3 \oplus |-|^6 \twoheadrightarrow \Theta^{\pm} \otimes |-|^3.$$

Thus Frobenius reciprocity with respect to $\overline{\mathscr{B}} \subset \mathscr{G}$ yields a nonzero $\mathscr{G} \times G$-module map

$$\mathcal{V}_7 \longrightarrow \mathrm{Ind}_{\overline{\mathscr{B}}}^{\mathscr{G}}(\Theta^{\pm} \otimes |-|^3).$$

Since $\mathscr{T}$ is the center of $\mathcal{M}$ it acts trivially on $\mathcal{V}_6$, thus $\mathrm{Ind}_{\overline{\mathscr{B}}}^{\mathscr{G}}(\Theta^{\pm} \otimes |-|^3) \cong i_{\overline{\mathscr{B}}}^{\mathscr{G}}(|-|^{\frac{5}{2}}) \otimes \Theta^{\pm}$. Note that $i_{\overline{\mathscr{B}}}^{\mathscr{G}}(|-|^{\frac{5}{2}}) \otimes \Theta^{\pm}$ is an irreducible $\mathscr{G} \times G$-module. Thus by Theorem 6.2, there is a surjective G-module map

$$i_Q^G(|-|^{\frac{5}{2}} \circ \varpi_4) \cong \Theta(i_{\overline{\mathscr{B}}}^{\mathscr{G}}(|-|^{\frac{5}{2}})) \twoheadrightarrow \Theta^{\pm}.$$

By applying Proposition 2.5 we get the following theorem.

Theorem 9.3. *There is a bijection between the irreducible G-modules $\{\Theta^+, \Theta^-\}$ and the two irreducible summands of the unique semisimple quotient of $i_Q^G(|-|^{\frac{5}{2}} \circ \varpi_4)$.*

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