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The Action of a Plane Singular Holomorphic Flow on a Non-invariant Branch

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Abstract. We study the dynamics of a singular holomorphic vector field at $(\mathbb{C}^2, 0)$. Using the associated flow and its pullback to the blow-up manifold, we provide invariants relating the vector field, a non-invariant analytic branch of curve, and the deformation of this branch by the flow. This leads us to study the conjugacy classes of singular branches under the action of holomorphic flows. In particular, we show that there exists an analytic class that is not complete, meaning that there are two elements of the class that are not analytically conjugated by a local biholomorphism embedded in a one-parameter flow. Our techniques are new and offer an approach dual to the one used classically to study singularities of holomorphic vector fields.

1 Introduction

The classical study of singularities of plane holomorphic vector fields, since the work of Seidenberg [17], has focused on the local structure of the associated foliation and its leaves (*i.e.*, the invariant curves of the vector field); index-like results [2], the Separatrix Theorem of Camacho and Sad [3], studies of holonomy [15] (being very succinct), etc. deal mainly with what one might call the *static* structure of the invariant sets of the vector field. Generalizations of these studies to higher dimension like [4] or [7] and even recent works on real-analytic singularities in dimension 3 like Cano, Moussu, and Sanz's study on the relation between oscillation and spiraling [6] or the related [5] apply the blow-up technique in such a way that the ensuing exceptional divisor becomes an invariant set but not composed of invariant points.

However, the dynamical nature of the vector field as the infinitesimal generator of a *flow* is lost the very moment the equation of the exceptional divisor is "factored out" from the local equation of the pullback of the vector field to the blow-up manifold. This prevents the classical techniques from applying to the study of the flow associated with a holomorphic vector field in the neighbourhood of an equilibrium point.

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In this work, we present what we deem to be the first study of these dynamics. Given a singular point of a plane holomorphic vector field, we study how its associated flow behaves after a finite chain of point blow-ups. Specifically, we study how that flow "moves" a non-invariant branch passing through the equilibrium point.

Consider a singular holomorphic vector field X defined in an open neighbourhood U of $(0,0) \in \mathbb{C}^2$ and an irreducible germ of analytic curve Γ contained in the same open set U, say $\Gamma \equiv (f = 0)$ for some $f \in \mathcal{O}_{(\mathbb{C}^2,0)}$. Let $\{\psi_s\}_{s\in\mathbb{C}}$ be the one-parameter group whose infinitesimal generator is X. Let $\epsilon \in \mathbb{C}$; define

$$\psi_{-\epsilon}(\Gamma) := \Gamma_\epsilon \equiv \big(f \circ \psi_\epsilon(x,y) = 0 \big).$$

By expanding $f \circ \psi_\epsilon$ as a Taylor power series in the variable ϵ we obtain

$$\Gamma_\epsilon \equiv \Big(\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n(f)(x,y) = 0 \Big), \tag{1.1}$$

where $X^0(f) = f$, and we set $X^{j+1}(f) = X(X^j(f))$ for $j \ge 0$ recursively. We shall call $\{\Gamma_\epsilon\}$ the *holomorphic deformation of* Γ *by* X (or by $\{\psi_\epsilon\}$). The coefficient of $x^i y^j$ for $f \circ \psi_\epsilon$ is an entire function of ϵ for any $i + j \ge 0$. Assume for simplicity that the tangent cone of Γ is not $x = 0$. The curve Γ_ϵ has a Puiseux parametrization of the form $(t^n, \sum_{j=n}^{\infty} a_j(\epsilon) t^j)$ for any ϵ in a small neighborhood of 0 in $\mathbb{C}$ where n is the multiplicity of Γ at $(0,0)$. If Γ is not invariant by X, there is a first index k such that $a_k(\epsilon)$ is not a constant function. We call $k = (X,\Gamma)_{(0,0)}$ the *contact order* between X and Γ and prove the following theorem.

Theorem 1.1 *We have*

$$k := (X,\Gamma)_{(0,0)} = \big(X(f), f\big)_{(0,0)} - n - c + 1, \tag{1.2}$$

where $(X(f), f)_{(0,0)}$ is the intersection multiplicity of f and $X(f)$, or in other words the tangency order of X with Γ, and c is the conductor of Γ. Moreover, the function $a_k : \mathbb{C} \to \mathbb{C}$ is locally injective and of the form $\lambda\epsilon + \mu$ or $\lambda e^{\gamma\epsilon} + \mu$ where $\lambda, \gamma \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. The function a_k is linear if X is nilpotent.

The previous theorem provides the value of $k = (X,\Gamma)_{(0,0)}$ and the form of the coefficient a_k. Since a_k is locally injective, we can interpret k as the vanishing multiplicity of the Lie derivative of the Puiseux parametrization of Γ with respect to X.

Property (1.2) is not obvious: a priori the value of $(X,\Gamma)_{(0,0)}$ could have depended on other terms of the Taylor power series expansion of $f \circ \psi_\epsilon$. For instance, consider the intersection multiplicity $(\Gamma, \Gamma_\epsilon)_{(0,0)}$. It is equal to $\min\{(X^n(f), f)_{(0,0)} : n \ge 1\}$ for $\epsilon \in \mathbb{C}^*$ in a small neighborhood of 0 by equation (1.1). The minimum can be realized for $n > 1$, as is the case for $\Gamma = (y^2 - x^3 = 0)$ and $X = x\frac{\partial}{\partial y}$ where $(X(f), f)_{(0,0)} = 5$ and $(X^2(f), f)_{(0,0)} = 4$.

Another novel concept is that of the *path shared* between X and Γ: the sequence $(P_i)_{i=0}^n$ of infinitely near points of Γ such that the pullback X_i is singular at all P_i except at P_n. This sequence exists because after a finite chain of blow-ups following a non-invariant curve, the vector field becomes non-singular (and hence, gives rise to a non-trivial flow on the exceptional divisor). We also provide a Noether-like formula

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for the sequence of contact exponents $(X_i, \Gamma_i)_{P_i}$, where Γ_i is the strict transform of Γ at P_i.

The previous discussion motivates the study of the action of one-parameter groups on irreducible curves. Consider an equivalence class $\mathcal{C}$ for the equivalence relation given by the analytic conjugacy of plane branches. We say that two curves $\Gamma_1, \Gamma_2 \in \mathcal{C}$ are *connected by a geodesic* if they are conjugated by the time 1 flow, $\exp(X)$, of a germ of holomorphic singular vector field. We say that $\mathcal{C}$ is *complete* if given any two curves $\Gamma_1, \Gamma_2 \in \mathcal{C}$, they are connected by a geodesic. The term complete is motivated by analogy with the case of finite dimensional Lie groups G that have a bi-invariant metric where geodesics are of the form $t \mapsto \exp(tX) \cdot g$ where X belongs to the Lie algebra of G, $g \in G$ and t varies in $\mathbb{R}$. An example of a complete class $\mathcal{C}$ is the class of smooth curves (Proposition 4.5).

One could define a notion of formal completeness in which X is a formal vector field, (a derivation of $\mathbb{C}[[x, y]]$ preserving its maximal ideal). The definitions are, in fact, equivalent.

Theorem 1.2 *Let $\mathcal{C}$ be a class of analytic conjugacy of plane branches. Then $\mathcal{C}$ is complete if and only if $\mathcal{C}$ is formally complete.*

Since unipotent biholomorphisms are always embedded in the one-parameter group of a formal vector field (*cf.* Remark 4.8), any conjugacy class modulo such diffeomorphisms is complete, by Theorem 1.2. Moreover, since any analytic conjugacy φ between curves Γ_1 and Γ_2 may be written in the form $D_0\varphi \circ \psi$, where $\psi := (D_0\varphi)^{-1} \circ \varphi$ has linear part equal to the identity (see Corollary 4.14), we deduce that Γ_1 and Γ_2 can be connected by two "segments of geodesic". More precisely, there exist germs of singular holomorphic vector fields X, Y such that $(\exp(Y) \circ \exp(X))(\Gamma_1) = \Gamma_2$ (Corollary 4.9). A class of analytic conjugacy $\mathcal{C}$ of a plane branch Γ is identified with the set of left cosets of $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)/\operatorname{Stab}(\Gamma)$, where $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ is the group of germs of holomorphic diffeomorphisms defined in a neighborhood of $0 \in \mathbb{C}^2$ and $\operatorname{Stab}(\Gamma) = \{\varphi \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0) : \varphi(\Gamma) = \Gamma\}$ is the stabilizer of Γ. There exist local biholomorphisms that cannot be embedded in the flow of a formal vector field ([16,20]), but to show that a class is not complete, we need to prove a stronger result: that there exists a left coset $\varphi \circ \operatorname{Stab}(\Gamma)$ in $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)/\operatorname{Stab}(\Gamma)$ such that none of its elements can be embedded in the flow of a formal vector field. We will show that there exist local biholomorphisms φ_0 such that any $\varphi \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ sharing the same second jet as φ_0 is not embedded in the flow of a formal vector field. Then we shall prove that there are plane branches Γ such that its stabilizer is small: any element of $\operatorname{Stab}(\Gamma)$ has second jet equal to the identity map. Combining these two results we obtain that no element of $\varphi_0 \circ \operatorname{Stab}(\Gamma)$ is embedded in the flow of a formal vector field. By following the previous ideas, we obtain the following proposition.

Proposition 1.3 *Let Γ be the plane branch with Puiseux parametrization $(t^6, t^7 + t^{10} + t^{11})$. Then the class $\mathcal{C}$ of analytic conjugacy of Γ is non-complete.*

We can provide a topology in the class $\mathcal{C}$ of a plane branch Γ by considering a topology in $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ and the corresponding quotient topology in the set

$\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)/\mathrm{Stab}(\Gamma)$. A natural choice is the Krull topology (also called $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[x,y]]$), where the sets $S_{k,\varphi}$ of elements of $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ whose k-jet coincides with the k-jet of φ provide a base of open sets of the topology by varying φ in $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ and k in $\mathbb{N}$. Proposition 1.3 can be reinterpreted as a genericity property in the class $\mathcal{C}$.

Proposition 1.4 *Let $\mathcal{C}$ be the analytic class of the plane branch Γ with Puiseux parametrization $(t^6, t^7+t^{10}+t^{11})$. Let*

$$\mathcal{C}' = \{\Gamma' \in \mathcal{C} : \Gamma \text{ and } \Gamma' \text{ are connected by a geodesic}\}.$$

Then $\mathcal{C} \setminus \mathcal{C}'$ contains an open set of $\mathcal{C}$ for the Krull topology. In other words, $\mathcal{C}'$ is not dense in $\mathcal{C}$.

The previous result does not hold for other natural topologies.

Proposition 1.5 *Let Γ be a plane branch and let $\mathcal{C}$ be its analytic class of conjugacy. Let $\Gamma' \in \mathcal{C}$. Then there exist a holomorphic deformation Γ'_ϵ of Γ' by a holomorphic vector field, defined in a neighborhood of $\epsilon = 0$, $\Gamma'_0 = \Gamma'$ and a simple continuous curve $\gamma\colon [0,1] \to \mathbb{C}$ such that $\gamma(0) = 0$ and Γ is connected by a geodesic to $\Gamma'_{\gamma(t)}$ for any $t \in (0,1]$.*

We obtain, in passing, a *flow-based* solution to the moduli problem for plane branches [19], solved by Hefez and Hernandes [11] using Lie-group methods.

Notice, finally, that our techniques are quite different from those of classical deformation theory [10]: in this, one is concerned with deformations by adding a "small" parameter to the equation of the curve, and the aim is to study the geometric and topological properties of the moduli so obtained. We are specifically concerned with deformations caused by flows, so that (in a rough sense) we are adding the parameter at all the orders of the equation.

2 Notation and Definitions

Our base ring is $\mathcal{O} = \mathcal{O}_P$, the ring of germs of holomorphic functions in a neighbourhood of a point P of a two-dimensional complex-analytic manifold, whose base "set" we shall usually denote, as is the custom, by $(\mathbb{C}^2,0)$. The maximal ideal of $\mathcal{O}$ will be denoted by $\mathfrak{m}_{0,P}$ or simply $\mathfrak{m}_0$ when no confusion arises. Assume $P = (0,0) \in \mathbb{C}^2$ for simplicity. We denote $\mathbb{C}[[x,y]]$ by $\widehat{\mathcal{O}}$ and its maximal ideal by $\mathfrak{m}$.

Definition 2.1 We say that $f, g \in \widehat{\mathcal{O}}$ have the same k-jet, and we let $j^k f = j^k g$ if $f - g \in \mathfrak{m}^{k+1}$.

Let $(f_k)_{k\geq 1}$ be a sequence in $\widehat{\mathcal{O}}$. Then it converges to $f \in \widehat{\mathcal{O}}$ in the $\mathfrak{m}$-adic topology (or also the Krull topology) if for any $l \geq 1$, there exists $k_0 \geq 1$ such that $j^l f_k = j^l f$ for any $k \geq k_0$.

Definition 2.2 We say that X is a *(local holomorphic) vector field* if is a $\mathbb{C}-$derivation $X : \mathcal{O}_P \to \mathcal{O}_P$ continuous for the $\mathfrak{m}_P-$adic topology. It is *singular* if $X(\mathfrak{m}_P) = \mathfrak{m}_P$ and *regular* otherwise. In the case $P = (0,0) \in \mathbb{C}^2$, we write

$$X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

where $A := X(x)$ and $B := X(y)$ belong to $\mathcal{O}$. Analogously, by replacing $\mathcal{O}$ and $\mathfrak{m}_0$ with $\widehat{\mathcal{O}}$ and $\mathfrak{m}$, we can define formal vector fields.

Remark For the sake of simplicity, a local holomorphic vector field will frequently be called simply a vector field. We never consider non-holomorphic vector fields.

Definition 2.3 Let X be a formal singular vector field. We say that X is *nilpotent* if its linear part is a nilpotent vector field.

We also say that P is a singular point for X (especially, but not only, when X can be understood as a vector field on a larger analytic manifold). Finally, X is *truly singular* at P (or P is a *true singularity* of X) if it is singular and there do not exist a regular vector field Y and a regular holomorphic function $f \in \mathcal{O}_P$ such that $X = f^m Y$ for some positive integer m (this is related to what is called a *strictly singular* point in [14]). Note that all these definitions are given for the local case; we shall be explicit when dealing with non-local situations.

The *multiplicity* of a formal vector field X is the largest non-negative integer m such that $X(\mathfrak{m}) \subset \mathfrak{m}^m$. Thus, a non-singular vector field has multiplicity 0, and in general, if $X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$, then the multiplicity of X is the smallest of the multiplicities of $a(x,y)$ and $b(x,y)$.

An analytic *branch* (simply branch) at P is any reduced and irreducible curve $\Gamma \subset (\mathbb{C}^2, 0)$. Unless otherwise specified, all our curves will be analytic branches, and they will be defined either by a reduced and irreducible holomorphic function $f \in \mathfrak{m}_P$ or by a Puiseux expansion $\varphi(t) = (x(t), y(t))$ when local coordinates at P are already chosen. All the results related to desingularisation of plane branches (and, as a requirement, finite sequences of point blow-ups, exceptional divisors, etc.) and their topological (not analytic) structure are assumed known; two good modern references are [8,18].

Consider a point P belonging to a two-dimensional complex analytic manifold $\mathcal{M}$. Denote by $\mathcal{M}_P$ the germ of $\mathcal{M}$ at P (which is, essentially, the same thing as $(\mathbb{C}^2, 0)$). As our work is based on the process of point blow-ups, we need the following definition.

Definition 2.4 Let X be a singular vector field at P and let $\pi\colon \mathcal{X} \to \mathcal{M}_P$ be the blow-up with centre P. The unique holomorphic vector field $\overline{X}$ on the whole $\mathcal{X}$ such that $\pi_*(\overline{X}) = X$ outside of the exceptional divisor $\pi^{-1}(P)$ is called the *pullback of X to $\mathcal{X}$*.

The fact that $\overline{X}$ exists is due to the singularity of X at P: otherwise, $\overline{X}$ is not defined (it has "poles" on the exceptional divisor).

Remark Notice that we are taking the "true" pullback of X on $\mathcal{X}$; we are interested in the *dynamics* of X, not just in the geometric structure of its integral curves. Thus, if (x,y) are local coordinates at P and one looks at the chart of π with equations

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$x = \overline{x}, y = \overline{xy}$ and

$$X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

for some $a(x,y), b(x,y) \in \mathfrak{m}_P$, then on the chart $(\overline{x}, \overline{y})$, the local equation of $\overline{X}$ is given by

$$\overline{X} = a(\overline{x}, \overline{xy})\frac{\partial}{\partial \overline{x}} + \frac{1}{\overline{x}}\big(-\overline{y}a(\overline{x}, \overline{xy}) + b(\overline{x}, \overline{xy})\big)\frac{\partial}{\partial \overline{y}},$$

an expression that shows why X must have a singularity at P in order to admit a pullback to $\mathcal{X}$. As the reader will have noticed, we do not eliminate the possible common factor $\overline{x}$ in the expression of $\overline{X}$. This implies that, usually, the pullback of a singular vector field will not be truly singular; it will have some true singularities on the exceptional divisor, but most of the points will be just equilibrium points such that, near them, $\overline{X}$ is of the form $\overline{x}^m Y$ for some non-negative integer m and non-singular vector field Y.

The reader familiar with the theory of plane holomorphic foliations will notice the similarity and the differences between our approach and the one common in those works. This difference is exactly what makes our technique useful for studying deformations.

Anyway, we can consider the desingularisation of the underlying foliation of a singular vector field. The following result is a restatement of the main one in [17].

Theorem 2.5 (*cf.* [17]) *Let X be a singular vector field at $P \in \mathcal{M}_P$. There is a finite sequence of blow-ups $\pi\colon \mathcal{X} \to \mathcal{M}_P$,*

$$\mathcal{X} = \mathcal{X}_N \xrightarrow{\pi_{N-1}} \mathcal{X}_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} \mathcal{X}_1 \xrightarrow{\pi_0} (\mathbb{C}^2, 0),$$

$\pi = \pi_0 \circ \cdots \circ \pi_{N-1}$, whose centres $(P_i)_{i=0}^{N-1}$ are singular points for the respective pullbacks of X and such that the pullback $\overline{X}$ of X on $\mathcal{X}$ has a finite number of true singularities and at any of these, say Q, $\overline{X}$ admits an expression of the form

$$x^a y^b\Big(\mu x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} + \text{h.o.t.}\Big),$$

where (x,y) are local coordinates at Q and the exceptional divisor is included in $xy = 0$, $\mu \neq 0$, and $\lambda/\mu \notin \mathbb{Q}_{>0}$. The shortest non-empty sequence of blow-ups for which this happens is called the minimal reduction of singularities of X.

Let Γ and X be an analytic branch and a singular vector field at $(\mathbb{C}^2, 0)$. Let $\mathcal{X}_0 = (\mathbb{C}^2, 0)$, $\overline{\Gamma}_0 = \Gamma$, and $P_0 = (0,0)$. We define recursively the blow-up $\pi_i\colon \mathcal{X}_{i+1} \to \mathcal{X}_i$ with centre P_i, where P_i is the intersection of the strict transform of Γ (by $\pi_0 \circ \cdots \circ \pi_{i-1}$) with the corresponding exceptional divisor for $i \geq 1$. The next result follows easily from the fact that Γ is analytic.

Proposition 2.6 *With the notation of the last paragraph, Γ is invariant by X if and only if P_i is a singular point of the pullback $\overline{X}_i$ of X to $\mathcal{X}_i$ for any $i \geq 0$. In particular, if Γ is not invariant by X then there exists $i_0 \geq 0$ such that P_i is a singular point of $\overline{X}_i$ of X for any $0 \leq i \leq i_0$, but P_{i_0+1} is a regular point of $\overline{X}_{i_0+1}$.*

This result provides the following definition.

Definition 2.7 The *path shared* by a non-invariant analytic branch Γ and a singular vector field X is the sequence $(P_0, P_1, \dots, P_{i_0+1})$ given by Proposition 2.6. Notice that we include in the shared path the point at which the pullback of X is non-singular.

Remark The last point shared by X and Γ could be a singular point of the strict transform of Γ; we only require it to be a regular point for the pullback of X.

The following result will be important in the study of the relation between a curve and its deformation.

Lemma 2.8 *Let $(P_i)_{i=0}^N$ be the shared path between Γ and a singular vector field X. The last point P_N is not a corner of the exceptional divisor.*

Proof This is because after blowing up a singular point, the exceptional divisor is always invariant for the pullback. If P_N were a corner, then the pullback $\overline{X}$ at P_N would possess at least two invariant curves: both components of the exceptional divisor. This would imply that P_N is singular for $\overline{X}$, which contradicts the definition. ∎

We now introduce our main object of study.

Definition 2.9 Given a singular germ of analytic vector field X and an irreducible germ of analytic plane curve Γ at $(\mathbb{C}^2, 0)$ with $\Gamma \equiv (f = 0)$ for $f \in \mathcal{O}$, we define the *deformation of* Γ *caused by* X or *by the flow associated with* X as the family

$$\Gamma_\epsilon \equiv \Big(\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n(f) = 0 \Big) \equiv \Big(f + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} X^n(f) = 0 \Big).$$

We will refer to the entire family as well as to each of its members as "the deformation of Γ".

Notice that, because X is singular, if its multiplicity is greater than 1, then the local equation of Γ_ϵ is, roughly speaking, a higher order deformation of the local equation of Γ, in the sense that the terms added to f are of order at least one more than the vanishing order of f. In any case, it is clear that the deformation of a non-singular analytic branch by a singular vector field is non-singular for any $\epsilon \in \mathbb{C}$.

The following consequence of the formula for the higher derivative of a product is what makes blow-ups a sensible tool for studying deformations caused by vector fields.

Lemma 2.10 *Let X be a singular vector field at $(\mathbb{C}^2, 0)$, $\pi\colon \mathcal{X} \to (\mathbb{C}^2, 0)$ be the blow-up with centre $(0,0)$ and $\overline{X}$ the pullback of X by π. If $\Gamma \equiv (f = 0)$ is an analytic branch through $(0,0)$ and $\overline{\Gamma}$ is its strict transform by π, then $\overline{\Gamma_\epsilon} = \overline{\Gamma}_\epsilon$; that is, the strict transform of the deformation of Γ by X is the deformation of the strict transform $\overline{\Gamma}$ by $\overline{X}$. This generalises to any finite sequence of blow-ups with centres singular points of X and its successive pullbacks.*

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Finally, the one-parameter group of biholomorphisms induced by a vector field and the one of its pullback are essentially the same object.

Lemma 2.11 *Let X be a singular vector field at $(\mathbb{C}^2, 0)$ and $\{\psi_{X,s}(z)\}_s$ its one-parameter group of germs of biholomorphisms. Let $\pi\colon \mathcal{X} \to (\mathbb{C}^2, 0)$ be a sequence of blow-ups whose centres are singular points of each pullback of X and let $\overline{X}$ be the pull-back of X to $\mathcal{X}$. The biholomorphism associated with $\overline{X}$ for the value s of the parameter is the unique holomorphic extension $\overline{\psi}_{\overline{X},s}$ to the whole $\mathcal{X}$ of the biholomorphism $\pi^{-1} \circ \psi_{X,s} \circ \pi$ defined on $\mathcal{X} \setminus \pi^{-1}(0,0)$.*

3 Main Results

The deformation Γ_ϵ of an analytic branch Γ caused by a singular vector field X (which is, obviously, analytically conjugated to Γ) has a nice behaviour due to the Cauchy–Kowalewski Theorem.

Proposition 3.1 *Let X be a singular analytic vector field at $(\mathbb{C}^2, 0)$ and let Γ be an analytic plane branch that is not invariant for X. Then Γ_ϵ and Γ share the same path with X except possibly the last point: for ϵ small enough, the last shared point is certainly different.*

Proof Let $(P_i)_{i=0}^N$ be the path shared by X and Γ. By Lemma 2.8, P_N is not a corner of the exceptional divisor. By definition, the vector field $\overline{X}$ is non-singular at P_N, and it is tangent to the exceptional divisor $E = \pi^{-1}(0,0)$. This implies that Γ_ϵ meets the exceptional divisor away from P_N for ϵ small enough. ∎

For the sake of clarity let us recall the definition of intersection multiplicity.

Definition 3.2 Let $\Delta \equiv (g(x,y) = 0)$ be an analytic curve in $(\mathbb{C}^2, 0)$ that does not contain Γ. The *intersection multiplicity* $(\Gamma \cap \Delta)_{(0,0)}$ (also denoted by $(f,g)_{(0,0)}$) of Γ and Δ at $(0,0)$ is the (finite) number

$$(\Gamma \cap \Delta)_{(0,0)} = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(f,g).$$

In the case we are dealing with, where Γ is a branch, this number can be computed as

$$(\Gamma \cap \Delta)_{(0,0)} = \operatorname{ord}_t \big(g(\varphi(t))\big),$$

where $\varphi(t)$ is any irreducible Puiseux parametrization of Γ. The sub-index $(0,0)$ is usually omitted.

The following corollary is a direct consequence of Proposition 3.1.

Corollary 3.3 *Let $\Gamma \equiv (f = 0)$ be a (possibly singular) analytic branch at $(\mathbb{C}^2, 0)$ that is not invariant by a singular analytic vector field X. If $n_0, n_1, \dots, n_N$ is the sequence of multiplicities of Γ at the points of the path it shares with X, then the intersection multiplicity of Γ and Γ_ϵ is given by*

$$(\Gamma, \Gamma_\epsilon)_{(0,0)} = \sum_{i=0}^{N-1} n_i^2 = \min\big\{(X^n(f), f)_{(0,0)} : n \geq 1\big\}$$

for $0 < |\epsilon| \ll 1$.

Proof As the sequence of infinitely near points shared by Γ and Γ_ϵ is the shared path between X and Γ except the last point (for $\epsilon \ll 1$), Noether's formula (see, for example, [8]) gives

$$(\Gamma, \Gamma_\epsilon)_{0,0} = \sum_{i=0}^{N-1} n_i n_{P_i}(\Gamma_\epsilon),$$

where $n_{P_i}(\Gamma_\epsilon)$ denotes the multiplicity of the strict transform of Γ_ϵ at P_i. Since Γ and Γ_ϵ are topologically equivalent (as they are analytically conjugated), their sequence of multiplicities at their infinitely near points are the same: $n_{P_i}(\Gamma_\epsilon) = n_i$, and the first equality follows. Notice that we need to set $\epsilon \ll 1$, because Γ and Γ_ϵ might share more points for ϵ not small enough. For the second equality, let $k = \min\{(X^n(f), f)_{(0,0)} : n \geq 1\}$. Certainly, $k \leq (\Gamma, \Gamma_\epsilon)_{(0,0)}$. Notice that

$$(\Gamma, \Gamma_\epsilon)_{(0,0)} = \operatorname{ord}_t \Big(\sum_{r \in T_k} \frac{\epsilon^r}{r!} a_r t^k + \text{h.o.t.} \Big),$$

where $T_k = \{r \in \mathbb{N} : (f, X^r(f))_{(0,0)} = k\}$ and a_r is the term of order k in $X^r(f)(\varphi(t))$, for a parametrization $\varphi(t)$ of $\Gamma \equiv (f = 0)$. If $(\Gamma, \Gamma_\epsilon)_{(0,0)}$ were strictly greater than k for some $\epsilon \neq 0$ in every pointed neighborhood of 0, then

$$\sum_{r \in T_k} \frac{a_r}{r!} \epsilon^r = 0$$

for all $\epsilon \in \mathbb{C}$ by the isolated zeros principle, so that $a_r = 0$ for all r, against the assumption. ∎

Definition 3.4 The *tangency order* between X and Γ is defined as $\operatorname{tang}_{(0,0)}(X, \Gamma) = (X(f), f)_{(0,0)}$ (see [1]).

Lemma 3.5 *We have*

$$\operatorname{tang}_{(0,0)}(X, \Gamma) = (\Gamma, \Gamma_\epsilon)$$

for $0 < \epsilon \ll 1$ *when* Γ *is non-singular.*

Proof We can assume that Γ is not invariant by X, since otherwise $\operatorname{tang}_{(0,0)}(X, \Gamma) = \infty$ and $\Gamma = \Gamma_\epsilon$ for any $\epsilon \in \mathbb{C}$. As Γ is non-singular and is not invariant for X, after a change of coordinates, we can assume that $f = y$ and $X(y) \notin (y)$. Writing

$$X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y},$$

we obtain $B(x, y) = a(x^k + \text{h.o.t.}) + y(\overline{B}(x, y))$ for some $k > 0$ and $a \neq 0$. As $A(0, 0) = 0$, an easy inductive argument implies that

$$\operatorname{ord}_x \big(X^k(y)(x, 0) \big) \geq k,$$

which is what we need, by Corollary 3.3. ∎

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3.1 Vector Fields, Differential Forms, and Curves

Now consider a branch Γ which, for the sake of simplicity, we assume tangent to the OX axis, so $\Gamma \equiv (f(x,y)=0)$ with $f(x,y) = y^n + \text{h.o.t.}$ It is well known that Γ admits what is called an *irreducible Puiseux parametrization*

$$\varphi(t) \equiv \big(x(t), y(t)\big) = \Big(t^n, \sum_{i \geq n} a_i t^i\Big),$$

where $\varphi(t)$ is not of the form $\varphi(t^k)$ for any $k \geq 2$; the greatest common divisor of n and the exponents appearing in $y(t)$ is 1. Up to replacing Γ with its conjugate by some local biholomorphism of the form $(x,y) \mapsto (x, y + a(x))$, one easily deduces that there exists what we shall call a *prepared Puiseux parametrization*.

Definition 3.6 A *prepared Puiseux parametrization* of Γ is an irreducible Puiseux parametrization such that $m > n$ and $n \nmid m$, where $m = \operatorname{ord}_t(y(t))$.

Before proceeding any further, let us recall some definitions.

Definition 3.7 The *semigroup* S_Γ (or simply S) associated with Γ is the set

$$S_\Gamma = \big\{(\Gamma \cap \Delta)_{(0,0)} : \Delta \equiv (f(x,y)=0), f(x,y) \in \mathbb{C}\{x,y\}, \Gamma \not\subset \Delta\big\}.$$

It is a sub-semigroup of $\mathbb{N}$. It is well known (due to the fact that Γ is a branch) that there is $c \in S_\Gamma$ such that $p \geq c$ implies $p \in S_\Gamma$. The least c satisfying this property is called the *conductor* of Γ.

Given a differential form $\omega \in \Omega^1_{\mathcal{O}}$, say $\omega = a(x,y)dx + b(x,y)dy$, the *contact* of ω with Γ is defined (as in [19]) as

$$v_\Gamma(\omega) = \operatorname{ord}_t \big(a(x(t),y(t))\dot{x}(t) + b(x(t),y(t))\dot{y}(t)\big) + 1,$$

which does not depend on the parametrization of Γ. On the other hand, given a vector field X, say $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$, let us calculate $\operatorname{tang}_{(0,0)}(X,\Gamma)$. We have $\frac{\partial f}{\partial y} \neq 0$ and (certainly) $\dot{x}(t) \neq 0$. Since $f(x(t),y(t)) = 0$, we deduce that

$$\frac{\partial f}{\partial x}\dot{x}(t) + \frac{\partial f}{\partial y}\dot{y}(t) = 0, \tag{3.1}$$

which can be rewritten as

$$\frac{\partial f}{\partial x} = -\frac{\dot{y}(t)}{\dot{x}(t)}\frac{\partial f}{\partial y},$$

so that, when computing the tangency order $\operatorname{tang}_{(0,0)}(X,\Gamma)$, one gets

$$X(f)\big(x(t),y(t)\big) = A\big(x(t),y(t)\big)\frac{\partial f}{\partial x}\big(x(t),y(t)\big) + B\big(x(t),y(t)\big)\frac{\partial f}{\partial y}\big(x(t),y(t)\big),$$

which, substituting (3.1), gives

$$X(f)\big(x(t),y(t)\big)\,\dot{x}(t) = \frac{\partial f}{\partial y}\big(x(t),y(t)\big)\big(-A(x(t),y(t))\dot{y}(t) + B(x(t),y(t))\dot{x}(t)\big),$$

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that leads to the following valuative formula:

$$\operatorname{ord}_t\left(X(f)(x(t),y(t))\right)+\operatorname{ord}_t(\dot{x}(t))=\operatorname{ord}_t\left(\frac{\partial f}{\partial y}(x(t),y(t))\right)+v_\Gamma(\overline{\omega})-1,$$

for $\overline{\omega}=B(x,y)dx-A(x,y)dy$. It is well known (see, for example, [19]) that

$$\operatorname{ord}_t\left(\frac{\partial f}{\partial y}(x(t),y(t))\right)=c+n-1,$$

where c is the conductor of S_Γ. Hence, we get

$$\operatorname{tang}_{(0,0)}(X,\Gamma)+(n-1)=c+(n-1)+v_\Gamma(\overline{\omega})-1,$$

that is,

$$\operatorname{tang}_{(0,0)}(X,\Gamma)=v_\Gamma(\overline{\omega})+c-1.$$

Thus, we might define $v_\Gamma(X):=\operatorname{tang}_{(0,0)}(X,\Gamma)-c+1$ and obtain, in a natural way, $v_\Gamma(X)=v_\Gamma(\overline{\omega})$. Dually, we obtain the following formula for the conductor.

Corollary 3.8 *Let $\Gamma\equiv(f=0)$ be a singular branch at $(\mathbb{C}^2,0)$ and let $X=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$ be any singular vector field. If $\omega=-B(x,y)dx+A(x,y)dy$, then*

$$c=\operatorname{tang}_{(0,0)}(X,\Gamma)-v_\Gamma(w)+1.$$

The next result follows from a simple (classical) computation.

Lemma 3.9 *Let ω be a singular differential form in $(\mathbb{C}^2,0)$ and let $\pi:\mathcal{X}\to(\mathbb{C}^2,0)$ be the blow-up of $(\mathbb{C}^2,0)$ with centre $(0,0)$ with equations $x=\overline{x}, y=\overline{xy}$. Let Γ be a branch (singular or not) at $(\mathbb{C}^2,0)$ whose tangent cone is not $x=0$. Consider the differential form on $\mathcal{X}$ given by $\overline{\omega}=(\pi^*\omega)/\overline{x}$ (which is the dual form of the pullback of X to $\mathcal{X}$) and the strict transform $\overline{\Gamma}$ of Γ whose intersection with $\pi^{-1}(0,0)$ is P. If n is the multiplicity of Γ at $(0,0)$, then*

$$v_\Gamma(\omega)=v_{\overline{\Gamma}}(\overline{\omega})+n.$$

Corollary 3.10 *Let Γ be an analytic branch at $(\mathbb{C}^2,0)$ that is not invariant by a singular analytic vector field $X=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$. Let $\omega=-B(x,y)dx+A(x,y)dy$ be the "dual" differential form of X. Then*

$$v_\Gamma(\omega)=n_{N-1}+\sum_{j=0}^{N-1}n_j,$$

where $n_0,n_1,\dots,n_N$ is the sequence of multiplicities of Γ at the points of the path it shares with X.

Notice that X (and hence ω) is only relevant to the formula, because its pullback must be nonsingular at P_N. In this sense, the formula is "independent" of X; it only depends on Γ and N.

Proof Let $(P_j)_{j=0}^N$ be the path shared by X and Γ and n_j the multiplicity of the strict transform $\overline{\Gamma}_j$ of Γ at P_j. For each $j = 0, \dots, N$, if we denote by $\overline{X}_j$ the pullback of X to the respective space (so that $\overline{X}_0 = X$) and $\overline{\omega}_j$ its dual form. We have

$$v_{\overline{\Gamma}_j}(\overline{\omega}_j) = v_{\overline{\Gamma}_{j-1}}(\overline{\omega}_{j-1}) - n_{j-1}, \text{ for } j = 1, \dots, N-1,$$

by Lemma 3.9. We can assume that $\overline{\Gamma}_{N-1} = (t^{n_{N-1}}, t^q + \text{h.o.t.})$ with $q > n_{N-1}$. As $\overline{X}_{N-1}$ does not preserve the tangent cone of $\overline{\Gamma}_{N-1}$ (because P_N is not a singular point of $\overline{X}_N$), the form $\overline{\omega}_{N-1}$ can be written as

$$\overline{\omega}_{N-1} = (ax + by + \text{h.o.t.})dx + (cx + dy + \text{h.o.t.})dy$$

with $a \neq 0$, which gives $v_{\Gamma_{N-1}}(\overline{\omega}_{N-1}) = 2n_{N-1}$ and, from Lemma 3.9, we get

$$v_{\overline{\Gamma}_N}(\overline{\omega}_N) = n_{N-1},$$

and then

$$n_{N-1} = v_{\overline{\Gamma}_N}(\overline{\omega}_N) = v_\Gamma(\omega) - n_0 - n_1 - \cdots - n_{N-1},$$

and the result follows. ∎

3.2 The Shared Path and Puiseux's Expansion

The concept of the path shared by a singular vector field and an analytic branch is deeply related to the Puiseux expansion of the branch and the contact between the branch and the vector field (or the branch and its deformation).

Start with an analytic branch Γ at $(\mathbb{C}^2, 0)$ that is not tangent to the OY axis, so that it admits a Puiseux expansion[1] of the form

$$\Gamma \equiv \varphi(t) = \Big(t^n, \sum_{i \geq n} a_i t^i\Big),$$

where Γ is regular if and only if $n = 1$. Let X be a singular vector field at $(\mathbb{C}^2, 0)$. We need a technical result.

Lemma 3.11 *Let $a(z) = \sum_{i\geq n} a_i(z) t^i$ be a power series with integer exponents such that $n \geq 1$ and each $a_i(z)$ is a holomorphic function in z (each with its own radius of convergence), with $a_n(0) \neq 0$. If $r(z) = \sum_{i\geq 1} r_i(z) t^i$ is such that $r(z)^n = a(z)$, then $r_i(z)$ are also holomorphic functions in z and $r_1(0) \neq 0$.*

Proof The proof is done by induction on i or, what amounts to the same, by the method of indeterminate coefficients. Actually, one can prove that there exist polynomials $P_j(z)$ in $j-1$ variables such that

$$r(z)^n = \Big(r_1(z)^n t^n + \sum_{j=2}^{\infty} \big(n r_1(z)^{n-1} r_j(z) + P_j(r_1(z), \dots, r_{j-1}(z))\big)\, t^{n+j-1}\Big)$$

from which the result follows. ∎

[1] We always assume the parametrizations to be irreducible.

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Definition 3.12 Assume that X is a singular vector field. Let P and Q be the points defined in the divisor of the blow-up of the origin by the tangent cone of Γ and $x = 0$ respectively. We say that X is *prepared* relative to Γ if either P or Q is a singular point of $\overline{X}$.

Remark 3.13 Given a vector field X we can assume that it is prepared up to a linear change of coordinates that preserves the tangent cone of Γ at $(0,0)$, which is assumed to be $y = 0$. The preparation guarantees that no curve of the form Γ_ϵ has the OY axis as its tangent cone. Given a vector field $X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$, it is prepared relative to Γ if and only if $\frac{\partial a}{\partial y}(0,0) = 0$ or $\frac{\partial b}{\partial x}(0,0) = 0$. The transform $\overline{X}$ of a singular vector field X is of the form

$$\overline{X} = xA(x,y)\frac{\partial}{\partial x} + \left(\mu_0 + \mu_1 y + \mu_2 y^2 + xB(x,y)\right)\frac{\partial}{\partial y},$$

and X is prepared if and only if $\mu_0 = 0$ or $\mu_2 = 0$.

Consider the deformation Γ_ϵ of Γ by X.

Proposition 3.14 *Assume that X is prepared relative to Γ. The deformation Γ_ϵ admits an irreducible Puiseux parametrization,*

$$\Gamma_\epsilon \equiv \varphi_\epsilon(t) = \left(t^n, \sum_{i\ge n} \widetilde{a}_i(\epsilon)t^i\right).$$

with $\widetilde{a}_i(\epsilon)$ being an entire function in ϵ and $\widetilde{a}_i(0) = a_i$ for all $i \ge n$.

Proof Let $\{\psi_s\}_{s\in\mathbb{C}}$ the one-parameter group associated with X. Let $(\gamma_1(t), \gamma_2(t)) = (t^n, \sum_{i\ge n} a_i t^i)$ be a Puiseux parametrization. We define the map

$$\left(\gamma_1(s,t), \gamma_2(s,t)\right) = \psi_s\left(\gamma_1(t), \gamma_2(t)\right).$$

It is well defined and holomorphic in a neighborhood of $t = 0$. As a consequence, $\gamma_1(s,t)$ and $\gamma_2(s,t)$ are of the form $\gamma_1(s,t) = \sum_{j=1}^{\infty} b_j(s)t^j$ and $\gamma_2(s,t) = \sum_{j=1}^{\infty} c_j(s)t^j$, where b_j and c_j are entire functions for any $j \ge 1$. Since the multiplicity of every curve Γ_s is equal to n for $s \in \mathbb{C}$, all coefficients b_j and c_j with $j < n$ are identically 0. Moreover, b_n is a nowhere vanishing entire function, otherwise the tangent cone of Γ_s would be $x = 0$ for some $s \in \mathbb{C}$. Lemma 3.11 implies that there exists $\beta(s,t) = \sum_{j=1}^{\infty} \widetilde{b}_j(s)t^j$ such that $\beta(s,t)^n = \gamma_1(s,t)$, where $\widetilde{b}_j$ is an entire function for any $j \ge 1$ and $\widetilde{b}_1$ is nowhere vanishing. Denote by $(s, \alpha(s,t))$ the inverse map of $(s, \beta(s,t))$. It is well-defined in a neighborhood of $t = 0$ since $\widetilde{b}_1$ is nowhere vanishing. The parametrization that we are looking for is $(t^n, \gamma_2(s, \alpha(s,t)))$. ∎

As a corollary we obtain the following result.

Corollary 3.15 *For ϵ small enough, the deformation Γ_ϵ admits an irreducible Puiseux parametrization:*

$$\Gamma_\epsilon \equiv \varphi_\epsilon(t) = \left(t^n, \sum_{i\ge n} \widetilde{a}_i(\epsilon)t^i\right).$$

with $\widetilde{a}_i(\epsilon)$ holomorphic in ϵ and $\widetilde{a}_i(0) = a_i$ for all $i \ge n$.

The algebraic counterpart to the geometric concept of the shared path is the *contact exponent*.

Definition 3.16 The *contact exponent* of a singular vector field X with an analytic branch Γ at a point P of a complex analytic surface $\mathcal{M}$, denoted by $(X,\Gamma)_P$, is the least i such that $\widetilde{a}_i(\epsilon)$ is not constant in Proposition 3.14 (for any irreducible Puiseux parametrization of Γ).

Remark 3.17 The contact exponent is independent of the choice of coordinates. Consider a local biholomorphism $\phi \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ such that the linear part $D_0\phi$ at the origin does not send the tangent line to Γ at 0 to the OY axis. It can be shown that $(X,\Gamma)_{(0,0)} = (\phi_* X, \phi(\Gamma))_{(0,0)}$ by a simple calculation.

Remark 3.18 Assume Γ has an irreducible Puiseux expansion $(t^n, at^m + \text{h.o.t.})$ with $a \neq 0$ and $n < m$. If $j = (X,\Gamma)_{(0,0)} < m$, then j must be a multiple of n; otherwise, the topological types of Γ and Γ_ϵ would be different, which is impossible, because they are analytically equivalent.

One has a formula analogous to that of Lemma 3.9, which provides the relation between the shared path and the contact exponent.

Lemma 3.19 *Assume Γ is not invariant for X and let $\pi\colon \mathcal{X} \to (\mathbb{C}^2,0)$ be the blow-up with centre $(0,0)$ and $\overline{\Gamma}$ the strict transform of Γ by π, which meets $\pi^{-1}(0,0)$ at P. Let $\overline{X}$ be the pullback of X to $\mathcal{X}$. Let n be the multiplicity of Γ at $(0,0)$ and $\overline{n}$ that of $\overline{\Gamma}$ at P. Then*

- *either $\overline{X}$ is non-singular at P and $(X,\Gamma)_{(0,0)} = n$,*
- *or $\overline{X}$ is singular at P and*

$$(X,\Gamma)_{(0,0)} = (\overline{X},\overline{\Gamma})_P + \overline{n}.$$

Proof If $\overline{X}$ is non-singular at P, the result is straightforward as X does not fix the tangent cone of Γ. Assume, then, that P is singular for $\overline{X}$.

Take a prepared irreducible Puiseux parametrization of Γ_ϵ,

$$\Gamma_\epsilon \equiv \varphi_\epsilon(t) = \Big(t^n, \sum_{m\le i<j} a_i t^i + \alpha_j(\epsilon)t^j + \text{h.o.t.}\Big) \tag{3.2}$$

with $j = (X,\Gamma)_{(0,0)}$, as in Proposition 3.14. Let $\pi\colon \mathcal{X} \to (\mathbb{C}^2,0)$ be the blow-up with centre $(0,0)$ with equations $x = \overline{x}$, $y = \overline{x}\overline{y}$, for which $\overline{\Gamma}$ meets $\pi^{-1}(0,0)$ at $\overline{y} = 0$. There are two cases.

- If $m \ge 2n$, the curve $\overline{\Gamma}_\epsilon$ has the same Puiseux parametrization as (3.2) except that the $\overline{y}$–coordinate has all the exponents subtracted by n. The multiplicity of Γ_ϵ (and hence $\overline{\Gamma}$) is n, and the result follows.
- If $n < m < 2n$, then, $a_m \neq 0$ and, by Remark 3.18, we have $j \ge m$ (otherwise, $j = n$ and $\overline{X}$ would not be singular at P), and we can write

$$\overline{\Gamma}_\epsilon \equiv \overline{\varphi}_\epsilon(t) = \Big(t^n, \sum_{m\le i<j} a_i t^{i-n} + a_j(\epsilon)t^{j-n} + \text{h.o.t.}\Big)$$

(with either $j > m$ and $a_m \neq 0$ or $j = m$ and $a_j(0) \neq 0$), which is not of irreducible Puiseux type (as $m - n < n$). In order to transform it to an irreducible Puiseux parametrization, one needs to extract $m - n$–th roots of the second coordinate:

$$u = \sqrt[m-n]{\sum_{i=m}^{\infty} a_i(\epsilon) t^{i-n}}.$$

Notice that such a root is a holomorphic function defined in a neighborhood of $(\epsilon, t) = (0,0)$ since $a_m(0) \neq 0$. We obtain

$$t = \sum_{1 \le i < j-m+1} \alpha_i u^i + \alpha_j(\epsilon) u^{j-m+1} + \text{h.o.t.},$$

where $\alpha_j(0) \neq 0$ if $j = m$. From this, an irreducible Puiseux parametrization of $\overline{\Gamma}_\epsilon$ is given by

$$\begin{aligned} \overline{\Gamma}_\epsilon &\equiv \overline{\varphi}_\epsilon(u) \\ &= \Big(\sum_{n \le i < j-(m-n)} \overline{a}_i u^i + \overline{a}_j(\epsilon) u^{j-(m-n)} + \text{h.o.t.}, u^{m-n} \Big). \end{aligned}$$

with $\overline{a}_j(\epsilon)$ not constant. Hence, $(X,\Gamma)_{(0,0)} = (\overline{X}, \overline{\Gamma})_P + m - n$, and $\overline{n}$ is, in this case, $m - n$, which finishes the proof. ∎

Lemma 3.19 states that if $P_0 = (0,0), P_1, \dots, P_N$ is the shared path between X and Γ, then

$$(X,\Gamma)_{(0,0)} = \begin{cases} \nu_0(\Gamma) & \text{if } N = 1, \\ \nu_{P_1}(\Gamma_1) + (\overline{X}_1, \Gamma_1)_{P_1} & \text{otherwise,} \end{cases}$$

where $\overline{X}_1$ and Γ_1 are, respectively, the pullback of X and the strict transform of Γ at P_1.

Corollary 3.20 *Let $P_0, \dots, P_N$ be the shared path between X and Γ. Then*

$$(X,\Gamma)_{(0,0)} = n_{N-1} + \sum_{j=1}^{N-1} n_j,$$

where $n_0, n_1, \dots, n_N$ is the sequence of multiplicities of Γ at the points of the path it shares with X. So that the contact order between X and Γ depends only on Γ and N.

Furthermore, the contact order of a vector field X with a branch Γ is essentially that of the dual differential form with Γ.

Theorem 3.21 *Let $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ be a singular vector field at $(\mathbb{C}^2, 0)$ and Γ be an analytic branch at $(\mathbb{C}^2, 0)$ with multiplicity n, not invariant for X. Let $\omega = -B(x,y)dx + A(x,y)dy$ be the "dual" differential form of X. Then*

$$\nu_\Gamma(\omega) = (X,\Gamma)_{(0,0)} + n.$$

The result is an immediate consequence of Corollaries 3.10 and 3.20.

Proposition 3.26 will essentially provide the analytic classification of plane branches except for Zariski's invariant, which requires a specific definition. Consider an analytic branch Γ having irreducible Puiseux expansion

$$\Gamma \equiv \varphi(t) = (x(t), y(t)) = \Big(t^n, \sum_{i \geq n} a_i t^i\Big)$$

and let X be a singular vector field at $(\mathbb{C}^2, 0)$ such that $(X, \Gamma)_{(0,0)} = j$. Now fix a chain of blow-ups

$$\mathcal{X} = \mathcal{X}_N \xrightarrow{\pi_{N-1}} \mathcal{X}_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} \mathcal{X}_1 \xrightarrow{\pi_0} (\mathbb{C}^2, 0)$$

with each π_i having centre P_i belonging to $E_i = \pi_{i-1}^{-1}(P_{i-1})$, the exceptional divisor corresponding to the blow-up of P_{i-1}. We call $\overline{X}_i$ the pullback of X to $\mathcal{X}_i$, which we assume is singular at P_i for $i = 0, \dots, N-1$ (writing $\overline{X}_0 = X$ and $P_0 = P$), and we assume P_N is a non-singular point of $\overline{X}_N$ in $E_N = \pi_{N-1}^{-1}(P_{N-1})$. We know that all the exceptional divisors $E_1, \dots, E_N$ are invariant for X_N. We let $\pi = \pi_0 \circ \cdots \circ \pi_{N-1}$. We require some lemmas. The next result can be seen as a corollary of the Poincaré–Hopf formula.

Lemma 3.22 *Let X be a singular analytic vector field at $(\mathbb{C}^2, 0)$ and let $\pi \colon \mathcal{X} \to (\mathbb{C}^2, 0)$ be a finite sequence of blow-ups whose centres are singular points for each pullback of X. If E is an irreducible component of the exceptional divisor in $\pi^{-1}(0,0)$ that is not composed of singular points of the pullback $\overline{X}$ of X by π, then there are exactly two singular points for $\overline{X}$ in E counting multiplicities.*

Lemma 3.23 *If the pullback $\overline{X}$ of X to $\mathcal{X}$ has a single singularity in E_N, then $\overline{X}_{|_{E_N}}$ is analytically conjugated to $\partial/\partial z$ in the chart not containing that singularity.*

Proof The vector field $\overline{X}_{|_{E_N}}$ has a singular point with multiplicity 2, by Lemma 3.22. It is analytically conjugated to $\partial/\partial z$, where z is a complex coordinate in the chart $\mathbb{P}^1\mathbb{C} \setminus \{\infty\}$ of $\mathbb{P}^1\mathbb{C}$. ■

Definition 3.24 A divisor E_i is *free* if either $i = 1$ or $P_{i-1} \in E_k$ implies $k = i - 1$. In other words, if E_i meets only one other exceptional divisor in $\mathcal{X}_i$, or what amounts to the same, if P_{i-1} is not the intersection of two exceptional divisors.

3.3 Action of a Flow on a Puiseux Parametrization

We can calculate the general form of the term of the Puiseux parametrization associated with the contact exponent. As a consequence, we can deduce that the Puiseux expansion of a singular branch may be simplified by "removing" coefficients by means of a holomorphic flow, which parallels Zariski's approach to the moduli problem [19].

Lemma 3.25 *With the setting above, assume that $N > 1$ (or $(X, \Gamma)_{(0,0)} > n$, which is the same thing). Let $\widetilde{\Gamma}$ be another singular branch at $(\mathbb{C}^2, 0)$, topologically equivalent to Γ, admitting a parametrization*

$$\widetilde{\Gamma} \equiv \widetilde{\varphi}(t) = (\widetilde{x}(t), \widetilde{y}(t)) = \Big(t^n, \sum_{i \geq n} \widetilde{a}_i t^i\Big).$$

Let $(P_i)_{i=0}^{N}$, $(\widetilde{P}_i)_{i=0}^{\widetilde{N}}$ be the paths shared by X and Γ, $\widetilde{\Gamma}$, respectively. Then there is an n-th root of unity $\xi \in \mathbb{C}$ with $\widetilde{a}_i = a_i \xi^i$ for all $n \le i < j = (X,\Gamma)_{(0,0)}$ (resp. $n \le i \le j$) if and only if $N = \widetilde{N}$ and $P_k = \widetilde{P}_k$ for $k = 0,\dots,N-1$ (resp. $k = 0,\dots,N$).

Proof The intersection multiplicity $(\Gamma,\widetilde{\Gamma})_{(0,0)}$ is given by Halphen's formula (*cf.* [10, Proposition 3.10 of Chapter 1]):

$$(\Gamma,\widetilde{\Gamma})_{(0,0)} = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \operatorname{ord}_t \left(y(e^{\frac{2\pi i j}{n}} t) - \widetilde{y}(e^{\frac{2\pi i k}{n}} t) \right).$$

We define

$$\Lambda(j) = \frac{1}{n}\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} \min\left(\operatorname{ord}_t (y(e^{\frac{2\pi i l}{n}} t) - y(e^{\frac{2\pi i k}{n}} t)), j \right).$$

For $\Gamma \ne \widetilde{\Gamma}$ let $j = j(\Gamma,\widetilde{\Gamma})$ be the natural number such that there exists $\xi \in \mathbb{C}$ with $\xi^n = 1$ satisfying $\widetilde{a}_l = a_l \xi^l$ for all $n \le l < j$, but no such ξ satisfies $\widetilde{a}_l = a_l \xi^l$ for all $n \le l \le j$. In such a case, Halphen's formula provides $(\Gamma,\widetilde{\Gamma})_{(0,0)} = \Lambda(j)$. Notice that $\Lambda(j)$ is a strictly increasing function of j, since for instance, $\operatorname{ord}_t(y(t) - y(t)) = \infty$. The intersection multiplicity $(\Gamma,\widetilde{\Gamma})_{(0,0)}$ is also a strictly increasing function of the number of common infinitely near points of Γ and $\widetilde{\Gamma}$ by Corollary 3.3.

Let $\{\psi_s\}_{s\in\mathbb{C}}$ be the one-parameter family associated with X. We let $\Gamma_s = \psi_s(\Gamma)$. Consider $s_0 \in \mathbb{C}^*$ close to 0. By construction $(\Gamma,\Gamma_{s_0})_{(0,0)}$ is equal to $\Lambda((X,\Gamma)_{(0,0)})$. Notice that exactly the first N infinitely near points of Γ and Γ_{s_0} coincide by Proposition 3.1. Therefore, the previous discussion implies $(\Gamma,\widetilde{\Gamma})_{(0,0)} = \Lambda((X,\Gamma)_{(0,0)})$ is equivalent to $N = \widetilde{N}$, $P_k = \widetilde{P}_k$ for $k = 0,\dots,N-1$ and $P_N \ne \widetilde{P}_N$ and also equivalent to $j(\Gamma,\widetilde{\Gamma}) = (X,\Gamma)_{(0,0)}$. Moreover, $(\Gamma,\widetilde{\Gamma})_{(0,0)} > \Lambda((X,\Gamma)_{(0,0)})$ is equivalent to $N = \widetilde{N}$, $P_k = \widetilde{P}_k$ for $k = 0,\dots,N$ and also equivalent to $j(\Gamma,\widetilde{\Gamma}) > (X,\Gamma)_{(0,0)}$. ∎

Next, we provide the general expression of the j-th term of the Puiseux parametrization of a curve Γ under the action of a local vector field X such that $(X,\Gamma)_{(0,0)} = j$. Such a term can be eliminated from a Puiseux parametrization of Γ (meaning that it is zero for one of the images of X by a biholomorphism in the one parameter flow of X) as long as X has a single singularity on the last divisor E_N of the shared path. It can never be removed if X has two singularities and E_N is not free.

Proposition 3.26 *With the same setting, let $\{\psi_s\}_{s\in\mathbb{C}}$ be the one parameter group associated with X. We let $\Gamma_s = \psi_s(\Gamma)$ and $j = (X,\Gamma)_{(0,0)}$. Then we have*

$$\Gamma_s \equiv \varphi_s(t) = (x_s(t), y_s(t)) = \left(t^n, \sum_{i\ge n} \widetilde{a}_i(s) t^i \right)$$

with $\widetilde{a}_i \equiv a_i$ for $n \le i < j$, and $\widetilde{a}_j$ is a locally injective function. Moreover $\widetilde{a}_j(s)$ is of the form $\lambda s + \mu$ ($\lambda \in \mathbb{C}^*, \mu \in \mathbb{C}$) if $\overline{X}$ has a single singularity in E_N, and of the form $\lambda e^{\gamma s} + \mu$ ($\lambda,\gamma \in \mathbb{C}^*, \mu \in \mathbb{C}$), otherwise. In particular, in the former case, there exists a unique $s_0 \in \mathbb{C}$ such that $\widetilde{a}_j(s_0) = 0$. In the latter case, we get $\mu = 0$ if E_N is non-free.

Proof By definition, $\widetilde{a}_j(s)$ is not a constant function. Let

$$R = \{\lambda^j : \lambda^n = 1 \text{ and } \lambda^k = 1 \text{ for any } n \le k < j \text{ such that } a_k \ne 0\}.$$

Notice that $R = \{1\}$, unless a_j is the coefficient associated with a Puiseux characteristic exponent. Since all the curves Γ_s are topologically conjugated, we deduce that $R \neq \{1\}$ implies that $\widetilde{a}_j$ is a nowhere vanishing function.

Consider $s, s' \in \mathbb{C}$. The N-th infinitely near points of Γ_s and $\Gamma_{s'}$ are equal if and only if $\frac{\widetilde{a}_j(s)}{\widetilde{a}_j(s')} \in R$, by Lemma 3.25 (the condition $\frac{\widetilde{a}_j(s)}{\widetilde{a}_j(s')} \in R$ should be interpreted as $\widetilde{a}_j(s) = \widetilde{a}_j(s')$ if $R = \{1\}$). Since $\overline{X}$ is regular at the point of E_N in the strict transform of Γ_s for any $s \in \mathbb{C}$, it follows that $\widetilde{a}_j$ is locally injective.

Assume that $\overline{X}$ has a single singularity at E_N. By Lemma 3.23, $\overline{X}_{|E_N}$ is globally analytically conjugated to $\partial/\partial z$ away from that singularity. Since the one-parameter flow $\{\eta_s\}_{s\in\mathbb{C}}$ of $\partial/\partial z$ satisfies $\eta_s(z) = s + z$, the map $s \mapsto \eta_s(z_0)$ is injective for every choice of $z_0 \in \mathbb{C}$. Hence, the Nth-infinitely near points of Γ_s and $\Gamma_{s'}$ are different if $s, s' \in \mathbb{C}$ and $s \neq s'$. Therefore, $\widetilde{a}_j$ is an injective function. It is well known that an injective holomorphic function of $\mathbb{C}$ is linear.

Suppose that $\overline{X}$ has two singularities at E_N. If we place those singularities at 0 and ∞ for some coordinate z in the Riemann sphere, the vector field $\overline{X}$ is of the form $\alpha z \frac{\partial}{\partial z}$ for some $\alpha \in \mathbb{C}^*$. Since the flow in time s of $\overline{X}$ is $e^{s\alpha} z$, we deduce that $\widetilde{a}_j(z + \frac{2\pi i}{\alpha})/\widetilde{a}_j(z) \equiv \beta \in R$ for any $z \in \mathbb{C}$ (or $\widetilde{a}_j(z + \frac{2\pi i}{\alpha}) = \widetilde{a}_j(z)$ if $R = \{1\}$). If β has order k, we obtain

$$\widetilde{a}_j(z + w) = \widetilde{a}_j(z) \Longleftrightarrow \exists l \in \mathbb{Z} \text{ such that } w = lk\frac{2\pi i}{\alpha}.$$

Hence, the function $\widehat{a}_j := \widetilde{a}_j \circ \frac{k \ln s}{\alpha}$ is an injective function of $\mathbb{C}^*$. Such a function has to be equal to either $\lambda s + \mu$ or $\frac{\lambda}{s} + \mu$, where $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. Therefore $\widetilde{a}_j$ is of the form $\lambda e^{\frac{\alpha s}{k}} + \mu$ or $\lambda e^{\frac{-\alpha s}{k}} + \mu$.

Suppose that E_N is non-free and $\overline{X}$ has two singularities in E_N that are necessarily its corners. Since the strict transform of Γ intersects E_N in a non-corner point, t^j is a monomial of $y(t)$ corresponding to a Puiseux characteristic exponent and cannot be erased by any ψ_s. Thus, $\widetilde{a}_j$ is nowhere vanishing, and hence $\mu = 0$. ∎

Remark 3.27 Consider the vector fields $Y = \lambda \frac{\partial}{\partial z}$ and $Z = \gamma(z - \mu)\frac{\partial}{\partial z}$ whose flows we denote by $\{\psi_s\}_{s\in\mathbb{C}}$ and $\{\eta_s\}_{s\in\mathbb{C}}$, respectively. We have $\lambda s + \mu = \psi_s(\mu)$ and $\lambda e^{\gamma s} + \mu = \eta_s(\lambda + \mu)$ for any $s \in \mathbb{C}$. Thus the values of $a_k(s)$ are obtained through the action of a holomorphic vector field in the Riemann sphere with one or two singular points, respectively.

The following technical results will be useful later on.

Lemma 3.28 *If Y is a nilpotent singular vector field at $(\mathbb{C}^2, 0)$ admitting two transverse non-singular invariant curves, then its multiplicity is strictly greater than* 1.

Proof Since Y has two transverse non-singular invariant curves, its linear part must be diagonalisable. Since this linear part is nilpotent by hypothesis, it must be zero; *i.e.*, Y has multiplicity at least 2. ∎

As a consequence, nilpotent vector fields become regular only at free divisors.

Lemma 3.29 *With the above notation, assume X is nilpotent. Then E_N (the divisor containing P_N, point at which $\overline{X}_N$ is regular) is a free divisor. Moreover, there is only one singularity of $\overline{X}_N$ in E_N and if $N > 1$, it is the intersection of E_N with the only other divisor it meets (actually, E_{N-1}).*

Proof If $N \leq 2$, then E_N is automatically free, and the statement holds. Assume then that $N > 2$. A simple computation shows that X_i has nilpotent linear part at P_i for all $i = 1, \dots, N-1$. If E_N were not free, then P_{N-1} would belong to E_{N-1} and another E_k for $k \neq N-1$. As all the exceptional divisors are invariant, P_{N-1} would be a singular point for $\overline{X}_{N-1}$ with two transverse non-singular invariant curves. We know that $\overline{X}_{N-1}$ has nilpotent linear part, hence X_{N-1} would have multiplicity at least 2, by Lemma 3.28. This prevents P_N from being regular for $\overline{X}_N$, as the multiplicity of a singular vector field decreases at most by one after a single blow-up.

The existence of a single singularity in E_N is a consequence of the existence of a single eigenvector for the linear part of $\overline{X}_{N-1}$ at P_{N-1}. If $N > 1$, then the intersection of E_N with the only other divisor it meets (which is, of necessity, E_{N-1}) must be a singular point for $\overline{X}_N$, as there are two invariant varieties passing through that point. ∎

A straightforward application of Lemma 3.29 gives the following proposition.

Proposition 3.30 *Let Γ be a branch through $(\mathbb{C}^2, 0)$ and X let be a nilpotent singular vector field at $(\mathbb{C}^2, 0)$ for which Γ is not invariant. Assume $(P_i)_{i=0}^N$ is the path shared by Γ and X. Then P_N is a non-singular point of E_N, $\overline{X}_N$ has a single singular point in E_N and if $N > 1$, then this singular point is $E_N \cap E_{N-1}$.*

Proof of Theorem 1.1 The formula for $k = (X, \Gamma)_{(0,0)}$ is a consequence of Corollary 3.8 and Theorem 3.21. The properties of a_k are a consequence of Proposition 3.26. If X is nilpotent, Propositions 3.30 and 3.26 imply that a_k is linear. ∎

Remark 3.31 Let Γ be an analytic branch with irreducible Puiseux parametrization

$$\Gamma \equiv \varphi(t) = (x(t), y(t)) = \Big(t^n, \sum_{i \geq n} a_i t^i\Big).$$

We can assume $a_n = 0$ up to replacing Γ with $\exp(-a_n x \frac{\partial}{\partial y})(\Gamma)$. Furthermore, since $(\Gamma, x^k)_{(0,0)} = (x^k \partial/\partial y, \Gamma)_{(0,0)} = kn$ for $k \geq 1$, we can also assume that the Puiseux parametrization of Γ is prepared by conjugating it with holomorphic diffeomorphisms embedded in the one-parameter groups of the nilpotent vector fields $x^2 \frac{\partial}{\partial y}, x^3 \frac{\partial}{\partial y}, \dots,$ by Propositions 3.26 and 3.30. So in order to transform Γ to normal form by using biholomorphisms embedded in the flows of nilpotent vector fields, we can assume that the Puiseux parametrization is prepared.

Hence, any exponent corresponding to the contact with a nilpotent vector field can be removed from the Puiseux expansion *by means of a flow*; or what is the same, the infinitely near point of Γ at the corresponding exceptional divisor can be "moved to the origin" of that divisor by a flow.

Corollary 3.32 *Let Γ be an analytic branch with prepared irreducible Puiseux parametrization*

$$\Gamma \equiv \varphi(t) = (x(t), y(t)) = \Big(t^n, \sum_{i\geq m} a_i t^i\Big)$$

and let X be a nilpotent singular analytic vector field at $(\mathbb{C}^2, 0)$. Assume the contact exponent $j = (X, \Gamma)_{(0,0)}$ between X and Γ is greater than m. Then Γ is analytically equivalent via a biholomorphism in the holomorphic flow associated with X to a branch $\widetilde{\Gamma}$ with parametrization

$$\widetilde{\Gamma} \equiv \widetilde{\varphi}(t) = (\widetilde{x}(t), \widetilde{y}(t)) = \Big(t^n, \sum_{i\geq m} \widetilde{a}_i t^i\Big)$$

with $\widetilde{a}_i = a_i$ for $i < j$ and $\widetilde{a}_j = 0$.

Any exponent corresponding to the semigroup of Γ is the contact with a nilpotent vector field and, hence, can be removed by a flow.

Corollary 3.33 *Let Γ and $\varphi(t)$ be as in Corollary 3.32. If $j > m$ is the intersection multiplicity of a singular analytic curve Δ with Γ, then there exists a singular vector field X, with vanishing linear part, such that $(X, \Gamma)_{(0,0)} = j$. In particular, the same conclusion as in Proposition 3.26 holds (*i.e.*, the term a_j can be eliminated from the parametrization φ without affecting the previous ones).*

Proof Consider $f \in \mathbb{C}\{x, y\}$ such that $j = (\Gamma, f)_{(0,0)}$. We obtain that the multiplicity at the origin is greater than 1, since otherwise $(\Gamma, f)_{(0,0)} \leq m$. The vector field $X = f(x, y)\frac{\partial}{\partial y}$ has vanishing linear part at $(0, 0)$. By Theorem 3.21, this X has contact exponent $(X, \Gamma)_{(0,0)} = j$. Applying Corollary 3.32, we are done. ∎

From all the previous discussions we know that a finite composition of local holomorphic diffeomorphisms embedded in flows (including a linear one, intended to make the tangent cone of Γ at $(0, 0)$ different from the OX axis), sends Γ to a curve that has a prepared Puiseux expansion of the form

$$\varphi(t) = (x(t), y(t)) = \Big(t^n, \sum_{i\geq m} a_i t^i\Big),$$

where $n < m$, $n \nmid m$, $a_m \neq 0$, and if i is in the semigroup associated with Γ and $m < i \leq c$, where c is the conductor of Γ, then $a_i = 0$, and we also have $a_i = 0$ if $i \leq c$ is the contact exponent with a nilpotent vector field.

Proposition 3.34 *Under the conditions of the last paragraph, let λ be the least exponent $\lambda > m$ such that $a_\lambda \neq 0$ and c the conductor of Γ. Assume $\lambda < c$. Let X be a non-nilpotent singular vector field. Then $(X, \Gamma)_{(0,0)} \leq \lambda$. Moreover, $(X, \Gamma)_{(0,0)} < \lambda$ implies that $(X, \Gamma)_{(0,0)}$ is of the form $(pn + qm) - n$, where $p \geq 0$, $q \geq 0$ and $m - n \neq (pn + qm) - n \geq n$.*

Proof Let $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$. Write

$$A(x, y) = a_{10}x + a_{01}y + \overline{A}(x, y) \quad \text{and} \quad B(x, y) = b_{10}x + b_{01}y + \overline{B}(x, y).$$

Consider $\varphi^*\omega$, where ω is the dual form $\omega = -B(x,y)dx + A(x,y)dy$:

$$\begin{aligned}\varphi^*\omega = -(b_{10}t^n + b_{01}(a_m t^m + a_\lambda t^\lambda + \text{h.o.t.}) + \overline{B}(\varphi(t)))nt^{n-1}dt \\ + (a_{10}t^n + a_{01}(a_m t^m + a_\lambda t^\lambda + \text{h.o.t.}) \\ + \overline{A}(\varphi(t)))(ma_m t^{m-1} + \lambda a_\lambda t^{\lambda-1} + \text{h.o.t.})dt.\end{aligned}$$

Let $j = (X,\Gamma)_{(0,0)}$, so that $j + n = \nu_\Gamma(\omega)$ by Theorem 3.21. We can assume that $j \notin \{n, m\}$, since $n = (2n + 0\cdot m) - n$ and $m = (n+m) - n$.

We have $b_{10} = 0$, since otherwise $\nu_\Gamma(\omega) = 2n$, and we would have $j = n$. The property $\nu_\Gamma(\omega) < n + m$ implies that $\nu_\Gamma(\omega)$ is a multiple of n. In particular, we get $\nu_\Gamma(\omega) \neq m$ and $j \neq m - n$. Moreover, j is a multiple of n greater or equal than n. So we can assume $\nu_\Gamma(\omega) \geq n + m$ from now on. Indeed, we obtain $\nu_\Gamma(\omega) > n + m$ and $j > m$, since $j \neq m$. This implies that $b_{01}n = a_{10}m$. Since X is non-nilpotent, it follows that $a_{10} \neq 0$ and $b_{01} \neq 0$. The pullback $\varphi^*\omega$ satisfies

$$t\varphi^*\omega = \big(g(t) + (a_{10}\lambda - b_{01}n)a_\lambda t^{n+\lambda} + O(t^{n+\lambda+1})\big)\,dt,$$

where $a_{10}\lambda - b_{01}n \neq 0$ and the exponents of all monomials with non-vanishing coefficients of the Taylor power series expansion of $g(t)$ belong to the semigroup

$$S' := \{pn + qm : p \geq 0,\ q \geq 0,\ p + q \geq 1\}.$$

We claim that $n + \lambda$ does not belong to S'. Otherwise, $n + \lambda = pn + qm$. If $p \geq 1$, then λ belongs to S, a contradiction. If $p = 0$, then $q \geq 2$ and λ is the contact order $(y^{q-1}\partial/\partial x, \Gamma)_{(0,0)}$ of Γ with a nilpotent vector field, again a contradiction.

Since $\lambda + n \notin S'$ and $a_{10}\lambda - b_{01}n \neq 0$, it follows that $m < j \leq \lambda$. Moreover, $j < \lambda$ implies $j \in S' - n$. ∎

Thus, we can transform the Puiseux expansion of Γ into one in which any exponent less than c, corresponding to the contact with a holomorphic flow is zero, except for λ, *by means of a composition of flows.*

Theorem 3.35 *Under the same conditions as above, if $j > \lambda$ is the contact exponent of Γ with an analytic vector field X, then the term of order j can be eliminated from a prepared Puiseux expansion via a biholomorphism in a nilpotent holomorphic flow.*

Proof Let X be a singular vector field such that $(X,\Gamma)_{(0,0)} = j$. Since $j > \lambda$, X is nilpotent by Proposition 3.34. Apply Corollary 3.32 to finish the proof. ∎

We end this section with a characterization of Zariski's λ invariant of a plane branch Γ in terms of tangency orders (or contact orders) of vector fields with Γ.

Theorem 3.36 (Zariski's λ invariant) *Under the conditions of Proposition 3.34, let λ be the least exponent $\lambda > m$ such that $a_\lambda \neq 0$ and c is the conductor of Γ. Then $\lambda + n = \upsilon_\Gamma(mydx - nxdy)$. Indeed if $\lambda < c$, then m and λ are the unique positive integers j such that j is the contact exponent of a singular vector field with Γ but is not the contact exponent of a nilpotent vector field with Γ. As a consequence, λ is an analytic invariant of Γ if $\lambda < c$.*

Proof Let $\lambda < \infty$ be as in the statement, and let $\omega = mydx - nxdy$. By direct substitution:

$$v_\Gamma(\omega) = \operatorname{ord}_t \Big(\Big(\sum_{i \ge m} mna_i t^{i+n-1} \Big) - \Big(\sum_{i \ge m} nia_i t^{i+n-1} \Big) \Big) + 1$$

so that

$$\begin{aligned} v_\Gamma(\omega) &= \Big(nx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}, \Gamma \Big)_{(0,0)} = \operatorname{ord}_t \big((mn - n\lambda) a_m t^{\lambda+n-1} + \text{h.o.t.} \big) + 1 \\ &= \lambda + n, \end{aligned}$$

which gives the first part of the statement. Moreover, m is also a contact exponent, since $m = (x\partial/\partial x, \Gamma)_{(0,0)}$. It cannot be expressed as a contact with a nilpotent vector field, since the coefficient of t^m in $y(t)$ can not be erased (Corollary 3.32).

Assume that there exists $j \notin \{m, \lambda\}$ satisfying the hypotheses. We obtain $j < \lambda$ and $j + n = (pn + qm) \ge 2n$ for some $p \ge 0$ and $q \ge 0$ with $(p,q) \notin \{(1,0),(0,1),(1,1)\}$ by Proposition 3.34. The vector field $x^{p-1}y^q\partial/\partial y$ is nilpotent if $p \ge 1$, since $(p,q) \ne (1,0)$ and $(p,q) \ne (1,1)$. It satisfies $(x^{p-1}y^q\partial/\partial y, \Gamma)_{(0,0)} = j$, and so we get a contradiction. Analogously, if $q \ge 1$ the vector field $x^p y^{q-1}\partial/\partial x$ is nilpotent and $(x^p y^{q-1}\partial/\partial x, \Gamma)_{(0,0)} = j$ holds, providing a contradiction. ■

4 Analytic Classes and Their Completeness

Now we focus on whether curves in the same class of analytic conjugacy of a given plane branch are conjugated by local biholomorphisms in a one-parameter group. Let us give some definitions.

Definition 4.1 We say that $\rho(t)$ (where $\rho(t)$ belongs to the maximal ideal $\mathfrak{m}_1$ of $\mathbb{C}[[t]]$) is a formal diffeomorphism if its linear part is non-vanishing (or in other words if $\rho(t) \in \mathfrak{m}_1 \setminus \mathfrak{m}_1^2$). If, in addition to the above properties, $\rho(t)$ belongs to $\mathbb{C}\{t\}$, then it is a local biholomorphism defined in the neighborhood of the origin by the inverse function theorem.

We say that $\psi(x,y) = (a(x,y), b(x,y)) \in \mathfrak{m} \times \mathfrak{m}$ is a formal diffeomorphism and write $\psi \in \widehat{\operatorname{Diff}}(\mathbb{C}^2, 0)$, if the linear part of ψ at the origin is a linear isomorphism.

Definition 4.2 The Krull topology for formal diffeomorphisms or vector fields is defined by considering them as n-tuples of formal power series and the induced product topology in $\mathbb{C}[[x,y]]^n$ (*cf.* Definition 2.1).

Definition 4.3 Let ψ be a formal diffeomorphism. We say that ψ is *unipotent* if its linear part is a unipotent linear map.

Definition 4.4 The time 1 flow $\exp(X)$ of a singular vector field X is

$$\exp(X)(x,y) = \Big(\sum_{j=0}^{\infty} \frac{X^j(x)}{j!}, \sum_{j=0}^{\infty} \frac{X^j(y)}{j!} \Big).$$

Given a formal vector field X, we use the above formula to define the formal diffeomorphism $\exp(X)$. The formula is well defined; indeed, if $(X_k)_{k\ge1}$ is a sequence of

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vector fields that converges to X in the $\mathfrak{m}$-adic topology, then $\exp(X_k)$ converges to $\exp(X)$ in the Krull topology when $k \to \infty$.

First let us provide an example of a complete class.

Proposition 4.5 *The class of analytic conjugacy of all smooth plane branches is complete.*

Proof Consider two smooth curves Γ, Γ'. Up to a change of coordinates, we can assume $\Gamma \equiv (y = 0)$ and that Γ' is not tangent to $x = 0$. Thus, Γ and Γ' admit Puiseux parametrizations $(t, 0)$ and $(t, a(t))$, respectively. As a consequence, the local biholomorphism $\psi(x, y) = (x, y + a(x))$ conjugates Γ and Γ'. Since $\psi = \exp(a(x)\partial/\partial y)$, we are done. ∎

The following theorem implies Theorem 1.2.

Theorem 4.6 *Let Γ and Γ' be two plane branches that are conjugated by the exponential $\exp(\widehat{X})$ of a singular formal vector field. Then they are conjugated by a local biholomorphism embedded in the flow of a singular holomorphic vector field X. Moreover, if $\widehat{X}$ is nilpotent we can take X to be nilpotent.*

Proof Assume $\Gamma \neq \Gamma'$, since the result is trivial otherwise. Up to a linear change of coordinates, we can assume that none of the tangent cones of the curves Γ and Γ' is the OY axis. The curves Γ and Γ' have Puiseux parametrizations $\alpha(t) = (t^n, a(t))$ and $\beta(t) = (t^n, b(t))$, respectively, where n is the common multiplicity at the origin. By hypothesis, there exists $\rho \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ such that $(\exp(\widehat{X}) \circ \alpha)(t) \equiv (\beta \circ \rho)(t)$.

Write $\widehat{X} = \widehat{A}(x, y)\frac{\partial}{\partial x} + \widehat{B}(x, y)\frac{\partial}{\partial y}$. Consider a sequence $(X_k)_{k\geq 1}$ of singular vector fields that converge to $\widehat{X}$ in the $\mathfrak{m}$-adic topology. For instance, this can be obtained by defining $X_k = A_k(x, y)\frac{\partial}{\partial x} + B_k(x, y)\frac{\partial}{\partial y}$, where A_k (resp. B_k) is the polynomial of degree less or equal than k such that $\widehat{A} - A_k \in \mathfrak{m}^{k+1}$ (resp. $\widehat{B} - B_k \in \mathfrak{m}^{k+1}$) for $k \geq 1$. Analogously, we choose a sequence $(\rho_k)_{k\geq 1}$ in $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}, 0)$ converging to ρ in the Krull topology. We define the curve Γ_k as $\exp(X_k)(\Gamma)$ and let $(x_k(t), y_k(t)) = (\exp(X_k) \circ \alpha \circ \rho_k^{-1})(t)$ for $k \geq 1$. The sequence $(x_k(t), y_k(t))_{k\geq 1}$ converges to $(t^n, b(t))$ in the Krull topology. Consider the holomorphic function $\sigma_k(t)$ such that $\sigma_k(t)^n \equiv x_k(t)$ and $(\sigma_k)'(0) = 1$ for $k \gg 1$. Since $(\sigma_k)'(0) \neq 0$, it is a local biholomorphism and its inverse σ_k^{-1} exists. The sequence $(\sigma_k)_{k\geq 1}$ converges to t in the Krull topology. Thus, $(t^n, b_k(t)) := (x_k, y_k) \circ \sigma_k^{-1}(t)$ is a parametrization of Γ_k that converges to $(t^n, b(t))$ in the Krull topology when $k \to \infty$.

Since Γ and Γ_k are conjugated by a local biholomorphism contained in a one-parameter flow, it suffices to show that for fixed $k \gg 1$, there exists a local biholomorphism $\theta \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ such that $\theta(\Gamma) = \Gamma$ and $\theta(\Gamma') = \Gamma_k$. Indeed, then $\exp(\theta^* X_k)(\Gamma) = \Gamma'$. Moreover, if $\widehat{X}$ is nilpotent, then X_k is nilpotent for any $k \geq 1$, since $\widehat{X}$ and X_k have the same linear part at $(0, 0)$, and $\theta^* X_k$ is nilpotent as a conjugate of X_k.

Thus, we need to prove that there exists θ with

$$(\theta \circ \alpha)(t) \equiv \alpha(t),$$
$$\theta(t^n, b(t)) \equiv (t^n, b_k(t)).$$

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Let $h(x,y)=0$ be a local (irreducible) equation of Γ and define

$$\theta(x,y)=\big(x,y+h(x,y)\gamma(x,y)\big).$$

If we prove that there exists a holomorphic function $\gamma(x,y)$ for which the conditions on θ are satisfied, we are done. The fact that $\theta(\alpha(t))\equiv\alpha(t)$ is obvious by construction. The other condition, $\theta(t^n,b(t))\equiv(t^n,b_k(t))$, is equivalent to

$$\gamma(t^n,b(t))\equiv\frac{b_k(t)-b(t)}{h(t^n,b(t))}. \tag{4.1}$$

The denominator $h(t^n,b(t))$ is not identically 0 since $\Gamma\neq\Gamma'$. The right-hand side of equation (4.1) converges to 0 in the Krull topology when $k\to\infty$. Thus there exists a solution of equation (4.1) for some $k>1$, which completes the proof. ∎

Corollary 4.7 *Let Γ,Γ' be two plane branches conjugated by a unipotent formal diffeomorphism $\psi\in\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$. Then Γ and Γ' are conjugated by a local biholomorphism embedded in a one-parameter group generated by a nilpotent vector field.*

We will use the following well-known result.

Remark 4.8 (*cf.* [9,13]) The exponential provides a bijection between the set of formal nilpotent singular vector fields and the set of unipotent formal diffeomorphisms. Moreover, it specializes to a bijection between the Lie algebra of formal vector fields with vanishing linear part at the origin and the group of formal diffeomorphisms with identity linear part.

Corollary 4.7 is an immediate consequence of Theorem 4.6 and Remark 4.8.

Corollary 4.9 *Let Γ and Γ' be two plane branches in the same class of analytic conjugacy. There exists a nilpotent vector field X and a linear vector field Y such that $(\exp(Y)\circ\exp(X))(\Gamma)=\Gamma'$.*

Proof Let $\psi\in\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ such that $\psi(\Gamma)=\Gamma'$. Then we have $\psi=L\circ\sigma$, where L is the linear part of ψ at the origin, and the linear part of σ at the origin is the identity. Let $\overline{\Gamma}=\sigma(\Gamma)$. We have $L(\overline{\Gamma})=\Gamma'$. There exists a nilpotent vector field X such that $\exp(X)(\Gamma)=\overline{\Gamma}$ by Corollary 4.7. Moreover, L is of the form $\exp(Y)$ for some linear vector field. Therefore, we obtain $(\exp(Y)\circ\exp(X))(\Gamma)=\Gamma'$. ∎

We have obtained the analytic reduction of holomorphic branches [11] to what Zariski calls "short parametrizations" [19, p. 19] (see Corollary 4.11, and the subsequent line).

Definition 4.10 Let Γ be a germ of plane branch. We denote by Λ the set of contact exponents between Γ and singular vector fields. Notice that $\Lambda+n$ is the set of orders of contact of Kähler differentials with Γ by Theorem 3.21.

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Corollary 4.11 *Let Γ be a branch at $(\mathbb{C}^2, 0)$ with prepared irreducible Puiseux parametrization*

$$\Gamma \equiv \varphi(t) = \left(t^n, \sum_{i \geq m}^{\infty} a_i t^i\right).$$

Let λ be its Zariski invariant (or $\lambda = \infty$) and let $c > \lambda$ be the conductor of the semigroup associated with Γ. There is a nilpotent singular vector field X such that

$$\exp(X)(\Gamma) \equiv \overline{\varphi}(t) = \left(t^n, a_m t^m + a_\lambda t^\lambda + \sum_{i > \lambda}^{c-1} \overline{a}_i t^i\right)$$

with $\overline{a}_i = 0$ for $i \in \Lambda$.

A parametrization as in Corollary 4.11 is called a *short parametrization*.

Proof In order to simplify Γ we remove step by step the coefficients of t^j in the second component of the Puiseux parametrization of Γ for $m < j < \lambda$ and $c \leq j$. The normalizing map is of the form $\exp(X_j)$, where $(X_j, \Gamma)_{(0,0)} = j$ and X_j is a nilpotent singular vector field by Corollary 3.33 and Theorem 3.36. Indeed, it is easy to see that we can assume that $X_j \to 0$ when $j \to \infty$ in the $\mathfrak{m}$-adic topology. Moreover the tangent cone of Γ defines a singular point P of $\overline{(X_j)}_1$ in the divisor E_1 of the blow-up of the origin for $j > n$, since otherwise $n < j = (X_j, \Gamma)_{(0,0)} = n$. We deduce that $\exp(X_j)$ is a unipotent biholomorphism whose linear part has matrix $\left(\begin{smallmatrix} 1 & c_j \\ 0 & 1 \end{smallmatrix}\right)$, where $c_j = 0$ if $j \gg 1$. The limit of the composition of these exponentials, in the appropriate order, is a well-defined formal unipotent diffeomorphism $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ conjugating Γ with a curve with Puiseux parametrization of the form $\overline{\varphi}(t)$. The result is a consequence of Corollary 4.7. ∎

Remark 4.12 The reduction to normal form in [11] is obtained via the action of unipotent biholomorphisms. We have just restated this fact in the context of holomorphic flows.

The expression of Corollary 4.11 can be simplified further by means of another flow (corresponding to a linear change of coordinates and a change of parameter).

Lemma 4.13 ([19]) *A branch Γ whose short parametrization is*

$$\Gamma \equiv \varphi(t) = \left(t^n, a_m t^m + a_\lambda t^\lambda + \sum_{i > \lambda}^{c-1} a_i t^i\right)$$

is analytically equivalent to

$$\Gamma' \equiv \left(t^n, t^m + t^\lambda + \sum_{i > \lambda}^{c-1} \overline{a}_i t^i\right),$$

where there exist $u, v \in \mathbb{C}^\star$ such that $\overline{a}_i = v^m u^{-i} a_i$.

Proof We define $\psi(x,y) = (u^n x, v^m y)$ for some $u, v \in \mathbb{C}^\star$ to be specified later on. We have

$$(\psi \circ \varphi)(t) = \Big(u^n t^n, v^m a_m t^m + v^m a_\lambda t^\lambda + \sum_{i>\lambda}^{c-1} v^m a_i t^i \Big).$$

Define the parameter $s = ut$. The curve $\psi(\Gamma)$ has parametrization

$$\Big(s^n, v^m u^{-m} a_m s^m + v^m u^{-\lambda} a_\lambda s^\lambda + \sum_{i>\lambda}^{c-1} v^m u^{-i} a_i s^i \Big).$$

It suffices to consider $u, v \in \mathbb{C}^\star$ such that $v^m u^{-m} = a_m^{-1}$ and $v^m u^{-\lambda} = a_\lambda^{-1}$. ∎

Combining Corollary 4.11 and Lemma 4.13 we obtain the following result.

Corollary 4.14 *Let Γ be a singular branch in $(\mathbb{C}^2, 0)$ having conductor c. Let (x,y) be a local system of coordinates. There exist a local biholomorphism ψ embedded in the flow of a nilpotent vector field, a linear map G and a reparametrization $\tau \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}, 0)$ such that*

$$\big(G \circ \psi(\Gamma) \equiv G \circ \psi \circ \varphi \circ \tau \big)(t) = \Big(t^n, t^m + t^\lambda + \sum_{\lambda<i<c} a_i t^i \Big),$$

where $\varphi(t)$ is the parametrization of Γ with $a_i = 0$ if $i < c$ and $i \in \Lambda \setminus \{\lambda\}$.

Proof There exists a linear automorphism $H(x,y)$ such that the tangent cone to $\Gamma' := H(\Gamma)$ at the origin is the axis $y = 0$. There exists a local biholomorphism $J(x,y) = (x, y + c(x))$ for some $c(x) \in \mathbb{C}\{x\}$ of vanishing order at least 2 such that $J(\Gamma')$ has a prepared irreducible Puiseux parametrization. We apply Corollary 4.11 to $J(\Gamma')$ to obtain a unipotent biholomorphism $\phi \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ such that $\phi(J(\Gamma'))$ has a short parametrization. Finally, we apply Lemma 4.13 to $\phi(J(\Gamma'))$ to obtain a linear isomorphism K such that $\Gamma'' := K(\phi(J(\Gamma')))$ has the desired parametrization. The biholomorphism $\phi \circ J$ is unipotent, since the linear part $D_0 J$ of J at the origin is the identity map. The conjugate $H^{-1} \circ (\phi \circ J) \circ H$ of $\phi \circ J$ is a unipotent biholomorphism $\rho \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2, 0)$ and then we obtain $\Gamma'' = (G \circ \rho)(\Gamma)$, where $G = K \circ H$ is a linear map. Since Γ and $\rho(\Gamma)$ are conjugated by a unipotent local biholomorphism, it follows that they are conjugated by a local biholomorphism ψ embedded in the flow of a nilpotent vector field. We obtain $\Gamma'' = (G \circ \psi)(\Gamma)$. ∎

The parametrization provided by Corollary 4.14 is called a *canonical parametrization* by Zariski [19] and the *normal form of* Γ by Hefez-Hernandes [11]. We shall use the latter terminology. Moreover, if $\overline{\Gamma}$ is another branch whose normal form has coefficients $\overline{a}_i$, one can prove (see [11, 19]) that they are analytically equivalent if and only if there exists u such that $u^{\lambda-m} = 1$ and $\overline{a}_i = u^{i-m} a_i$, which describes the complete moduli of Γ.

5 Non-complete Analytic Classes

We provide examples of non-complete analytic classes. Whether or not a single formal diffeomorphism is embedded in the flow of a formal singular vector field is deeply related to the spectrum of its linear part and more precisely to the resonances among its eigenvalues. For the sake of completenes, we recall these concepts along with some results. We work in dimension 2, because that is the case we are interested in, but the results concerning resonances are valid for any dimension (*cf.* [12]).

Definition 5.1 Consider a formal singular vector field X whose linear part is in Jordan normal form; in particular, X is of the form

$$X = \Big(\lambda_1 x + \delta y + \sum_{i+j\geq 2} a_{ij}x^i y^j\Big)\frac{\partial}{\partial x} + \Big(\lambda_2 y + \sum_{i+j\geq 2} b_{ij}x^i y^j\Big)\frac{\partial}{\partial y}.$$

We say that the monomial $x^i y^j \partial/\partial x$ with $i \geq 0$, $j \geq 0$, $i+j \geq 1$ and $(i,j) \neq (1,0)$ is resonant if $i\lambda_1 + j\lambda_2 = \lambda_1$. Analogously, we say that the monomial $x^i y^j \partial/\partial y$ with $i \geq 0$, $j \geq 0$, $i+j \geq 1$ and $(i,j) \neq (0,1)$ is resonant if $i\lambda_1 + j\lambda_2 = \lambda_2$.

Definition 5.2 Consider a formal diffeomorphism ψ whose linear part is in Jordan normal form; in particular, ψ is of the form

$$\psi(x,y) = \Big(\lambda_1 x + \delta y + \sum_{i+j\geq 2} a_{ij}x^i y^j, \lambda_2 y + \sum_{i+j\geq 2} b_{ij}x^i y^j\Big).$$

We say that the monomial $x^i y^j e_1 := (x^i y^j, 0)$ with $i \geq 0$, $j \geq 0$, $i+j \geq 1$ and $(i,j) \neq (1,0)$ is resonant if $\lambda_1^i \lambda_2^j = \lambda_1$. Analogously, we say that the monomial $x^i y^j e_2 := (0, x^i y^j)$ with $i \geq 0$, $j \geq 0$, $i+j \geq 1$ and $(i,j) \neq (0,1)$ is resonant if $\lambda_1^i \lambda_2^j = \lambda_2$. A formal diffeomorphism is non-resonant if there are no resonant monomials.

Remark 5.3 The property of being non-resonant depends only on the eigenvalues of the linear part.

The next result is Poincaré's linearisation map for formal diffeomorphisms. As is customary, we denote by $D_0\psi$ the linear part of a formal diffeomorphism ψ.

Proposition 5.4 (*cf.* [12, Theorem 4.21]) *Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be a non-resonant formal diffeomorphism. Then ψ is conjugated by a formal diffeomorphism to $(x,y) \mapsto (\lambda_1 x, \lambda_2 y)$, where λ_1 and λ_2 are the eigenvalues of the linear part $D_0\psi$ of ψ at $(0,0)$.*

Corollary 5.5 *Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be a non-resonant formal diffeomorphism. Then there exists a formal singular vector field X such that $\psi = \exp(\widehat{X})$.*

Proof The formal diffeomorphism ψ is formally conjugated to a linear diagonal map by Proposition 5.4. Since the latter map is embedded in the flow of a singular vector field, it follows that ψ is embedded in the flow of a formal vector field. ∎

Let us consider the problem of embedding formal resonant diffeomorphisms in formal flows. Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ and assume for simplicity that $(D_0\psi)(x,y) = (\lambda_1 x, \lambda_2 y)$. The equation $\psi = \exp(X)$ implies $D_0\psi = \exp(D_0 X)$. Notice that if $\lambda_1 \neq \lambda_2$, then the choice of the eigenvalues $\log\lambda_1, \log\lambda_2$ completely determines $D_0 X$.

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Definition 5.6 Consider the above setting. We say that a resonance $x^i y^j e_1$ (resp. $x^i y^j e_2$) of ψ is *strong* if the monomial $x^i y^j \partial/\partial x$ (resp. $x^i y^j \partial/\partial y$) is a resonant monomial of the vector field $\log\lambda_1 x\partial/\partial x + \log\lambda_2 y\partial/\partial y$, *i.e.*, if $\lambda_1^i\lambda_2^j = \lambda_1$ and $i\log\lambda_1 + j\log\lambda_2 = \log\lambda_1$ (resp. $\lambda_1^i\lambda_2^j = \lambda_2$ and $i\log\lambda_1 + j\log\lambda_2 = \log\lambda_2$).

A resonance of ψ that is not *strong* will be called *weak*.

We will use the following special case of [16, Proposition 1.5].

Proposition 5.7 *Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be such that $(D_0\psi)(x,y) = (\lambda_1 x, \lambda_2 y)$. Let $B\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a linear map such that* $\exp(B) = D_0\psi$. *Assume that $j^k\psi = D_0\psi + f_k$, where both components of f_k are homogeneous polynomials of degree k. Furthermore, assume that f_k contains non-vanishing weakly resonant monomials. Then ψ is not embedded in the flow of any formal vector field X such that $D_0X = B$.*

We can return to the problem of determining non-complete classes. The results regarding completeness of analytic classes $\mathcal{C}$ depend on the topology that we consider for the infinite-dimensional space $\mathcal{C}$. First, we see that in some sense, being connected by a geodesic is a dense property.

Proof of Proposition 1.5 Let ψ be a local biholomorphism conjugating Γ and Γ'. Up to a linear change of coordinates, we can assume that the linear part $D_0\psi$ of ψ at the origin is in Jordan normal form; in particular, its matrix is of the form $\begin{pmatrix} u & w \\ 0 & v \end{pmatrix}$, where $u, v \in \mathbb{C}^\star$. Consider the family $\sigma_\epsilon(x,y) = (e^{\epsilon a}x, e^{\epsilon b}y)$ for some $a, b \in \mathbb{C}$ that are linearly independent over $\mathbb{Q}$. The map σ_ϵ converges to Id when $\epsilon \to 0$. Let us define the family (Γ'_ϵ) by $\Gamma'_\epsilon = (\sigma_\epsilon \circ \psi)(\Gamma)$. The map $D_0(\sigma_\epsilon \circ \psi)$ has eigenvalues $ue^{\epsilon a}$ and $ve^{\epsilon b}$.

Let $F_{p,q}(\epsilon) = u^p v^q e^{(ap+bq)\epsilon} - 1$ and $T_{p,q} = F_{p,q}^{-1}(0)$ for $(p,q) \in \mathbb{Z}\times\mathbb{Z}$. Resonances between the eigenvalues of $D_0(\sigma_\epsilon \circ \psi)$ are obtained when there exists $(p,q) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ such that $(ue^{\epsilon a})^p(ve^{\epsilon b})^q = 1$. This equation is equivalent to $\epsilon \in T_{p,q}$. Since $ap + bq \neq 0$, the function $F_{p,q}$ is not constant and $T_{p,q}$ is a countable closed set for any $(p,q) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. We deduce that $T := \cup_{(p,q)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} T_{p,q}$ is countable and hence there exists $\epsilon_0 \in \mathbb{C}^*$ such that $\{t\epsilon_0 : t \in \mathbb{R}^*\} \cap T = \varnothing$. We define the path $\gamma\colon [0,\infty) \to \mathbb{C}$ by $\gamma(t) = t\epsilon_0$. The map $\sigma_\epsilon \circ \psi$ is embedded in the flow of a formal vector field for any $\epsilon \notin T$ by Corollary 5.5. Therefore, Γ and Γ'_ϵ are connected by a geodesic for any $\epsilon \in \gamma(0,\infty)$ by Theorem 4.6. ∎

Let us show that the analytic class $\mathcal{C}_0$ of the plane branch Γ_0 with Puiseux parametrization $(t^6, t^7 + t^{10} + t^{11})$ is non-complete. First, we study the stabilizer group $\mathrm{Stab}(\Gamma_0) = \{\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0) : \psi(\Gamma_0) = \Gamma_0\}$ of Γ_0.

Lemma 5.8 *The linear part at the origin of any element ψ of* $\mathrm{Stab}(\Gamma_0)$ *is the identity map.*

Proof The linear part $D_0\psi$ is a map of the form $(x,y) \mapsto (ax + by, cx + dy)$. Since $D_0\psi$ preserves the tangent cone of Γ_0, we deduce that $c = 0$. In particular $ad \neq 0$

because ψ is a formal diffeomorphism. We have

$$\psi(t^6, t^7 + t^{10} + t^{11}) \equiv \left(at^6 + bt^7 + bt^{10} + bt^{11} + O(t^{12}), dt^7 + dt^{10} + dt^{11} + O(t^{12})\right).$$

Consider a formal power series $\sigma(t)$ such that $\sigma(t)^6 \equiv (x \circ \psi)(t^6, t^7 + t^{10} + t^{11})$. It must admit the expression $\sigma(t) \equiv a^{1/6}t + (b/6)a^{-5/6}t^2 + O(t^3)$. Moreover, it is a formal diffeomorphism in one variable and its inverse σ^{-1} satisfies $\sigma^{-1}(t) \equiv a^{-1/6}t - (b/6)a^{-4/3}t^2 + O(t^3)$. A simple calculation leads us to

$$\psi(t^6, t^7 + t^{10} + t^{11}) \circ \sigma^{-1}(t) \equiv \left(t^6, da^{-7/6}t^7 - \frac{7}{6}bda^{-7/3}t^8 + O(t^9)\right).$$

Since ψ belongs to $\mathrm{Stab}(\Gamma_0)$, $bda^{-7/3}$ vanishes. We deduce $b = 0$ as a consequence of $ad \neq 0$. Thus, the formal diffeomorphisms σ and σ^{-1} are of the form $t \mapsto a^{1/6}t + O(t^7)$ and $t \mapsto a^{-1/6}t + O(t^7)$, respectively. We obtain

$$\psi(t^6, t^7 + t^{10} + t^{11}) \circ \sigma^{-1}(t) \equiv \left(t^6, a^{-7/6}dt^7 + a^{-10/6}dt^{10} + a^{-11/6}dt^{11} + O(t^{12})\right).$$

Since $\psi(\Gamma_0) = \Gamma_0$ there exists $\xi \in \mathbb{C}$ such that $\xi^6 = 1$ and $a^{-7/6}d = \xi^7$, $a^{-10/6}d = \xi^{10}$, and $a^{-11/6}d = \xi^{11}$. We get $a^{1/6} = \xi^{-1}$ by dividing the last two equations and then $a = (a^{1/6})^6 = \xi^{-6} = 1$. By plugging $a^{-1/6} = \xi$ into $a^{-7/6}d = \xi^7$, we get $d = 1$. Hence, $D_0\psi$ is the identity map. ∎

Proposition 5.9 *Let X be a formal vector field that preserves Γ_0. Then X has vanishing second jet.*

Proof Since X preserves Γ_0, it follows that the time s flow $\exp(sX)$ of X preserves Γ_0 for any $s \in \mathbb{C}$. All the formal diffeomorphisms $\exp(sX)$ in the one-parameter group of X have identity linear part at the origin by Lemma 5.8. In particular, X has vanishing linear part. We write

$$X = \Big(\sum_{i+j\geq 2} a_{ij}x^i y^j\Big)\frac{\partial}{\partial x} + \Big(\sum_{i+j\geq 2} b_{ij}x^i y^j\Big)\frac{\partial}{\partial y}.$$

Consider the dual form

$$\omega = -\Big(\sum_{i+j\geq 2} b_{ij}x^i y^j\Big)dx + \Big(\sum_{i+j\geq 2} a_{ij}x^i y^j\Big)dy.$$

Since X preserves Γ_0, it follows that $(t^6, t^7 + t^{10} + t^{11})^*\omega \equiv 0$. We have

$$\nu_{\Gamma_0}(x^2dx) = 18, \qquad \nu_{\Gamma_0}(xydx) = 19, \qquad \nu_{\Gamma_0}(y^2dx) = 20,$$
$$\nu_{\Gamma_0}(x^2dy) = 19, \qquad \nu_{\Gamma}(xydy) = 20, \qquad \nu_{\Gamma}(y^2dy) = 21,$$

and $\nu_\Gamma(x^i y^j dx) \geq 24 \leq \nu_\Gamma(x^i y^j dy)$ for $i + j \geq 3$. Since $(t^6, t^7 + t^{10} + t^{11})^*\omega \equiv 0$, we deduce $b_{20} = 0$. We get

$$\begin{aligned}&-6b_{11}t^{11}(t^7 + t^{10} + t^{11}) - 6b_{02}t^5(t^7 + t^{10} + t^{11})^2\\ &\quad + a_{20}t^{12}(7t^6 + 10t^9 + 11t^{10}) + a_{11}t^6(t^7 + t^{10} + t^{11})(7t^6 + 10t^9 + 11t^{10})\\ &\qquad + a_{02}(t^7 + t^{10} + t^{11})^2(7t^6 + 10t^9 + 11t^{10}) + O(t^{23}) = 0.\end{aligned}$$

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We write the linear system of equations satisfied by the coefficients of t^{18}, t^{19}, t^{20}, t^{21}, and t^{22}:

$$\begin{array}{rcrcrcrcrcl} -6b_{11} & & & + & 7a_{20} & & & & & = & 0 \\ & & -6b_{02} & & & + & 7a_{11} & & & = & 0 \\ & & & & & & & + & 7a_{02} & = & 0 \\ -6b_{11} & & & + & 10a_{20} & & & & & = & 0 \\ -6b_{11} & - & 12b_{02} & + & 11a_{20} & + & 17a_{11} & & & = & 0. \end{array}$$

The matrix of the system is regular, hence $b_{11} = b_{02} = a_{20} = a_{11} = a_{02} = 0$. In particular, X has a vanishing second jet. ∎

Proposition 5.10 *Let $\psi \in \mathrm{Stab}(\Gamma_0)$. Then ψ and the identity map have the same second jet.*

Proof The linear part of ψ is the identity map by Lemma 5.8. Thus, ψ is of the form $\exp(X)$ for some unique formal nilpotent vector field X (in fact X has vanishing linear part) by Remark 4.8. Let $f = 0$ be an irreducible equation of Γ_0. Notice that $f \circ \exp(sX) = \sum_{j=0}^{\infty} \frac{s^j}{j!} X^j(f)$ by Taylor's formula and that $X^j(f) \in \mathfrak{m}^{j+1}$ for any $j \geq 1$. Therefore, $f \circ \exp(sX)$ belongs to $\mathbb{C}[s][[x,y]]$, and then

$$G(s,t) := f \circ \exp(sX) \circ (t^6, t^7 + t^{10} + t^{11})$$

belongs to $\mathbb{C}[s][[t]]$. Moreover, $G(s,t)$ vanishes for $s \in \mathbb{Z}$ since $\{\exp(sX) : s \in \mathbb{Z}\}$ is the cyclic group $\langle \psi \rangle$ and $\langle \psi \rangle$ is contained in $\mathrm{Stab}(\Gamma_0)$. Since the coefficients of t^j of $G(s,t)$ are polynomials that vanish at $\mathbb{Z}$, we deduce that $G \equiv 0$. In particular the elements of the one-parameter group generated by X preserve Γ_0 and hence X preserves Γ_0. By Proposition 5.9, the vector field X has vanishing second jet and hence $j^2\psi \equiv Id$. ∎

We just completed the first step of the proof of Proposition 1.3. Now we want to construct 2-jets of biholomorphisms such that any formal diffeomorphism with such a 2-jet is not embedded in the flow of a formal vector field.

Lemma 5.11 *Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ such that its second jet is equal to $(x,y) \mapsto (x + x^2 + y^2, -y)$. Then ψ is not embedded in the flow of a formal vector field.*

Proof Assume, aiming for contradiction, that ψ is of the form $\exp(X)$ for some formal vector field. The eigenvalues of the linear part of X at the origin are α and β with $e^\alpha = 1$ and $e^\beta = -1$. We claim that for any choice of α and β, at least one of the resonances $x^2 e_1$ or $y^2 e_1$ is weak. Otherwise, we obtain

$$2\alpha - \alpha = 0 \text{ and } 2\beta - \alpha = 0 \implies \alpha = \beta = 0 \implies e^\beta = 1$$

and since $e^\beta = -1$ this is a contradiction. Hence, the formal diffeomorphism ψ is not embedded in a formal flow by Proposition 5.7. ∎

Lemma 5.12 *Let $\psi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be such that its second jet is equal to $(x,y) \mapsto (e^{2\pi i/3}x + y^2, e^{4\pi i/3}y + x^2)$. Then ψ is not embedded in the flow of a formal vector field.*

Proof Assume, aiming for contradiction, that ψ is of the form $\exp(X)$ for some formal vector field. The eigenvalues of the linear part of X at the origin are α and β with $e^{\alpha} = e^{2\pi i/3}$ and $e^{\beta} = e^{4\pi i/3}$. We claim that for any choice of α and β, at least one of the resonances $x^2 e_2$ or $y^2 e_1$ is weak. Otherwise, we have $2\alpha = \beta$ and $2\beta = \alpha$. This implies $\alpha = \beta = 0$, contradicting $e^{\alpha} = e^{2\pi i/3}$. Therefore, the formal diffeomorphism ψ is not embedded in a formal flow by Proposition 5.7. ■

Proof of Proposition 1.3 Consider the biholomorphism

$$\psi(x,y) = (x + x^2 + y^2, -y) \text{ or } \psi(x,y) = (e^{2\pi i/3}x + y^2, e^{4\pi i/3}y + x^2)$$

and the curve $\Gamma = \psi(\Gamma_0)$. Any formal diffeomorphism σ conjugating Γ_0 and Γ is of the form $\psi \circ \rho$, where $\rho \in \mathrm{Stab}(\Gamma_0)$. Since $j^2\rho \equiv Id$ by Proposition 5.10, we deduce $j^2(\sigma \circ \rho) \equiv j^2\psi$, and hence $\sigma \circ \rho$ is not embedded in the flow of a formal vector field for any $\rho \in \mathrm{Stab}(\Gamma_0)$ by Lemmas 5.11 and 5.12. Therefore, the analytic class $\mathcal{C}_0$ is non-complete. ■

Proof of Proposition 1.4 Consider the subset T of $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ of diffeomorphisms whose second jet is equal to $(x + x^2 + y^2, -y)$ (instead we could choose $(e^{2\pi i/3}x + y^2, e^{4\pi i/3}y + x^2)$ too). The set T is open in the Krull topology. Moreover, since $\mathrm{Stab}(\Gamma_0)$ consists of formal diffeomorphisms with trivial second jet, it follows that T is a union of left cosets of $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)/\mathrm{Stab}(\Gamma_0)$. As a consequence, its projection $\widetilde{T}$ in $\mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)/\mathrm{Stab}(\Gamma_0) \sim \mathcal{C}_0$ is an open set in the induced quotient topology. Every plane branch Γ in $\widetilde{T}$ is of the form $\sigma(\Gamma_0)$, where $\sigma \in \mathrm{Diff}_{\mathcal{O}}(\mathbb{C}^2,0)$ satisfies $j^2\sigma \equiv j^2(x + x^2 + y^2, -y)$. Therefore, Γ_0 is not connected to Γ by a geodesic by the proof of Proposition 1.3. We just obtained an open subset $\widetilde{T}$ of $\mathcal{C}_0$ whose elements are not connected to Γ_0 by a geodesic. ■

Remark 5.13 Notice that in the examples in the proof of Proposition 1.3, the curves Γ_0 and Γ have the same tangent cone.

Remark 5.14 Let us focus in the case where $\psi(x,y) = (x + x^2 + y^2, -y)$. Zariski's λ invariant of Γ_0 is $\lambda = 10$. Let Γ be the curve of parametrization

$$\psi(t^6, t^7 + t^{10} + t^{11}) = (t^6 + O(t^{12}), -t^7 - t^{10} - t^{11}).$$

Up to a change of parameter $t \mapsto ut$ with $u^6 = 1$, the curve Γ is of the form

$$\left(t^6 + O(t^{12}), -\frac{t^7}{u^7} - \frac{t^{10}}{u^{10}} - \frac{t^{11}}{u^{11}}\right),$$

and the coefficient of t^7 is equal to 1 if and only if $u = -1$. Then

$$\left(t^6 + O(t^{12}), t^7 - t^{10} + t^{11}\right)$$

parametrises Γ. Every parametrization of Γ of the form $(t^6, t^7 + ct^{10} + O(t^{11}))$ satisfies $c = -1$. Thus, the curves Γ_0 and Γ are not connected by a geodesic but have the same tangent cone and their parametrizations coincide up to (but not including) the term corresponding to Zariski's λ invariant.

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