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(KK)-PROPERTIES, NORMAL STRUCTURE AND FIXED POINTS OF NONEXPANSIVE MAPPINGS IN ORLICZ SEQUENCE SPACES

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In this paper we investigate Orlicz sequence spaces with regard to certain geometric properties that have proved to be important in fixed point theory. In particular, we shall consider various Kadec-Klee type properties, and weak and weak* normal structure. It turns out that many of these properties, though generally distinct, coincide in Orlicz sequence spaces and that all of them are intimately related to the so-called Δ_2-condition. Some of our results extend to vector-valued Orlicz sequence spaces. For example, we prove a rather powerful theorem on the preservation of weak normal structure under the formation of substitution spaces. There is also a fixed point theorem: the Orlicz sequence space h_M has the fixed point property if the complementary Orlicz function M^* satisfies the Δ_2-condition. Another one of our results implies that, under this assumption on M^*, h_M has weak normal structure if and only if M also satisfies the Δ_2-condition. Thus all Orlicz functions M such that M^* satisfies Δ_2 but M does not (such functions are easy to construct) provide illustrations of the (known) fact that weak normal structure is not necessary for the fixed point property to hold.

We now fix our terminology and recall some notions needed later. For the definition and standard facts about Orlicz sequence spaces, in particular the Δ_2-condition and the dualities $h_M^* \approx l_{M^*}$ and $l_M^* \approx h_{M^*}^{**}$, we refer to [**12**]. Let us just mention that we shall always assume Orlicz functions to be nondegenerate, and therefore strictly increasing. We shall consider the following Kadec-Klee type properties (see [**8**], and in particular [**6**] for the connections with normal structure and Chebyshev centers). A Banach space X is said to be *Kadec-Klee* (KK) if

$$\left.\begin{array}{l} x_n \xrightarrow{w} x \\ \|x_n\| \to \|x\| \end{array}\right\} \Rightarrow x_n \to x \quad \text{(in norm)},$$

and *uniformly Kadec-Klee* (UKK) if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

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$$(*) \qquad \left.\begin{array}{l} \|x_n\| \leqq 1 \ (n = 1, 2, \dots) \\ x_n \xrightarrow{w} x \\ \operatorname{sep}(x_n) > \epsilon \end{array}\right\} \Rightarrow \|x\| \leqq 1 - \delta$$

($\operatorname{sep}(x_n)$ is defined as $\inf\{ \|x_n - x_m\|:m \neq n\}$), and *weakly uniformly Kadec-Klee* (WUKK) if there exists some pair (ϵ, δ) with $0 < \epsilon < 1$ and $\delta > 0$ such that $(*)$ holds.

If X is a dual Banach space with w^*-sequentially compact unit ball, or a subspace thereof, then (KK*), (UKK*) and (WUKK*) denote the properties obtained from the above by replacing weak by w^*-convergence. For every Orlicz function M the space l_M is canonically isomorphic to $h^*_{M^*}$. Clearly, the unit ball of l_M (regarded as a subset of $h^*_{M^*}$) is closed for the w^*-topology of $h^*_{M^*}$, so that l_M is itself isometric to a dual Banach space. Hence the properties (KK*), (UKK*) and (WUKK*) are meaningful for l_M and its subspace h_M.

If (X_n) is a sequence of Banach spaces and M an Orlicz function, then

$$\left(\sum_{n=1}^{\infty} \oplus X_n\right)_{h_M}$$

denotes the Banach space of all sequences $x = (x_n)$ with $x_n \in X_n \quad (n = 1, 2, \dots)$ and $(\|x_n\|) \in h_M$, with norm

$$\|x\| = \|(\|x_n\|)\|_M.$$

The support ($\operatorname{supp} x$) of a sequence $x = (x_n)$ is $\{n \in \mathbf{N}:x_n \neq 0\}$. By $\operatorname{supp} x < \operatorname{supp} y$ we shall mean that $\max \operatorname{supp} x < \min \operatorname{supp} y$. A Banach space X (resp. a dual Banach space or subspace thereof) will be said to have *weak* (resp. w^*) *normal structure* if every weakly (resp. w^*) compact convex subset C of X contains a *non-diametral point* (i.e., a point x such that

$$\sup\{ \|x - y\|:y \in C\} < \operatorname{diam} C.)$$

It is well known ([7]) that X has weak normal structure if and only if there exists no sequence (x_n) in X such that

$$x_n \xrightarrow{w} 0, \ \|x_n\| \leqq 1 \ (n = 1, 2, \dots), \ \operatorname{diam}\{x_n:n \in \mathbf{N}\} = 1,$$

$$\lim_{n\to\infty} \|x_n\| = 1 \text{ and } \lim_{k\to\infty} d(x_{k+1}, \operatorname{co}\{x_1, \dots, x_k\}) = 1.$$

Such sequences will be called *w-diametral*. X has the *fixed point property* (FPP) if for every weakly compact convex set $C \subset X$ and for every nonexpansive map $T:C \to C$ there exists an $x \in C$ with $Tx = x$. It was proved in [**10**] that weak normal structure implies (FPP). The converse is false ([**9**]).

Our notation will be standard. E.g. co denotes convex hull, diam stands for diameter, and d denotes distance.

We begin with a simple and well-known fact that will be useful later.

LEMMA 1. *Let M be an Orlicz function satisfying the Δ_2-condition. Then*

$$(1) \qquad \forall t_0 > 0 \quad \lim_{\lambda \to 1} \frac{M(\lambda \cdot)}{M(\cdot)} = 1, \quad \textit{uniformly on } [0, t_0].$$

More precisely,

$$(2) \qquad \forall t_0 > 0 \ \exists K = K(t_0) < \infty \ \forall \lambda \geqq 1 \ \forall t \in \left(0, \frac{t_0}{\lambda}\right] \frac{M(\lambda t)}{M(t)} \leqq \lambda^K.$$

Proof. It clearly suffices to prove (2). We fix $t_0 > 0$ and, using the Δ_2-condition, choose $K < \infty$ such that

$$(3) \qquad \frac{M(2t)}{M(t)} \leqq K \quad \text{for all } t \in (0, t_0].$$

Recall that M is the integral of its right derivative p and that the latter is nondecreasing. Hence

$$tp(t) \leqq \int_t^{2t} p(s)ds = M(2t) - M(t) \leqq KM(t),$$

so

$$(4) \qquad \frac{p(t)}{M(t)} \leqq \frac{K}{t} \quad \text{for all } t \in \left(0, t_0\right].$$

For all $\lambda \geqq 1$ and $t \in (0, \frac{t_0}{\lambda}]$ we thus have

$$\log \frac{M(\lambda t)}{M(t)} = \int_t^{\lambda t} \frac{p(s)}{M(s)} ds \leqq \int_t^{\lambda t} \frac{K}{s} ds = K \log \lambda.$$

This proves (2).

Let X be a Banach space with a Schauder basis (x_n). We shall say that (x_n) satisfies the condition (C) if

$$(5) \qquad \forall c > 0 \exists \delta = \delta(c) > 0 \forall x \in X \forall n \in \mathbf{N}$$
$$[\,\|P_n x\| = 1 \wedge \|(I - P_n)x\| \geqq c \Rightarrow \|x\| \geqq 1 + \delta],$$

where P_n is the projection onto $[x_k]_{k=1}^n$ with kernel $[x_k]_{k=n+1}^{\infty}$. This notion was introduced by J. P. Gossez and E. Lami Dozo ([7]) who proved that it implies weak normal structure.

PROPOSITION 1. *Let M be an Orlicz function. Then M satisfies the Δ_2-condition if and only if the condition* (C) *holds for the standard basis* (e_n) *of* h_M.

Proof. We first assume the Δ_2-condition and derive (C). Let $c > 0$ be given and let s satisfy $M(s) = 1$. Using (2) we first choose $\alpha > 0$ so that

$$(6)\qquad M\left(\frac{1}{2}ct\right) \geqq \alpha M(t) \quad \text{for all } t \in (0, s]$$

and then pick δ so that

$$(7)\qquad 0 < \delta \leqq 1$$

and

$$(8)\qquad M\left(\frac{t}{1+\delta}\right) \geqq \left(1 - \frac{1}{2}\alpha\right)M(t) \quad \text{for all } t \in (0, s].$$

We claim that this δ satisfies (5) for the given c. Indeed, fix $x \in h_M$ and $n \in \mathbf{N}$ such that

$$\|P_n x\| = 1 \quad \text{and} \quad \|(I - P_n)x\| = c,$$

i.e.

$$(9)\qquad \sum_{k=1}^{n} M(|x_k|) = \sum_{k=n+1}^{\infty} M\left(\frac{|x_k|}{c}\right) = 1.$$

Notice that $|x_k| \leqq s$ for $k = 1, \dots, n$ and $|x_k|/c \leqq s$ for $k > n$. (7), (6) and (9) now imply that

$$(10)\qquad \sum_{k=n+1}^{\infty} M\left(\frac{|x_k|}{1+\delta}\right) \geqq \sum_{k=n+1}^{\infty} M\left(\frac{1}{2}|x_k|\right) \geqq \alpha \sum_{k=n+1}^{\infty} M\left(\frac{|x_k|}{c}\right) = \alpha.$$

Also, by (8) and (9) we have

$$(11)\qquad \sum_{k=1}^{n} M\left(\frac{|x_k|}{1+\delta}\right) \geqq \left(1 - \frac{1}{2}\alpha\right) \sum_{k=1}^{n} M(|x_k|) = 1 - \frac{1}{2}\alpha.$$

Adding (10) and (11) we conclude that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{1+\delta}\right) > 1,$$

i.e.,

$$\|x\| > 1 + \delta.$$

Clearly the same conclusion holds if $\|(I - P_n)x\| \geqq c$, so that (C) is established.

Now let us assume the Δ_2-condition does not hold. For arbitrary $\delta > 0$

and $k \in \mathbf{N}$ we shall then define an $x \in h_M$ such that, for some $n \in \mathbf{N}$,

$$x = P_{kn}x, \ \|x\| < 1 + \delta,$$
$$\|P_n x\| = \|(P_{2n} - P_n)x\| = \ldots = \|(P_{kn} - P_{(k-1)n})x\| \geqq 1.$$

Clearly, this means that (C) fails in a strong sense. Fix $\delta > 0$ and $k \in \mathbf{N}$. By the failure of the Δ_2-condition we can choose $t > 0$ so that

(12) $M(t) < 1$

and

(13) $M\left(\dfrac{t}{1 + \delta}\right) \leqq \dfrac{1}{2k}M(t).$

We now pick $n \in \mathbf{N}$ satisfying

(14) $\dfrac{1}{M(t)} \leqq n \leqq \dfrac{1}{M(t)} + 1$

and define $x \in h_M$ by

$$x_i := \begin{cases} t \text{ if } 1 \leqq i \leqq kn \\ 0 \text{ if } i \geqq kn + 1. \end{cases}$$

Then, by the first inequality in (14),

$$\sum_{i=(l-1)n+1}^{ln} M(|x_i|) = nM(t) \geqq 1 \quad \text{for } l = 1, \ldots, k,$$

so

$$\|P_n x\| = \|(P_{2n} - P_n)x\| = \ldots = \|(P_{kn} - P_{(k-1)n})x\| \geqq 1.$$

On the other hand (13), the second inequality in (14) and (12) yield

$$\sum_{i=1}^{\infty} M\left(\frac{|x_i|}{1 + \delta}\right) = knM\left(\frac{t}{1 + \delta}\right) \leqq \frac{1}{2}nM(t)$$
$$\leqq \frac{1}{2}(M(t) + 1) < 1,$$

so $\|x\| < 1 + \delta$.

Remark 1. If X has a basis satisfying condition (C), then it is known (cf. [7]) that X has weak normal structure. Moreover, it was shown in [5] that the same conclusion holds if the given norm $\|\cdot\|$ on X is replaced by an equivalent norm of the form $\|\cdot\| + |||\cdot|||$, where $|||\cdot|||$ is any seminorm on X satisfying $|||\cdot||| \leqq \gamma\|\cdot\|$ for some $\gamma < \infty$. By Proposition 1 therefore, if M satisfies the Δ_2-condition and $|||\cdot|||$ is any seminorm on h_M dominated by

$\|\cdot\|_M$ then

$$(h_M, \|\cdot\|_M + |||\cdot|||)$$

has weak normal structure. Let us already remark at this point that the Δ_2-condition also implies w^* normal structure for h_M, as we shall presently see.

We shall now show the equivalence of the Δ_2-condition for M with several Kadec-Klee type properties for h_M. We first state and prove the part of this result that carries over to substitution spaces.

PROPOSITION 2. *Let X_n be a* (KK) *space for each $n \in \mathbf{N}$ and let M be an Orlicz function satisfying the Δ_2-condition. Then*

$$X: = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_{h_M}$$

is (KK).

Proof. Let x, x^k $(k = 1, 2, \dots)$ be unit vectors in X such that

$$w - \lim_{k\to\infty} x^k = x.$$

Then clearly,

$$w - \lim_{k\to\infty} x_n^k = x_n \quad \text{for each } n \in \mathbf{N},$$

so

$$(15)\quad \liminf_{k\to\infty} \|x_n^k\| \geqq \|x_n\| \quad (n = 1, 2, \dots).$$

It is easily seen also that

$$(16)\quad \limsup_{k\to\infty} \|x_n^k\| \leqq \|x_n\| \quad (n = 1, 2, \dots).$$

Indeed, if not, then by passing to a subsequence if necessary, we may assume that for some $n_0 \in \mathbf{N}$ and $\epsilon > 0$ we have

$$(17)\quad \|x_{n_0}^k\| > \|x_{n_0}\| + \epsilon \quad \text{for all } k \in \mathbf{N}.$$

But since $\|x^k\| = \|x\| = 1$ $(k = 1, 2, \dots)$, (15) and (17) lead to the contradiction

$$1 = \liminf_{k\to\infty} \sum_{n=1}^{\infty} M(\|x_n^k\|)$$

$$\geqq \sum_{n=1}^{\infty} \liminf_{k\to\infty} M(\|x_n^k\|) \geqq M(\|x_{n_0}\| + \epsilon) + \sum_{n\neq n_0} M(\|x_n\|)$$

$$> \sum_{n=1}^{\infty} M(\|x_n\|) = 1.$$

It follows from (15) and (16) that

$$\lim_{k\to\infty} \|x_n^k\| = \|x_n\|,$$

hence

(18) $$\lim_{k\to\infty} x_n^k = x_n \quad (n = 1, 2, \ldots),$$

since every X_n is (KK).

We now show that

$$\lim_{k\to\infty} x^k = x.$$

Only in this part of the proof the Δ_2-condition is needed. Let $0 < \epsilon < \frac{1}{2}$ be arbitrary and let $s = M^{-1}(1)$. By the Δ_2-condition there exists a $K < \infty$ such that

(19) $$M\left(\frac{t}{\epsilon}\right) \leqq KM\left(\frac{1}{2}t\right) \text{ whenever } 0 \leqq t \leqq 2s.$$

Since

$$\sum_{n=1}^{\infty} M(\|x_n\|) = 1$$

there exists an $n_0 \in \mathbf{N}$ such that

$$\sum_{n=1}^{n_0} M(\|x_n\|) > 1 - \frac{\epsilon}{K}$$

and therefore

(20) $$\sum_{n=n_0+1}^{\infty} M(\|x_n\|) < \frac{\epsilon}{K}.$$

Next, using (18) and the fact that also

$$\sum_{n=1}^{\infty} M(\|x_n^k\|) = 1 \quad \text{for all } k,$$

we pick $k_0 \in \mathbf{N}$ so that

$$(21) \quad \sum_{n=1}^{n_0} M(\|x_n^k - x_n\|) < \frac{\epsilon}{K} \quad \text{for } k \geqq k_0,$$

$$\sum_{n=1}^{n_0} M(\|x_n^k\|) > 1 - \frac{\epsilon}{K} \quad \text{for } k \geqq k_0,$$

and therefore also

$$(22) \quad \sum_{n=n_0+1}^{\infty} M(\|x_n^k\|) < \frac{\epsilon}{K} \quad \text{for } k \geqq k_0.$$

It now follows from (19), (21), the convexity of M, (22) and (20) that for all $k \geqq k_0$ we have

$$\sum_{n=1}^{\infty} M\left(\frac{\|x_n^k - x_n\|}{\epsilon}\right) \leqq K \sum_{n=1}^{\infty} M\left(\frac{1}{2}\|x_n^k - x_n\|\right)$$
$$= K \sum_{n=1}^{n_0} M\left(\frac{1}{2}\|x_n^k - x_n\|\right)$$
$$+ K \sum_{n=n_0+1}^{\infty} M\left(\frac{1}{2}\|x_n^k - x_n\|\right)$$
$$\leqq K \cdot \frac{\epsilon}{K} + K\left[\frac{1}{2} \sum_{n=n_0+1}^{\infty} M(\|x_n^k\|) + \frac{1}{2} \sum_{n=n_0+1}^{\infty} M(\|x_n\|)\right]$$
$$\leqq \epsilon + K\left[\frac{1}{2}\frac{\epsilon}{K} + \frac{1}{2}\frac{\epsilon}{K}\right] = 2\epsilon < 1.$$

This means that $\|x^k - x\| < \epsilon$ for $k \geqq k_0$ and so the proof is complete.

Remark 2. We cannot replace (KK) by (UKK) or (WUKK) in Proposition 2. Even if all X_n are uniformly convex, then X need not be (WUKK), as was shown in [**6**, Example (f)].

However, in the scalar case the situation is most satisfactory:

PROPOSITION 3. *Let M be an Orlicz function. Then the following are equivalent*

(i) *M satisfies the Δ_2-condition*
(ii) *h_M is* (KK*), (ii)′ *l_M is* (KK*)
(iii) *h_M is* (UKK*), (iii)′ *l_M is* (UKK*)
(iv) *h_M is* (WUKK*), (iv)′ *l_M is* (WUKK*).

Proof. Since (iii) trivially implies (ii) and (iv), the equivalence of (i), (ii), (iii) and (iv) will be established once we have proved the implications (i) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i). The properties (ii)′, (iii)′ and (iv)′ present no problem: each of them is stronger than the corresponding property for h_M, whereas the Δ_2-condition is equivalent to $h_M = l_M$.

(i) $\Rightarrow$ (iii): Fix $\epsilon > 0$. Let us assume there is no $\delta > 0$ satisfying the definition of (UKK*) for this ϵ and work towards a contradiction. Recall that by Proposition 1 the Δ_2-condition is equivalent to the condition (C) for the standard basis (e_n) of h_M. For each $c > 0$ let $\delta(c) > 0$ be the largest δ satisfying (5). Now we choose $\alpha < 1$ so large that

$$(23) \quad \alpha\left(1 + \delta\left(\frac{\epsilon}{2}\right)\right) > 1.$$

The assumption implies the existence of a sequence (x^k) in the unit ball of h_M with $\text{sep}(x^k) \geqq \epsilon$ that w^*-converges to an $x \in h_M$ with $\|x\| > \alpha$. Pick $n_0 \in \mathbf{N}$ so that $\|P_{n_0}x\| > \alpha$ and then, using the coordinatewise convergence of (x^k) to x, a $k_0 \in \mathbf{N}$ such that

$$\|P_{n_0}x^k\| > \alpha \quad \text{for } k \geqq k_0.$$

Since $\text{sep}(x^k) > \epsilon$, it is also clearly possible to choose $k_1, k_2 \geqq k_0$ so that

$$\|(I - P_{n_0})(x^{k_1} - x^{k_2})\| > \epsilon.$$

For at least one of these indices, say k_1, we then must have

$$\alpha < \|P_{n_0}x^{k_1}\| \leqq 1 \quad \text{and} \quad \|(I - P_{n_0})x^{k_1}\| > \frac{\epsilon}{2}$$

and therefore

$$\left\|P_{n_0}\frac{x^{k_1}}{\|P_{n_0}x^{k_1}\|}\right\| = 1, \quad \left\|(I - P_{n_0})\frac{x^{k_1}}{\|P_{n_0}x^{k_1}\|}\right\| > \frac{\epsilon}{2}.$$

This implies that

$$\left\|\frac{x^{k_1}}{\|P_{n_0}x^{k_1}\|}\right\| \geqq 1 + \delta\left(\frac{\epsilon}{2}\right)$$

and so, by (23),

$$\|x^{k_1}\| \geqq \|P_{n_0}x^{k_1}\|\left(1 + \delta\left(\frac{\epsilon}{2}\right)\right) > \alpha\left(1 + \delta\left(\frac{\epsilon}{2}\right)\right) > 1,$$

which contradicts $\|x^{k_1}\| \leqq 1$.

(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i): Let us assume M fails to satisfy the Δ_2-condition. Choose $n_0 \in \mathbf{N}$ and $x \in h_M$ so that $\|x\| = 1$ and $P_{n_0}x = x$. Select numbers ϵ_k with $0 < \epsilon_k < 1 \quad (k = 1, 2, \ldots)$ so that

$$\lim_{k \to \infty} \epsilon_k = 0.$$

We shall define inductively a sequence of elements y^k in h_M with finite supports so that the following holds for all k:

$$(24) \quad \begin{cases} \text{supp } x < \text{supp } y^1 < \text{supp } y^2 < \ldots < \text{supp } y^k \\ 1 - \epsilon_k < \|y^k\| \leqq 1, \ \|x + y^k\| < 1 + \epsilon_k. \end{cases}$$

Suppose $y^1, \ldots, y^{k-1}$ have been defined for some $k \in \mathbf{N}$ and satisfy the requirements (24). Now choose $t_k > 0$ so that

$$(25) \quad M(t_k) < \epsilon_k$$

and

$$(26) \quad M\left(\frac{t_k}{1 + \epsilon_k}\right) \leqq \gamma_k M(t_k),$$

where

$$(27) \quad \gamma_k := \sum_{n=1}^{n_0} M(|x_n|) - \sum_{n=1}^{n_0} M\left(\frac{|x_n|}{1 + \epsilon_k}\right).$$

(This choice of t_k is possible by the failure of the Δ_2-condition.) Let $m_k \in \mathbf{N}$ satisfy

$$(28) \quad \frac{1}{M(t_k)} - 1 \leqq m_k \leqq \frac{1}{M(t_k)}$$

and put $n_{k-1} := \max \text{supp } y^{k-1}$. We now define $y^k \in h_M$ by

$$(29) \quad y_n^k = \begin{cases} t_k \text{ if } n_{k-1} + 1 \leqq n \leqq n_{k-1} + m_k \\ 0 \text{ if } n \leqq n_{k-1} \text{ or } n \geqq n_{k-1} + m_k + 1. \end{cases}$$

We then have, by (26), (28) and (27),

$$\begin{aligned} \sum_{n=1}^{\infty} M\left(\frac{|x_n + y_n^k|}{1 + \epsilon_k}\right) &= \sum_{n=1}^{n_0} M\left(\frac{|x_n|}{1 + \epsilon_k}\right) + m_k M\left(\frac{t_k}{1 + \epsilon_k}\right) \\ &\leqq \sum_{n=1}^{n_0} M\left(\frac{|x_n|}{1 + \epsilon_k}\right) + m_k \gamma_k M(t_k) \\ &\leqq \sum_{n=1}^{n_0} M\left(\frac{|x_n|}{1 + \epsilon_k}\right) + \gamma_k \\ &= \sum_{n=1}^{n_0} M(|x_n|) = 1, \end{aligned}$$

so

(30) $\|x + y^k\| \leqq 1 + \epsilon_k.$

Since, by (28),

$$\sum_{n=1}^{\infty} M(|y_n^k|) = m_k M(t_k) \leqq 1,$$

and, also by (25)

$$\sum_{n=1}^{\infty} M\left(\frac{|y_n^k|}{1 - \epsilon_k}\right) = m_k M\left(\frac{t_k}{1 - \epsilon_k}\right) \geqq \frac{1}{1 - \epsilon_k} m_k M(t_k)$$

$$\geqq \frac{1}{1 - \epsilon_k}(1 - M(t_k)) > \frac{1}{1 - \epsilon_k}(1 - \epsilon_k) = 1,$$

we have

$$1 - \epsilon_k < \|y^k\| \leqq 1.$$

This completes the inductive construction of (y^k).

Clearly (24) implies

$$\lim_{k \to \infty} \|x + y^k\| = \|x\| = 1 \quad \text{and}$$

$$w^* - \lim_{k \to \infty} (x + y^k) = x,$$

but not

$$\lim_{k \to \infty} (x + y^k) = x.$$

Hence h_M is not (KK*). To show that also (WUKK*) fails, consider the normalized sequence

$$\left(\frac{x + y^k}{\|x + y^k\|}\right).$$

Let $0 < \epsilon < 1$ be arbitrary. Evidently, by (24),

$$\text{sep}\left(\frac{x + y^k}{\|x + y^k\|}\right)_{k=k_0}^{\infty} > \epsilon$$

for sufficiently large k_0, whereas on the other hand the normalized sequence still w^*-converges to the unit vector x. This contradicts (WUKK*) and thus completes the proof.

We now aim for a characterization of the Δ_2-condition in terms of the non-existence of certain diametral sequences (Proposition 5). A corollary of this result will be that the Δ_2-condition for M is equivalent to l_M having w^*-normal structure.

We first deal with the case of substitution spaces.

PROPOSITION 4. *Let M be an Orlicz function satisfying the Δ_2-condition* and let X_n *be* (UKK) *for every $n \in \mathbf{N}$. Then*

$$X: = \left(\sum_{n=1}^{\infty} \oplus X_n\right)_{h_M}$$

has weak normal structure.

Proof. If not, then, by a well-known argument of M. S. Brodskii and D. P. Milman ([**4**], [**7**]) there exists in X a w-diametral sequence, i.e., a sequence (x^k) with the following properties:

$$(31)\quad \begin{cases} x^k \xrightarrow{w} 0, \ \|x^k\| \leqq 1 \quad (k = 1, 2, \dots), \ \operatorname{diam}\{x^k : k \in \mathbf{N}\} = 1, \\ \lim\limits_{k\to\infty} \|x^k\| = 1 \text{ and } \lim\limits_{k\to\infty} d(x^{k+1}, \operatorname{co}\{x^1, \dots, x^k\}) = 1. \end{cases}$$

This will lead to a contradiction. From

$$w - \lim_{k\to\infty} x^k = 0$$

it follows that

$$w - \lim_{k\to\infty} x_n^k = 0$$

in X_n for every $n \in \mathbf{N}$. Our first objective is to show that in fact

$$(32)\quad \lim_{k\to\infty} x_n^k = 0 \quad \text{for every } n \in \mathbf{N}.$$

Suppose not. Then we may assume by passing to a subsequence if necessary, that there exist $n \in \mathbf{N}$ and $\epsilon > 0$ such that

$$(33)\quad \|x_n^k\| \geqq \epsilon \quad (k = 1, 2, \dots)$$

and

$$(34)\quad \operatorname{sep}(x_n^k)_{k=1}^{\infty} \geqq \epsilon.$$

Now let $\delta = \delta(\epsilon)$ be chosen as in the definition of (UKK) for X_n, let $K < \infty$ satisfy (2) in Lemma 1 for $t_0 = 2M^{-1}(1)$, and let γ satisfy

$$(35)\quad 0 < \gamma < \frac{1}{2} \quad \text{and} \quad \left[1 + M\left(\frac{\delta\epsilon}{1-\gamma}\right)\right](1-\gamma)^K > 1.$$

We now pick $k_0 \in \mathbf{N}$ so that

$$(36)\quad 1 \geqq \|x^k\| > 1 - \gamma \quad \text{for } k \geqq k_0.$$

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The sequence $(x_m^{k_0} - x_m^k)_{k=k_0+1}^\infty$ converges weakly to $x_m^{k_0}$ for each m, so

(37) $\liminf_{k\to\infty} \|x_m^{k_0} - x_m^k\| \geqq \|x_m^{k_0}\| \quad (m = 1, 2, \dots).$

Again passing to a subsequence if necessary, we may further assume that

(38) $L: = \lim_{k\to\infty} \|x_n^{k_0} - x_n^k\|$ exists.

Notice that, by (31) and (34),

(39) $0 < \epsilon \leqq L \leqq 1.$

Now the normalized sequence

$$\left(\frac{x_n^{k_0} - x_n^k}{\|x_n^{k_0} - x_n^k\|}\right)_{k=k_0+1}^\infty$$

converges weakly to $x_n^{k_0}/L$ and has separation constant $\geqq\epsilon$ (by (34) and since $\|x_n^{k_0} - x_n^k\| \leqq 1$ for all k), so by the definition of δ we get

$$\left\|\frac{x_n^{k_0}}{L}\right\| \leqq 1 - \delta,$$

or

(40) $\|x_n^{k_0}\| \leqq (1 - \delta)L.$

We now claim that

(41) $$\liminf_{k\to\infty} \|x^{k_0} - x^k\| \geqq \inf\left\{\rho > 0: \sum_{m\neq n} M\left(\frac{\|x_m^{k_0}\|}{\rho}\right) + M\left(\frac{L}{\rho}\right) \leqq 1\right\}.$$

Indeed, let us denote the right member of (41) by ρ_0 and let $\rho < \rho_0$ be arbitrary. Then

$$\sum_{m\neq n} M\left(\frac{\|x_m^{k_0}\|}{\rho}\right) + M\left(\frac{L}{\rho}\right) > 1,$$

so for some $l \geqq n$,

$$\sum_{\substack{m=1\\ m\neq n}}^{l} M\left(\frac{\|x_m^{k_0}\|}{\rho}\right) + M\left(\frac{L}{\rho}\right) > 1.$$

Using (37) and (38) it then follows that for sufficiently large k,

$$\sum_{m=1}^{l} M\left(\frac{\|x_m^{k_0} - x_m^k\|}{\rho}\right) > 1,$$

so a fortiori

$$\liminf_{k\to\infty} \|x^{k_0} - x^k\| \geqq \rho$$

and this proves (41).

We shall now show that $\rho_0 > 1$. This contradicts (41) (since the left member of (41) is $\leqq 1$) and therefore proves (32). Indeed, by (40), (39), the choice of K, (36) and (35), we have

$$\begin{aligned}
&\sum_{m\neq n} M(\|x_m^{k_0}\|) + M(L) \\
&= \sum_{m\neq n} M(\|x_m^{k_0}\|) + M((1-\delta)L + \delta L) \\
&\geqq \sum_{m\neq n} M(\|x_m^{k_0}\|) + M(\|x_n^{k_0}\| + \delta\epsilon) \\
&\geqq \sum_{m\neq n} M(\|x_m^{k_0}\|) + M(\|x_n^{k_0}\|) + M(\delta\epsilon) \\
&= \sum_{m=1}^{\infty} M(\|x_m^{k_0}\|) + M(\delta\epsilon) \\
&\geqq \left[\sum_{m=1}^{\infty} M\left(\frac{\|x_m^{k_0}\|}{1-\gamma}\right) + M\left(\frac{\delta\epsilon}{1-\gamma}\right)\right](1-\gamma)^K \\
&> \left[1 + M\left(\frac{\delta\epsilon}{1-\gamma}\right)\right](1-\gamma)^K > 1.
\end{aligned}$$

Hence $\rho_0 > 1$.

In the remainder of the proof we assign different meanings to γ and δ. Let $\delta > 0$ satisfy (5) for $c = \frac{1}{2}$, where (x_n) is the standard basis (e_n) of h_M (recall that by Proposition 1 (5) holds for (e_n)) and choose $\gamma > 0$ so that

(42) $\dfrac{1-3\gamma}{1-2\gamma} > \dfrac{1}{2}$ and

(43) $(1-2\gamma)(1+\delta) > 1.$

Again let $k_0 \in \mathbf{N}$ be so large that

(44) $\|x^k\| > 1 - \gamma$ for $k \geqq k_0$.

Fix $n_0 \in \mathbf{N}$ so that

(45) $\|(I - P_{n_0})x^{k_0}\| < \gamma$ and

(46) $\|P_{n_0}x^{k_0}\| > 1 - \gamma.$

Using (32) we now pick $k_1 > k_0$ so that

(47) $\|P_{n_0}x^{k_1}\| < \gamma$

and therefore, by (44),

(48) $\|(I - P_{n_0})x^{k_1}\| > 1 - 2\gamma.$

(46) and (47) now yield

(49) $\|P_{n_0}(x^{k_0} - x^{k_1})\| \geqq \|P_{n_0}x^{k_0}\| - \|P_{n_0}x^{k_1}\| > 1 - 2\gamma$

and (48) and (45) that

(50) $$\|(I - P_{n_0})(x^{k_0} - x^{k_1})\| \geqq \|(I - P_{n_0})x^{k_1}\| - \|(I - P_{n_0})x^{k_0}\| > 1 - 3\gamma.$$

Thus, by (42)

(51) $$\left\|P_{n_0}\left(\frac{x^{k_0} - x^{k_1}}{1 - 2\gamma}\right)\right\| > 1 \quad \text{and}$$
$$\left\|(I - P_{n_0})\left(\frac{x^{k_0} - x^{k_1}}{1 - 2\gamma}\right)\right\| > \frac{1 - 3\gamma}{1 - 2\gamma} > \frac{1}{2}.$$

The definition of δ now implies that

$$\frac{\|x^{k_0} - x^{k_1}\|}{1 - 2\gamma} \geqq 1 + \delta,$$

so, by (43),

$$\|x^{k_0} - x^{k_1}\| \geqq (1 - 2\gamma)(1 + \delta) > 1.$$

This contradicts the fact that

$$\operatorname{diam}\{x^k : k \in \mathbf{N}\} = 1,$$

so the proof is complete.

Remark 3. Other results are known about the preservation of normal structure under the formation of sums ([**2**], [**11**]), but either they concern finite sums, or all summands are required to be uniformly convex. Let us also observe that by Proposition 4 the space in Example (f) of [**6**] has weak normal structure, although it is not (WUKK).

PROPOSITION 5. *An Orlicz function M satisfies the Δ_2-condition if and only if there does not exist a sequence (x^k) in h_M with the following properties*:

(52) $$\begin{cases} x^k \xrightarrow{w^*} 0, \ \|x^k\| \leqq 1 \quad (k = 1, 2, \dots), \ \operatorname{diam}\{x^k : k \in \mathbf{N}\} = 1, \\ \lim_{k\to\infty} \|x^k\| = 1 \ \textit{and} \ \lim_{k\to\infty} d(x^{k+1}, \operatorname{co}\{x^1, \dots, x^k\}) = 1. \end{cases}$$

Proof. We shall call a sequence satisfying (52) a w^*-diametral sequence. The "only if" part of the assertion is immediate from the final part of the proof of Proposition 4. (Note that in this scalar case the coordinatewise convergence (32) is given.)

Let us now assume that the Δ_2-condition fails to hold for M. We shall inductively define a w^*-diametral sequence in h_M. We begin by choosing three sequences $(a_k)_{k=1}^{\infty}$, $(b_k)_{k=0}^{\infty}$ and $(c_k)_{k=1}^{\infty}$ of positive numbers, all strictly increasing to 1 such that

(53) $b_k < a_k < c_k \quad (k = 1, 2, \dots)$

and

(54) $$\frac{1}{a_{k+1}} = \alpha_k \frac{1}{c_{k+1}} + (1 - \alpha_k)\frac{1}{b_k} \quad (k = 0, 1, 2, \dots)$$

where the α_k are numbers satisfying

(55) $3/5 < \alpha_k < 1$

(it is easily verified that such sequences exist).

Next we define inductively a sequence (x^k) in h_M with the following properties for all $k \in \mathbf{N}$:

(56) $b_{k-1} \leqq \|x^k\| < a_k$

(57) $b_k \leqq \operatorname{diam}\{x^1, \dots, x^{k+1}\} \leqq c_{k+1}$

(58) $b_k \leqq d(x^{k+1}, \operatorname{co}\{x^1, \dots, x^k\}) \leqq c_{k+1}$

(59) $\operatorname{supp} x^k < \operatorname{supp} x^{k+1}$.

Observe that (59) and the boundedness of (x^k) imply that

$$w^* - \lim_{k\to\infty} x^k = 0,$$

so that (x^k) is a w^*-diametral sequence. Put

$$x^1 := \frac{1}{2}(b_0 + a_1)e_1.$$

Assume that $x^1, \dots, x^k$ have been chosen and satisfy the above requirements. Fix $n_0 \in \mathbf{N}$ so that

$$x^i = P_{n_0} x^i \quad \text{for } i = 1, \dots, k.$$

Since by (56) and (53)

$$\|\lambda_1 x^1 + \dots + \lambda_k x^k\| < a_k < c_{k+1}$$

whenever $\lambda_1, \dots, \lambda_k \geqq 0$ and

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$$\sum_{i=1}^{k} \lambda_i = 1,$$

we have

$$(60) \quad d_k := \sup\left\{ \sum_{n=1}^{n_0} M\left(\frac{\lambda_1 x_n^1 + \ldots + \lambda_k x_n^k}{c_{k+1}}\right) : \lambda_1, \ldots, \lambda_k \geqq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} < 1.$$

Using the failure of Δ_2 we now pick $\delta > 0$ so that

$$(61) \quad M\left(\frac{\delta}{b_k}\right) \leqq 1$$

and

$$(62) \quad \frac{M\left(\dfrac{\delta}{a_{k+1}}\right)}{M\left(\dfrac{\delta}{c_{k+1}}\right)} \geqq \max\left\{\frac{1}{1 - d_k}, 3\right\}.$$

From (54) and the convexity of M we get

$$M\left(\frac{\delta}{a_{k+1}}\right) \leqq \alpha_k M\left(\frac{\delta}{c_{k+1}}\right) + (1 - \alpha_k) M\left(\frac{\delta}{b_k}\right).$$

Hence

$$1 \leqq \alpha_k \frac{M\left(\dfrac{\delta}{c_{k+1}}\right)}{M\left(\dfrac{\delta}{a_{k+1}}\right)} + (1 - \alpha_k) \frac{M\left(\dfrac{\delta}{b_k}\right)}{M\left(\dfrac{\delta}{a_{k+1}}\right)}$$

and thus, by (62),

$$1 \leqq \frac{1}{3}\alpha_k + (1 - \alpha_k) \frac{M\left(\dfrac{\delta}{b_k}\right)}{M\left(\dfrac{\delta}{a_{k+1}}\right)}.$$

Consequently, using (55), we find

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$$\frac{M\left(\dfrac{\delta}{b_k}\right)}{M\left(\dfrac{\delta}{a_{k+1}}\right)} \geqq \frac{1 - \alpha_k/3}{1 - \alpha_k} > 2.$$

From this and (61) it follows that

$$\frac{1}{M\left(\dfrac{\delta}{a_{k+1}}\right)} - \frac{1}{M\left(\dfrac{\delta}{b_k}\right)} > \frac{1}{M\left(\dfrac{\delta}{b_k}\right)} \geqq 1.$$

Hence there exists an $m \in \mathbf{N}$ such that

$$\frac{1}{M\left(\dfrac{\delta}{b_k}\right)} \leqq m < \frac{1}{M\left(\dfrac{\delta}{a_{k+1}}\right)},$$

or

$$(63) \quad M\left(\frac{\delta}{a_{k+1}}\right) < \frac{1}{m} \leqq M\left(\frac{\delta}{b_k}\right).$$

We now define x^{k+1} by

$$x_n^{k+1} := \begin{cases} \delta \text{ if } n_0 + 1 \leqq n \leqq n_0 + m \\ 0 \text{ if } 1 \leqq n \leqq n_0 \text{ or } n_0 + m + 1 \leqq n. \end{cases}$$

This x^{k+1} clearly satisfies (59). We verify the other requirements.

$$\begin{aligned} \|x^{k+1}\| &= \inf\left\{\rho > 0: \sum_{n=1}^{\infty} M\left(\frac{|x_n^{k+1}|}{\rho}\right) \leqq 1\right\} \\ &= \inf\left\{\rho > 0: M\left(\frac{\delta}{\rho}\right) \leqq \frac{1}{m}\right\}, \end{aligned}$$

so (63) yields (56). Furthermore, for every choice of $\lambda_1, \dots, \lambda_k \geqq 0$ with

$$\sum_{i=1}^{k} \lambda_i = 1,$$

we have

$$\begin{aligned} &\|x^{k+1} - (\lambda_1 x^1 + \dots + \lambda_k x^k)\| \\ &= \inf\left\{\rho > 0: \sum_{n=1}^{n_0} M\left(\frac{|\lambda_1 x_n^1 + \dots + \lambda_k x_n^k|}{\rho}\right) + mM\left(\frac{\delta}{\rho}\right) \leqq 1\right\}. \end{aligned}$$

Since, by (60), (62) and (63),

$$\sum_{n=1}^{n_0} M\left(\frac{|\lambda_1 x_n^1 + \ldots + \lambda_k x_n^k|}{c_{k+1}}\right) + mM\left(\frac{\delta}{c_{k+1}}\right)$$

$$\leqq d_k + mM\left(\frac{\delta}{c_{k+1}}\right)$$

$$\leqq d_k + m(1 - d_k)M\left(\frac{\delta}{a_{k+1}}\right) < d_k + m(1 - d_k)\frac{1}{m} = 1,$$

we conclude that

$$\|x^{k+1} - (\lambda_1 x^1 + \ldots + \lambda_k x^k)\| < c_{k+1}.$$

Since

$$\|x^{k+1}\| \leqq \|x^{k+1} - (\lambda_1 x^1 + \ldots \lambda_k x^k)\|$$

by (59), and we already know that $\|x^{k+1}\| \geqq b_k$, this establishes (58). Finally, (57) is immediate from (58) and the induction hypothesis, so the proof is complete.

COROLLARY 1. *An Orlicz function M satisfies the Δ_2-condition if and only if l_M (or h_M) has w^*-normal structure and if and only if Chebyshev centers with respect to w^*-compact convex sets in l_M (or h_M) are (norm) compact.*

Proof. If M satisfies Δ_2 then by Proposition 3 l_M is (WUKK*) and even (UKK*). The "only if" now follows from the proofs of Theorem 3 and Theorem 4 in [**6**]. Now suppose the Δ_2-condition fails. Then there exists a w^*-diametral sequence (x^k) in h_M. The construction in the proof of Proposition 5 shows also that we may assume the supports of the x^k to be mutually disjoint. Now let C be the w^*-closed convex hull of this sequence in l_M. Clearly, diam $C = 1$. Let $x \in C$ be arbitrary. Then

$$x = w^* - \lim \sum_{k=1}^{\infty} \lambda_k^{(n)} x^k,$$

where for each $n \in \mathbf{N}$ we have $\lambda_k^{(n)} \geqq 0$ for all k and

$$\sum_{k=1}^{\infty} \lambda_k^{(n)} = 1.$$

It follows that

$$x = \sum_{k=1}^{\infty} \lambda_k x^k$$

with $\lambda_k \geqq 0$ for all k and

$$\sum_{k=1}^{\infty} \lambda_k \leqq 1.$$

Hence

$$\|x - x^k\| \geqq (1 - \lambda_k) \|x^k\| \to 1 \quad \text{as } k \to \infty.$$

This shows that every $x \in C$ is diametral. Since evidently $C \subset h_M$, we have now proved that h_M fails to have w^* normal structure.

Our final result is a fixed point theorem. We shall prove that h_M has the fixed point property if h_M^* is separable, or, equivalently, if the complementary function M^* satisfies the Δ_2-condition. This will follow from a result of J. M. Borwein and B. Sims ([**3**]): every weakly orthogonal Banach lattice X with Riesz angle $\alpha(X) < 2$ has the (FPP). We recall that X is weakly orthogonal if

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| |x_n| \wedge |x_m| \| = 0$$

whenever (x_n) is a weak null sequence, and that the Riesz angle $\alpha(X)$ of X is defined as

$$\sup\{ \| |x| \vee |y| \| : \|x\| \leqq 1, \|y\| \leqq 1\}.$$

PROPOSITION 6. *Let M be an Orlicz function and let M^* satisfy the Δ_2-condition. Then h_M has the fixed point property.*

Proof. We begin by observing that every Orlicz sequence space h_M is weakly orthogonal. This follows directly from the trivial fact that the map $y \to |x| \wedge |y|$ is weak-norm continuous (for every fixed x). By the result quoted above it therefore remains only to show that $\alpha(h_M) < 2$. For the proof of this we shall need that M^* satisfies Δ_2 if and only if

$$\liminf_{t\to 0} \frac{tp(t)}{M(t)} > 1.$$

This fact is proved in [**12**, p. 148] under the assumption that p is continuous, but it holds in general. Indeed, it is not very difficult to show that for every Orlicz function M there exists an equivalent Orlicz function M_1 with continuous derivative p_1 and such that

$$\liminf_{t\to 0} \frac{tp_1(t)}{M_1(t)} = \liminf_{t\to 0} \frac{tp(t)}{M(t)}.$$

Since, by the equivalence of M and M_1, h_M^* is separable (i.e., M^* satisfies Δ_2) if and only if $h_{M_1}^*$ is separable (i.e., M_1^* satisfies Δ_2), the general statement is now clear from the special case mentioned above.

Our objective is to show the existence of a δ such that $0 < \delta < 1$ and

(64) $M(t) \leqq \frac{1}{2}M((2 - \delta)t)$ if $0 \leqq t \leqq s: = M^{-1}(1)$.

Once this is done it follows immediately that

$$\alpha(h_M) \leqq 2 - \delta.$$

Indeed, if $\|x\|, \|y\| \leqq 1$ then, by (64),

$$\sum_{n=1}^{\infty} M\left(\frac{|x_n| \vee |y_n|}{2 - \delta}\right) \leqq \sum_{n=1}^{\infty} \left[M\left(\frac{|x_n|}{2 - \delta}\right) + M\left(\frac{|y_n|}{2 - \delta}\right)\right]$$

$$\leqq \frac{1}{2} \sum_{n=1}^{\infty} [M(|x_n|) + M(|y_n|)] \leqq 1,$$

so $\| |x| \vee |y| \| \leqq 2 - \delta$. Since

$$\liminf_{t \to 0} \frac{tp(t)}{M(t)} > 1,$$

there exist numbers $\epsilon > 0$ and $t_0 > 0$ such that

(65) $tp(t) \geqq (1 + \epsilon)M(t)$ if $0 \leqq t \leqq t_0$.

We first deduce from (65) that

(66) $M(t) \leqq \dfrac{M(\alpha t)}{\alpha^{1+\epsilon}}$ whenever $\alpha > 1$ and $t \in \left[0, \dfrac{t_0}{\alpha}\right]$.

Indeed, for $\alpha > 1$ and $0 \leqq t \leqq t_0/\alpha$ we have, by (65),

$$\log \frac{M(\alpha t)}{M(t)} = \log M(\alpha t) - \log M(t) = \int_t^{\alpha t} \frac{p(s)}{M(s)} ds$$

$$\geqq \int_t^{\alpha t} \frac{1 + \epsilon}{s} ds = \log(\alpha^{1+\epsilon}),$$

so (66) follows.

Let us now consider the case $\alpha > 1$ and $t \in (t_0/\alpha, s]$. We write t as a convex combination of t_0/α and αt,

(67) $$t = \frac{\alpha - 1}{\alpha - \dfrac{t_0}{\alpha t}} \cdot \frac{t_0}{\alpha} + \left(1 - \frac{\alpha - 1}{\alpha - \dfrac{t_0}{\alpha t}}\right) \alpha t.$$

Note that by (66) and the convexity of M we have

$$M\left(\frac{t_0}{\alpha}\right) \leqq \frac{M(t_0)}{\alpha^{1+\epsilon}} = \frac{M\left(\dfrac{t_0}{\alpha t} \cdot \alpha t\right)}{\alpha^{1+\epsilon}} \leqq \frac{t_0}{\alpha t} \cdot \frac{1}{\alpha^{1+\epsilon}} M(\alpha t).$$

Hence (67) yields

$$(68) \quad M(t) \leqq \left(\frac{\alpha - 1}{\alpha - \frac{t_0}{\alpha t}} \cdot \frac{t_0}{\alpha^{2+\epsilon} t} + 1 - \frac{\alpha - 1}{\alpha - \frac{t_0}{\alpha t}} \right) M(\alpha t).$$

Let us denote the factor in brackets by $F(\alpha, t)$. Observe that $F(\alpha, t)$ is uniformly continuous for

$$(\alpha, t) \in \left[\frac{3}{2}, 2\right] \times \left[\frac{t_0}{2}, s\right]$$

and that

$$F(\alpha, t) < \frac{1}{\alpha}$$

(replace ϵ by 0), so in particular, $F(2, t) < \frac{1}{2}$. Hence for suitably small $\delta < \frac{1}{2}$ we have

$$(69) \quad F(2 - \delta, t) \leqq \frac{1}{2} \quad \text{for } t \in \left[\frac{t_0}{2}, s\right].$$

We can also choose δ small enough so that

$$(70) \quad \frac{1}{(2 - \delta)^{1+\epsilon}} < \frac{1}{2}.$$

Substituting (69) and (70) in (68) and (66), respectively, we arrive at (64), thus completing the proof.

Sacrificing some generality, we may now summarize our main results in the scalar case as follows:

COROLLARY 2. *Let M be an Orlicz function such that M^* satisfies the Δ_2-condition. Then h_M has* (FPP). *Moreover, h_M has weak normal structure if and only if M also satisfies Δ_2. The Δ_2-condition for M is also equivalent to each of the properties* (KK), (UKK) *and* (WUKK) *for h_M.*

Proof. Observe that if M^* satisfies Δ_2, then

$$h_{M^*} = l_{M^*} \approx h_M^*,$$

so that the w^* topology on h_M coincides with the weak topology. The assertions are now clear from Propositions 3, 5 and 6.

Remark 4. It is known (cf. [**12**]) that M satisfies Δ_2 if and only if

$$\limsup_{t \to 0} \frac{tp(t)}{M(t)} < \infty.$$

Therefore any Orlicz function M such that

$$\limsup_{t \to 0} \frac{tp(t)}{M(t)} = \infty \quad \text{and} \quad \liminf_{t \to 0} \frac{tp(t)}{M(t)} > 1$$

(it is easy to construct even piecewise linear functions of this kind) furnishes an example of a space with the (FPP) but without weak normal structure. Other such spaces are known of course (**[9]**, **[1]**, **[13]**) but except for c_0 all of them seem to be rather artificial.

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