ON THE LOCATION OF SINGULARITIES OF A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN FOUR VARIABLES

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1. Introduction. In this paper we shall investigate the singular behaviour of the solutions to the elliptic equation

$$T_4[\Psi] \equiv \frac{\partial^2\Psi}{\partial x_\mu\,\partial x_\mu} + A(r^2)x_\mu\frac{\partial\Psi}{\partial x_\mu} + C(r^2)\Psi = 0, \tag{1.1}$$

where $A(r^2)$, $C(r^2)$ are entire functions of the complex variable

$$r^2 = x_\mu x_\mu \ (\mu = 1, 2, 3, 4).$$

(Here repeated indices mean the summation convention is used.) The three-variable analogue $T_3[\Psi] = 0$ has been investigated extensively by Bergman **(3–8)** and others **(27; 28; 22; 23; 32)**.

Fundamental to this procedure is the use of an integral operator $\mathbf{B}_4[f]$ which maps holomorphic functions of three complex variables onto harmonic functions of four variables **(13–18)**:

$$H(\mathbf{X}) = \mathbf{B}_4[f], \qquad \mathbf{B}_4[f] \equiv -\frac{1}{4\pi^2}\int_{\Gamma^2}\int f(u,\eta,\xi)\,\frac{d\eta}{\eta}\frac{d\xi}{\xi}, \tag{1.2}$$

$$u = x_1\left(1+\frac{1}{\eta\xi}\right) + ix_2\left(1-\frac{1}{\eta\xi}\right) + x_3\left(\frac{1}{\xi}-\frac{1}{\eta}\right) + ix_4\left(\frac{1}{\xi}+\frac{1}{\eta}\right),$$

$\|\mathbf{X}-\mathbf{X}^0\| < \epsilon$, $\mathbf{X} \equiv (x_1, x_2, x_3, x_4)$, $\mathbf{X}^0$ is an initial point of definition, Γ^2 is a 2-cycle. (We frequently take as a 2-cycle the product of a simple contour C_ξ in the ξ-plane and one, C_η, in the η-plane. However, it is possible to extend our integral operator to the case where we use 2-chains instead.) The operator $\mathbf{B}_4[f]$ maps the analytic function,

$$f(u,\eta,\xi) = \sum_{n=0}^{\infty}\sum_{m,p=0}^{n} a_{nmp}\,u^n\eta^m\xi^p, \tag{1.3}$$

onto the harmonic function,

$$H(\mathbf{X}) = \sum_{n=0}^{\infty}\sum_{m,p=0}^{n} a_{nmp}\,H_n^{mp}(\mathbf{X}), \tag{1.4}$$

where the $H_n^{mp}(\mathbf{X})$ are harmonic, homogeneous polynomials of degree n which are defined by **(12)**,

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(1.5) $$u^n = \sum_{\kappa, l=0}^{n} H_n^{\kappa, l}(\mathbf{X})\xi^{-\kappa}\eta^{-l} \qquad (n = 0, 1, 2, \ldots).$$

These polynomials form a complete set with respect to the class of harmonic functions regular about the origin **(12, 13)**.

Another operator **(18)**, which bears close resemblance to the one Bergman **(3)** introduced for the three-variable case, transforms holomorphic functions of three complex variables into solutions of $T_4[\Psi] = 0$. (Recently, Gilbert and Howard **(19)** have extended this operator to the case where we have $p + 2$ variables; in other words the operator $\Omega_{p+2}[f]$ generates solutions to the partial differential equation $T_{p+2}[\Psi] = 0$.)

(1.6) $$\Psi(\mathbf{X}) = \Omega_4[f] \equiv -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{t=-1}^{+1} E(r, t)\, f(u[1-t^2]; \eta, \xi)dt,$$

where

$$E(r, t) \equiv \exp\left(-\frac{1}{2}\int_0^r A r dr\right)H(r, t);$$

$H(r, t)$, $|t| \leqslant 1$ is a solution of

(1.7) $$(1 - t^2)H r t - t^{-1}(t^2 + 1)H_r + rt\left(H_{rr} + \frac{3}{r}H_r + BH\right) = 0,$$

where $B = \frac{1}{2}rA_r - 2A - \frac{1}{4}r^2A^2 + C$, and H_r/rt is continuous at $r = t = 0$.

Using these operators it is possible to "transplant" properties of holomorphic functions to certain classes of solutions to $T_4[\Psi] = 0$ and $\Box\Psi = 0$ respectively. In particular, as Bergman has done in the three-variable situation **(3)**, we may obtain results concerning the location of singularities **(14; 15; 16)** and information concerning the growth of solutions **(19)**.

In this paper we shall investigate further the location of singularities for solutions of $T_4[\Psi] = 0$. This will be done by making use of a recent result by Bergman **(9)** extending the Hadamard **(21)** and Mandelbrojt **(25; 26)** theorems (on locations of singularities) to the case of several complex variables. We shall also employ the **(29)** and Behnke–Thullen **(2)** theorems along with the author's envelope method **(14; 18)** to obtain representations for the singularity manifold.

2. The envelope method for location singularities. In earlier papers **(14; 15)** the author proved the following theorems.

THEOREM 2.1. *Let $H(\mathbf{X})$ be a harmonic function generated by the integral operator $\mathbf{B}_4[f]$. Furthermore, let the singularity manifold of $\eta^{-1}\xi^{-1}f(u, \eta, \xi)$ be represented by*

$$\{\Phi(u, \eta, \xi) \equiv S(\mathbf{X}; \eta, \xi) = 0\}.$$

Then, the only possible singularities of $H(\mathbf{X})$ must be contained in the intersection

$$\{\mathbf{X}|S = 0\} \cap \left\{\mathbf{X}\left|\frac{\partial S}{\partial \eta} + \frac{\partial S}{\partial \xi}\frac{d\Pi(\eta)}{d\eta} = 0\right.\right\}, \tag{2.1}$$

where $\xi = \Pi(\eta)$ is an arbitrary analytic function.

THEOREM 2.2. *Let $H(\mathbf{X})$ be a harmonic function generated as above; furthermore, let the associated holomorphic function be rational, that is, let*

$$F(u, \eta, \xi) \equiv \frac{p(u, \eta, \xi)}{q(u, \eta, \xi)} = \frac{P(X; \eta, \xi)}{Q(X; \eta, \xi)}, \tag{2.2}$$

where p, q, P, Q are polynomials. Then, the only possible singularities of $H(\mathbf{X})$ must be contained in the intersection

$$\{\mathbf{X}|Q = 0\} \cap \{\mathbf{X}|\partial Q/\partial \eta = 0\} \cap \{\mathbf{X}|\partial Q/\partial \xi = 0\}. \tag{2.3}$$

In **(15)** we argued as follows. Since $Q(\mathbf{X}; \eta, \xi)$ is a polynomial in $\mathbf{X} \equiv (x_1, x_2, x_3, x_4)$, η, and ξ, then for each fixed $\eta = \eta^0$, $Q(\mathbf{X}; \eta^0, \xi)$ has a decomposition of the form

$$Q(\mathbf{X}; \eta^0, \xi) = [\xi - A_1(\mathbf{X}; \eta^0)]^{m_1} \dots [\xi - A_r(\mathbf{X}; \eta^0)]^{m_r},$$

where the m_k ($k = 1, 2, \dots, r$) are non-negative integers.

The criterium for a multiple pole singularity of the integrand is that there be an $m_k > 1$, i.e. that $\partial Q/\partial \xi = 0$ for some $\xi = A_k(\mathbf{X}; \eta^0)$.

For a point $\mathbf{X}$ to be a singularity of the harmonic function,

$$H(\mathbf{X}) = \mathbf{B}_4[F(u, \eta, \xi)],$$

$\mathbf{X}$ must correspond to a singularity of the integrand on the domain of integration which cannot be avoided by continuously deforming the domain of integration; see **(13; 14; 15)** for further details. Such points $\mathbf{X}$ must be contained in the set of points which correspond to two roots

$$\xi_k = A_k(\mathbf{X}; \eta^0), \ \xi_j = A_j(\mathbf{X}; \eta^0)$$

coinciding in the ξ-plane, since in this case a contour C_ξ may be "pinched" between ξ_k and ξ_j.

According to the argument used by Hadamard (in his theorem on the "multiplication of singularities" **(21)**) the only instance where it may not be possible to deform C_ξ in order to avoid a singularity crossing it (as we vary $\mathbf{X}$) is when two such ξ_k and ξ_j "pinch" C_ξ between them.

Interchanging the roles of η and ξ and combining this result with that of Theorem (2.1) we obtain Theorem (2.2), as is shown in **(15)**. An alternative proof of this theorem may be found in a recent paper by the author with H. C. Howard **(20)**.

We consider first the class of functions $\mathbf{M}(\mathfrak{C}^3)$ which are meromorphic in finite $\mathfrak{C}^3$. As Poincaré **(2; 30; 31)** has shown, if $f \in \mathbf{M}(\mathfrak{C}^3)$, then there exist

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two entire functions g, h, whose simultaneous zeros are relatively prime, and such that $f = g/h$. Furthermore, from the theorems of Behnke–Thullen **(2)** and Oka **(29)** one may prescribe the analytic sets $\{h_m(z_1, z_2, z_3)\}_{m=1}^{\infty}$ on which an $f \in \mathbf{M}(\mathfrak{C}^3)$ has pole-like singularities, provided no $h_n \equiv 0$ and a finite number of the g_m vanish in any given region $\mathfrak{B} \subset \mathfrak{C}^3$. Oka showed that it is always possible to find a meromorphic function having pole-like singularities on a given set of analytic surfaces provided the holomorphy domain is univalent. Hence in any finite region we can represent f as the following quotient of relatively prime entire functions:

$$f = \frac{g_1^{\alpha_1} \ldots g_p^{\alpha_p}}{h_1^{\beta_1} \ldots h_q^{\beta_q}}.$$

For the sake of simplicity we consider the case where f is a meromorphic function whose decomposition into a quotient of entire functions (as above) yields a single relatively prime factor in the denominator. That is, we assume the $\mathbf{B}_4$-associate of $H(\mathbf{X})$ has the form

$$(2.4)\quad f(u, \eta, \xi) = \frac{g(u, \eta, \xi)}{h(u, \eta, \xi)} \equiv \frac{g_1^{\alpha_1} \ldots g_p^{\alpha_p}}{h(u, \eta, \xi)} = \frac{G_1(\mathbf{X}; \eta, \xi)^{\alpha_1} \ldots G_p(\mathbf{X}; \eta, \xi)^{\alpha_p}}{\theta(\mathbf{X}; \eta, \xi)}.$$

Now for each fixed $\mathbf{X}$, $\theta(\mathbf{X}; \eta, \xi)$ is an entire holomorphic function of the two complex variables η, ξ in finite $\mathfrak{C}^2$. Consequently, by the Weierstrass preparation theorem **(2; 10)**, we can expand $\theta(\mathbf{X}, \eta, \xi)$ in a neighbourhood $\mathfrak{N}(\eta_0, \xi)$ about any zero $(\eta_0, \xi_0) \equiv (\eta_0[\mathbf{X}], \xi_0[\mathbf{X}])$ in a unique manner, say

$$(2.5)\qquad \theta(\mathbf{X}; \eta, \xi) = (\eta - \eta_0)^{\kappa}\Psi(\mathbf{X}; \eta, \xi)\Omega(\mathbf{X}; \eta, \xi),$$

where

$$\Psi(\mathbf{X}; \eta, \xi) \equiv (\xi - \xi_0)^m + A_1(\mathbf{X}; \eta)(\xi - \xi_0)^{m-1} + \ldots + A_m(\mathbf{X}; \eta).$$

$\Omega(\mathbf{X}; \eta, \xi)$ is regular and nowhere vanishes in $\mathfrak{N}$, and the $A_i(\mathbf{X}; \eta)$ are regular in a neighbourhood of η_0 and vanish at η_0. (We refer to the term $\Psi(\mathbf{X}; \eta, \xi)$ as a pseudo-polynomial.) We notice that the "constants" m, k actually vary with $\mathbf{X}$; however, as can be shown, $m(\mathbf{X})$, $k(\mathbf{X})$ are constant for certain regions of $\mathbf{R}^4$ and only change in value by an integral amount as $\mathbf{X}$ passes over certain distinguished hypersurfaces.

From Theorem 2.1, we realize that $H(\mathbf{X}) = \mathbf{B}_4[f]$ is regular at $\mathbf{X}$ provided

$$\mathbf{X} \notin \{\mathbf{X} | \theta(\mathbf{X}; \eta, \xi) = 0\} \cap \{\mathbf{X} | \partial\theta/\partial\eta + \Pi'(\eta)\partial\theta/\partial\xi = 0\},$$

where $\Pi(\eta)$ is an arbitrary analytic function of η. Since, $\theta = (\eta - \eta_0)^{\kappa}\Psi\Omega$, we have for the first variation of θ with respect to η,

$$(2.6)\qquad \delta_\eta[\theta] = \{\partial[(\eta - \eta_0)^{\kappa}\Psi\Omega]/\partial\eta + (\eta - \eta_0)^{\kappa}\Pi'(\eta)[\Psi_\xi\,\Omega + \Psi\Omega_\xi]\}\delta\eta.$$

Since Ψ is a pseudo-polynomial in ξ, the necessary and sufficient condition for a double root $\xi = \xi_\nu[\mathbf{X}; \eta]$ to exist is that Ψ_ξ vanish for some ξ, say ξ_ν. In order for the integral in the representation $H(\mathbf{X}) = \mathbf{B}_4[f]$ to exist we must

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exclude second-order poles of the integrand coinciding with the domain of integration **(15)**.) However, in order for $\theta = (\eta - \eta_0)^{\kappa}\Psi\Omega$ to vanish for $\xi = \xi_\nu[\mathbf{X}; \eta]$ and *all* η in the neighbourhood of η_0, we must have $\Psi[\mathbf{X}; \eta, \xi_\nu] \equiv 0$. These two facts imply that $\partial\theta[\mathbf{X}; \eta, \xi]/\partial\xi = 0$, which, in turn, for $\Pi(\xi)$ an arbitrary *analytic* function, implies that $\partial\theta[\mathbf{X}; \eta, \xi]/\partial\eta = 0$. By interchanging the roles played by η, ξ in (2.5) and proceeding in the above manner we find that a double η-root also implies that $\theta_\xi = \theta_\eta = 0$. This conclusion leads us to the following extensions of Theorems 2.1 and 2.2.

THEOREM 2.3. *Let $H(\mathbf{X})$ be a harmonic function generated by the operator $\mathbf{B}_4[f]$, where $f(u, \eta, \xi)$ is a meromorphic function with a decomposition into a quotient of entire functions of the type* (2.4). *Then $H(\mathbf{X})$ is regular provided that*

$$\mathbf{X} \notin \{\mathbf{X}|\theta(\mathbf{X}; \eta, \xi) = 0\} \cap \{\mathbf{X}|\partial\theta/\partial\xi = 0\} \cap \{\mathbf{X}|\partial\theta/\partial\eta = 0\}. \tag{2.7}$$

THEOREM 2.4. *Let $\Psi(\mathbf{X}) = \Omega_4[f]$ be a solution of the partial differential equation $\mathbf{T}_4[\Psi] = 0$, where the coefficients $A(r^2)$, $C(r^2)$ of $\mathbf{T}_4[\Psi]$ are entire. Furthermore, let $f(u, \eta, \xi)$ be as above a meromorphic function in finite $\mathfrak{C}^3$. Then $\Psi(\mathbf{X})$ is regular provided that $\mathbf{X}$ is not contained in the intersection* (2.7).

3. Bergman's extension of the Hadamard–Mandelbrojt criteria. In a recent paper, Bergman **(9)** extended the Hadamard–Mandelbrojt theorems concerning the number and location of the singularities of analytic functions of a single complex variable to the case of functions of two complex variables. His results may be modified to the case of three complex variables which we discuss later in this work.

Using Bergman's results, it is possible to obtain the analytic sets, which form the singularity manifold of meromorphic function of two complex variables. For instance, if

$$f(z_1, z_2) \equiv \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu\nu}\, z_1^{\mu} z_2^{\nu}, \tag{3.1}$$

then we may obtain as the singular sets

$$\mathbf{U}_{\mu\nu} \equiv \{(z_1, z_2)|\ |z_1| = \rho_\nu(\alpha), \arg z_1 = \phi_{\nu\mu}(\alpha); \alpha = z_2/z_1\}. \tag{3.2}$$

The $\rho_\nu(\alpha)\, exp[i\phi_{\nu\mu}(\alpha)]$ are the singular points of $F_\alpha(z_1) = f(z_1, \alpha z_1)$ for each fixed value of α; here we have ordered the singular points of each circle $\rho = \rho(\alpha)$ such that $\phi_{\nu,\mu}(\alpha) < \phi_{\nu,\mu+1}(\alpha)$. For the case where there is only one singular point of $F_\alpha(z_1)$ in and on the circle $\rho = \rho(\alpha)$ we have the following representation given by **(9)**:

$$\rho(\alpha) = \limsup_{N\to\infty} \left[\left| \sum_{k=0}^{N} a_{n-k,k}\, \alpha^k \right|^{1/N} \right], \tag{3.3}$$

$$\phi(\alpha) = \cos^{-1}[R^{+\prime}(0, \alpha)], \tag{3.4}$$

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where

$$R(h, \alpha) = \limsup_{N\to\infty} [|d_N(h, \alpha)|^{1/N}], \qquad h > 0, \tag{3.5}$$

$$d_N(h, \alpha) = \sum_{k=0}^{N} \left[\binom{N}{k} \rho^k(\alpha) h^{N-k} \left(\sum_{\mu=0}^{N-k} a_{N-k-\mu,\mu}\, \alpha^{\mu} \right) \right], \tag{3.6}$$

and $R^{+\prime}(0, \alpha)$ is the right-hand derivative

$$R^{+\prime}(0, \alpha) = \lim_{h\to 0^+} \left[h^{-1} \left(\limsup_{N\to\infty} |d_N(h, x)|^{1/N} - 1 \right) \right]. \tag{3.7}$$

Following Bergman **(9)**, we can determine the number of singular sets, $\mathbf{U}_{\mu\nu}{}^2$. Hence, it is possible to determine the representation for each singular set by subtracting successively the "meromorphic part" of $f(z_1, z_2)$, and considering two functions $G(z_1, z_2)$, $H(z_1, z_2)$ equivalent if their difference has at most a removable singularity in finite $\mathfrak{C}^2$. We assume, therefore, in the following that the sets $\mathbf{U}_{\mu\nu}{}^2$ are known.

Since the singularities of a holomorphic function of two complex variables lie on an analytic set, the function $\Psi(\alpha) \equiv \hat{\rho}(\alpha) e^{i\phi(\alpha)}$ (where $\alpha = z_2/z_1$) must be, with the exception of a "thin set" in $\mathfrak{C}^1$, an analytic function of α **(1)**. Consequently, with the exception of such a thin set, we may represent the singularity manifold of $f(z_1, z_2)$ in the form

$$\mathbf{U}^2 \equiv \sum_{\mu} \sum_{\nu} \{(z_1, z_2) | z_1 = \Psi_{\mu\nu}(z_2/z_1) \equiv \rho_\nu(z_2/z_1) \exp[i\phi_{\mu\nu}(z_2/z_1)]\}. \tag{3.8}$$

This representation of the singular sets may be used to compute the possible singular points of certain classes of harmonic functions whose $\mathbf{B}_4$-associates can be written as double series; for instance, those of the form

$$f(u, \eta, \xi) \equiv \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu\nu} (u\eta\xi)^{\mu} (\eta^k \xi^l)^{\nu}, \tag{3.9}$$

where k, l are constants. In this case, one may represent the singularity manifold of (3.9) as

$$S_{\mu\nu}(\mathbf{X}; \eta, \xi) \equiv \eta\xi Y + \eta Z + \xi Z^* + Y^* - \rho_\nu \left(\frac{\eta^k \xi^l}{u\eta\xi} \right) \exp\left[i\Phi_{\mu\nu} \left(\frac{\eta^k \xi^l}{u\eta\xi} \right) \right] = 0, \tag{3.10}$$

where the notation

$$Y = x_1 + ix_2, \quad Y^* = x_1 - ix_2, \quad Z = x_3 + ix_4, \quad Z^* = -(x_3 - ix_4)$$

is introduced in **(24)**. Combining this observation with Theorem 2.3 we obtain the following result.

THEOREM 3.1. *Let $H(\mathbf{X}) = \mathbf{B}_4[f]$ be a harmonic function of four variables with an associate of the form* (3.9). *Then $H(\mathbf{X})$ is regular at all points $\mathbf{X}$ not included in the intersections*

$$\{\mathbf{X}|S_{\mu\nu} = 0\} \cap \{\mathbf{X}|\partial S_{\mu\nu}/\partial\eta = 0\} \cap \{\mathbf{X}|\partial S_{\mu\nu}/\partial\xi = 0\},$$

where $S_{\mu\nu}$ is given by (3.10).

It is clear that this result has a simple extension to solutions of $\mathbf{T}_4[\Psi] = 0$, and also to other $\mathbf{B}_4$-associates, which may be represented as formal, double power-series.

We mention that in the case where $f(u, \eta, \xi)$ is a function of just one complex variable, we may use the original Hadamard–Mandelbrojt criteria to obtain the following result.

THEOREM 3.2. *Let $H(\mathbf{X})$ be a harmonic function of four variables defined by a series representation*

$$H(\mathbf{X}) = \sum_{n=0}^{\infty} a_n H_{mn}{}^{pn+k,qn+l}(\mathbf{X}), \tag{3.11}$$

where m, p, k, q, l are constants, such that $p, q < m$. Furthermore, if

$$\lim_{j\to\infty}\left(\frac{l_j}{l_{j-1}}\right) = 0, \qquad l_p = \overline{\lim_{n\to\infty}}|D_n{}^{(p)}|^{1/n},$$

where

$$D_n{}^{(p)} = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+p} \\ a_{n+1} & a_{n+2} & \dots & a_{n+p+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n+p} & a_{n+p+1} & \dots & a_{n+2p} \end{vmatrix},$$

then in general the only singularities of $H(\mathbf{X})$ lie on two-dimensional algebraic manifolds.

The $\mathbf{B}_4$-associate of (3.11) is

$$\tilde{f}(u, \eta, \xi) = \sum_{n=0}^{\infty} a_n\, u^{mn}\eta^{pn+k}\xi^{qn+l}. \tag{3.12}$$

We consider the simplest form of an associate of this type since it will be clear from this case what the more general situation is. We suppose that

$$f(u, \eta, \xi) = \sum_{n=0}^{\infty} a_n (u\eta\xi)^n \eta^k \xi^l \equiv F(u\eta\xi)\eta^k\xi^l, \tag{3.13}$$

where $F(v)$ has singularities at the points $\alpha_\nu = \rho_\nu e^{i\phi_\nu} (\nu = 1, 2, \dots)$ (which we can locate by the ordinary Hadamard–Mandelbrojt criteria). (A necessary and sufficient condition for $F(u\eta\xi)$ to be meromorphic is that

$$\lim_{j\to\infty}\frac{l_j}{l_{j-1}} = 0.$$

In this case all of the finite singularities are isolated and we may consider the associated singularities of $H(\mathbf{X})$ separately **(21**; **25**; **4**; also **13)**.). Here, the singularity manifold of the integrand may be represented as

$$\sum_\nu \{S_\nu(X;\eta,\xi) \equiv \eta\xi Y + \eta Z + \xi Z + Y - \alpha_\nu = 0\}. \tag{3.14}$$

Consequently, Theorem 2.3 implies that $H(\mathbf{X})$ is regular at those points not contained in the set $\sum_\nu \{YY^* - zz^* - \alpha_\nu Y = 0\}$, which may be written as the sum of spheres,

$$\sum_\nu [\{(x_1 - \tfrac{1}{2}\operatorname{Re}\alpha_\nu)^2 + (x_2 + \tfrac{1}{2}\operatorname{Im}\alpha_\nu)^2 + x_3^2 + x_4^2 = |\alpha_\nu/2|^2\} \\ \cap \{x_1 \operatorname{Im}\alpha_\nu + x_2 \operatorname{Re}\alpha_\nu = 0\}]. \tag{3.15}$$

For the associate (3.12) the singularity manifold may be represented as

$$\sum_\nu \{S_\nu(\mathbf{X};\eta,\xi) \equiv (\eta\xi Y + \eta Z + \xi Z^* + Y^*)^m - \alpha_\nu \eta^{(m-p)}\xi^{(m-q)} = 0\}.$$

It is clear that in general one obtains upon eliminating η, ξ from

$$S_\nu = 0, \qquad \partial S_\nu/\partial\eta = 0, \qquad \partial S_\nu/\partial\xi = 0$$

a two-dimensional algebraic manifold. This proves our result.

There are a number of other generalizations of the Hadamard approach to the coefficient problem for solutions to $\mathbf{T}_4[\Psi] = 0$, and for harmonic functions with expansions of the form (3.11). These are similar to the results obtained by S. Bergman for the case of harmonic functions of three variables **(7**; **8**; **9)**.

4. The general coefficient problem for harmonic functions of four variables. Let $H(\mathbf{X})$ be an arbitrary harmonic function of four variables regular about the origin with the expansion

$$H(\mathbf{X}) = \sum_{n=0}^{\infty} \sum_{m,p=0}^{n} a_{nmp} H_n^{m,p}(\mathbf{X}), \qquad ||\mathbf{X}|| < \rho. \tag{4.1}$$

The "general" coefficient problem is the determination of the singularities of $H(\mathbf{X})$ in terms of the coefficients $\{a_{nmp}\}$. It is possible for us to approach this problem by using a three-variable analogue of Bergman's theorem in connection with our Theorem 2.3. We proceed as follows.

Let $\mathfrak{S}$ be the space of three complex variables $\mathfrak{C}^3$. Then through each point $(z_1^0, z_2^0, z_3^0) \in \mathfrak{S}$ (except 0) there passes a unique "pair of analytic planes,"

$$\mathfrak{P}^{(4)}(\alpha) \equiv \{z_2 = \alpha z_1\}, \qquad \mathfrak{P}^{(4)}(\beta) \equiv \{z_3 = \beta z_1\}, \tag{4.2}$$

$\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}\beta \geqslant 0$, where $\alpha = z_2^0/z_1^0$, $\beta = z_3^0/z_1^0$. We introduce next the two-dimensional intersection $\mathfrak{P}^{(2)}(\alpha,\beta) = \mathfrak{P}^{(4)}(\alpha) \cap \mathfrak{P}^{(4)}(\beta)$, which is a complex linear manifold (of complex dimension 1). Following Bergman **(9)** we require that the origin of $\mathfrak{S}$ coincide with the origin of $\mathfrak{P}^{(2)}(\alpha,\beta)$ and that the positive x-axis is that part of $\mathfrak{P}^2(\alpha,\beta) \cap \{y_1 = 0\}$ on which $x_2 > 0$. Hence, we may introduce polar co-ordinates $\rho(\alpha,\beta)$, $\phi(\alpha,\beta)$ in each "plane" $\mathfrak{P}^{(2)}(\alpha,\beta)$, and associate in this manner each point of $\mathfrak{S}$ with the three complex numbers α, β, $\rho e^{i\phi}$.

Let

$$f(z_1, z_2, z_3) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\sigma=0}^{\infty} a_{\mu\nu\sigma}\, z_1^{\mu} z_2^{\nu} z_3^{\sigma} \tag{4.3}$$

and let $F_{\alpha,\beta}(z_1)$ be defined by

$$\begin{aligned} F_{\alpha,\beta}(z_1) &\equiv f(z_1, \alpha z_1, \beta z_1) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\sigma=0}^{\infty} a_{\mu\nu\sigma}\, z_1^{\mu+\nu+\sigma} \alpha^{\nu} \beta^{\sigma} \\ &= \sum_{N=0}^{\infty} z_1^{N} \Big(\sum_{\mu+\nu=0}^{N} a_{N-(\mu+\nu),\mu,\nu}\, \alpha^{\mu} \beta^{\nu} \Big). \end{aligned} \tag{4.4}$$

After Bergman, we introduce certain functions

$$\hat{\rho}(\alpha, \beta) \equiv \limsup_{N\to\infty} \Big[\Big| \sum_{\mu+\nu=0}^{N} a_{N-(\mu+\nu),\mu,\nu}\, \alpha^{\mu}\beta^{\nu} \Big|^{1/N} \Big], \tag{4.5}$$

$$d_N(h; \alpha, \beta) \equiv \sum_{k=0}^{N} \binom{N}{K} \hat{\rho}(\alpha, \beta)^{k} h^{N-k} \Big(\sum_{\mu+\nu=0}^{N} a_{N-(\mu+\nu)-k,\mu,\nu}\, \alpha^{\mu}\beta^{\nu} \Big), \qquad h > 0, \tag{4.6}$$

$$R(h; \alpha, \beta) \equiv \limsup_{N\to\infty} [|d_N(h; \alpha, \beta)|^{1/N}], \tag{4.7}$$

and finally the right-hand derivative of $R(h; \alpha, \beta)$ with respect to h at 0,

$$R^{+\prime}(0; \alpha, \beta) = \lim_{h\to 0^+} \Big[h^{-1} \Big(\limsup_{N\to\infty} |d_N(h; \alpha, \beta)|^{1/N} - 1 \Big) \Big]. \tag{4.8}$$

It is possible, then, to obtain a three-variable analogue of Bergman's lemma concerning the domain of regularity of $f(z_1, z_2, z_3)$. The proof differs from Bergman's in that one "pulls a bi-cylinder away" from a boundary point instead of a disk before proceeding to employ Hartogs' theorem. The reader is referred to the original paper of Bergman **(9)**.

LEMMA 4.1. *Let $f(z_1, z_2, z_3)$ be a holomorphic function of three complex variables with a Taylor expansion* (4.3). *Furthermore, for each (α_0, β_0) let there exist a neighbourhood $\mathfrak{N}^4(\alpha_0, \beta_0) = \mathfrak{N}^2(\alpha_0) \times \mathfrak{N}^2(\beta_0)$ and an $h > 0$, such that*

$$\hat{\rho}(\alpha + \beta)[R^{+\prime}(0; \alpha, \beta) + h - 1] \geqslant \delta > 0, \quad \textit{for every } (\alpha, \beta) \in \mathfrak{N}^4(\alpha_0, \beta_0).$$

Then $f(z_1, z_2, z_3)$ is regular in $\mathfrak{K} \cup \mathfrak{l}^5$, where

$$\mathfrak{K} \equiv \{ |z_1| < \rho(\alpha, \beta), |\alpha| \leqslant \infty, |\beta| \leqslant \infty \},$$
$$\mathfrak{l}^5 = \{ |z_1| = \rho(\alpha, \beta), |\phi| \leqslant \Phi(\alpha, \beta), |\alpha| \leqslant \infty\ |\beta| \leqslant \infty \},$$
$$\Phi(\alpha, \beta) \equiv \liminf_{\substack{\alpha_1\to\alpha \\ \alpha_2\to\beta}} |\cos^{-1} R^{+\prime}(0; \alpha_1, \alpha_2)|.$$

In the same way as in Section III, one may obtain representations for the four-dimensional singular sets $\mathbf{U}^4$:

$$\mathbf{U}^4_{\mu\nu} \equiv \Big\{ (z_1, z_2, z_3) | z_1 = \Psi_{\mu\nu}\Big(\frac{z_2}{z_1}, \frac{z_3}{z_1}\Big) \equiv \rho_{\nu}\Big(\frac{z_2}{z_1}, \frac{z_3}{z_1}\Big) \exp\Big[i\phi_{\mu\nu}\Big(\frac{z_2}{z_1}, \frac{z_3}{z_1}\Big)\Big]\Big\}. \tag{4.9}$$

Here $\Psi_{\mu\nu}(\alpha, \beta)$ is a holomorphic function of the two complex variables α, β with the exception of certain points on a thin set in $\mathfrak{C}^2$. Then again using Theorem 2.3 we obtain a general representation theorem for the possible singularities of an arbitrary harmonic function of four variables.

THEOREM 4.2. *Let $H(\mathbf{X})$ be an arbitrary harmonic function of four variables regular about the origin, with the representation*

$$H(\mathbf{X}) = \sum_{n=0}^{\infty} \sum_{m,p=0}^{n} a_{nmp} H_n^{m,p}(\mathbf{X}).$$

Then the only possible singularities of $H(\mathbf{X})$ lie on the intersections $(\mu, \nu = 1, 2, \ldots)$

$$\text{(4.10)} \quad \{\mathbf{X} | S_{\mu\nu}(\mathbf{X}; \eta, \xi) \equiv u - \rho_\nu(\eta u^{-1}, \xi u^{-1}) \exp[i\phi_{\mu\nu}(\eta u^{-1}, \xi u^{-1})] = 0\} \cap \{\mathbf{X} | \partial S_{\mu\nu}/\partial\eta = 0\} \cap \{\mathbf{X} | \partial S_{\mu\nu}/\partial\xi = 0\}.$$

COROLLARY 4.3. *The solution of $\mathbf{T}_4[\Psi] = 0$ (where A, C are entire) that has the representation*

$$\Psi(\mathbf{X}) = \sum_{n=0}^{\infty} \sum_{m,p=0}^{n} a_{nmp} \Psi_n^{m,p}(\mathbf{X}),$$

where

$$\Psi_n^{m,p}(\mathbf{X}) \equiv \int_{-1}^{+1} E(r, t) H_n^{m,p}(\mathbf{X}[1 - t^2]) dt,$$

may be singular only on the sets (4.10).

REFERENCES

1. H. Behnke and H. Grauert, *Analysis in non-compact complex spaces* (in *Analytic functions*), No. 24 (Princeton, 1960), pp. 11–44.
2. H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen* (Berlin, 1934).
3. S. Bergman, *Integral operators in the theory of linear partial differential equations* (Berlin, 1960).
4. ——— *Zur Theorie der algebraischen Potential Funktionen des dreidimensional Raumes*, Math. Ann., *99* (1929), 629–659, and *101* (1929), 534–538.
5. ——— *Multivalued harmonic functions in three variables*, Comm. Pure Appl. Math., *9* (1956), 327–338.
6. ——— *Operators generating solutions of certain differential equations in three variables and their properties*, Scripta Math., *26* (1961), 5–31.
7. ——— *Some properties of a harmonic function of three variables given by its series development*, Arch. Rational Mech. Anal., *8* (1961), 207–222.
8. ——— *Integral operators in the study of an algebra and a coefficient problem in the theory of three dimensional harmonic functions*, Duke Math. J., *30* (1963), 447–460.
9. ——— *On the coefficient problem on the theory of a system of linear partial differential equations*, J. Analyse Math., *11* (1963), 249–274.
10. S. Bochner and W. T. Martin, *Several complex variables* (Princeton, 1948).
11. C. Carathéodory, *Function theory*, Vols I, II (New York, 1950).

12. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions* II (New York, 1953).
13. R. P. Gilbert, *Singularities of three-dimensional harmonic functions*, Pacific J. Math., *10* (1960), 1243–1255.
14. ——— *Singularities of solutions to the wave equation in three dimensions*, J. Reine Angew. Math., *205* (1960), 75–81.
15. ——— *On harmonic functions of four variables with rational P_4-associates*, Pacific J. Math., **13** (1963), 79–96.
16. ——— *Harmonic functions in four variables with algebraic and rational P_4-associates*, Ann. Polon. Math., *15* (1964), 273–287.
17. ——— *Multivalued harmonic functions in four variables*, to appear in J. Analyse Math.
18. ——— *On a class of elliptic partial differential equations in four variables*, Pacific J. Math., *14* (1964), 1223–1236.
19. R. P. Gilbert and H. C. Howard, *On a class of elliptic partial differential equations*, Technical Note BN-344, December 1963.
20. ——— *Integral operator methods for generalized axially symmetric potentials in $(n + 1)$-variables*, to appear in J. Austral. Math. Soc.
21. J. Hadamard, *Essai sur l'étude des fonctions données par leurs développements de Taylor*, J. Math., *4*, *8* (1892), 101–186.
22. E. Kreyszig, *Coefficient problems in systems of partial differential equations*, Arch. Rational Mech. Anal., *1* (1958), 283–294.
23. ——— *On singularities of solutions of partial differential equations in three variables*, Arch. Rational Mech. Anal., *2* (1958), 151–159.
24. ——— *Kanonische integral Operatoren zur Erzeugung harmonischer Funktionen von vier Veränderlichen*, Arch. Math., *14* (1963), 193–203.
25. S. Mandelbrojt, *Théorème général fournissant l'argument des points singuliers situés sur le cercle de convergence d'une série de Taylor*, C. R. Acad. Sci. Paris, *204* (1937), 1456–1458.
26. R. de Mises, *La base géométrique du théorème de M. Mandelbrojt sur les points singuliers d'une fonction analytique*, C.R. Acad. Sci. Paris, *205* (1938), 1353–1355.
27. J. Mitchell, *Integral theorems for harmonic vectors in three real variables*, Math. Z., *82* (1963), 314–334.
28. ——— *Representation theorems for solutions of linear partial differential equations in three variables*, Arch. Rational Mech. Anal., *3* (1959), 439–459.
29. K. Oka, *Sur les fonctions analytiques de plusieurs variables* (Tokyo, 1960).
30. W. F. Osgood, *Lehrbuch der Funktionentheorie*, Vol. 2 (2nd ed., Leipzig, 1929).
31. H. Poincaré, *Sur les fonctions de deux variables*, Acta Math., *2* (1883), 64–72.
32. A. White, *Singularities of harmonic functions of three real variables generated by Whittaker-Bergman operators*, Ann. Polon. Math., *10* (1961), 82–100.

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