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ON RAMSEY GRAPH NUMBERS FOR STARS AND STRIPES

BY

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1. **Introduction.** Any term or symbol undefined in this paper is defined in [5]. For graphs F and G, $G \succ F$ means G contains a subgraph isomorphic to F and $E(G)$ denotes the edge set of G. If $E \subseteq E(G)$, $\langle E\rangle$ is the subgraph of G whose edge set is E and whose vertex set is that subset of vertices of G which are incident with edges in E.

Let $G_1, \ldots, G_t$ be given graphs. There exists a smallest integer $r(G_1, G_2, \ldots, G_t)$ such that for all edge partitions $E_1, \ldots, E_t$ of K_n where $n \geq r(G_1, \ldots, G_t)$, for at least one $i \in \{1, \ldots, t\}$, $\langle E_i\rangle \succ G_i$. The value of $r(G_1, \ldots, G_t)$ is called the Ramsey Number of the sequence of graphs $G_1, \ldots, G_t$.

Ramsey graph theory was formulated in [3] from the well-known theorem of Ramsey [7]. Some properties of the numbers $r(G_1, \ldots, G_t)$ were mentioned in [4]. There has been considerable interest in this topic recently. See Harary [6] and Burr [1] for extensive bibliographies.

In this paper we calculate Ramsey Numbers for certain cases when G_i is either a "star"-graph $K_{1,m}$ or a "stripe"-graph mP_2. These are illustrated for $m=5$.

FIGURE 1

2. **Determination of** $r(K_{1,m_1}, \ldots, K_{1,m_{t-1}}, sP_2)$. In order to determine these Ramsey Numbers, we shall require the following theorem proved by Burr and Roberts [2] and independently by the present authors.

THEOREM 1. *Let* $R=r(K_{1,m_1}, \ldots, K_{1,m_t})$ *and* $Z=\sum_{i=1}^{t}(m_i-1)$. *If* Z *is even and some* m_i *is even, then* $R=Z+1$, *otherwise* $R=Z+2$.

An acceptable t-colouring of K_n *will mean a partition* $E_1, \ldots, E_t$ *of* $E(K_n)$ *such that for each* $i=1, \ldots, t-1$, $\langle E_i\rangle \not\succ K_{1,m_i}$ *and* $\langle E_t\rangle s \not\succ P_2$. M *and* $\sum$ *will denote* $r(K_{1,m_i}, \ldots, K_{1,m_{t-1}}, sP_2)$ *and* $\sum_{i=1}^{t-1}(m_i-1)$ *respectively.*

THEOREM 2. *If* $\sum < s$, $M=2s$.

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Proof. The partition $\phi, \ldots, \phi, E(K_{2s-1})$ of $E(K_{2s-1})$ is an acceptable t-colouring. Hence $M > 2s-1$.

Suppose, contrary to the theorem, that K_{2s} has an acceptable t-colouring $E_1, \ldots, E_t$. Then for each $i=1, \ldots, t-1$, the degree of each vertex in $\langle E_i \rangle$ is less than m_i. Therefore the degree of each vertex in $\langle E_t \rangle$ is greater than or equal to $\lambda = 2s-1-\sum$. Since the colouring is acceptable, the maximal matching (see [5] page 96) of $\langle E_t \rangle$ has k independent edges where $k<s$. Let P be the set of k pairs of vertices incident with these k edges, let V be the set of these $2k$ vertices and W be the set of those vertices not in V. We note that $k<s$ implies $|W| \geq 2$. No edge incident with two vertices in W is in E_t or there would be $k+1$ independent edges in $\langle E_t \rangle$. Hence any two vertices w_1, w_2 are each incident with at least λ edges in E_t whose other vertices are in V. Suppose for each $j=1, \ldots, \lambda$, $[w_1, x_j]$ is an edge in E_t where $x_j \in V$. If y_j is the vertex paired with x_j in P, for each $j=1, \ldots, \lambda$ the edge $[y_j, w_2]$ is not in E_t. Otherwise the maximal matching in $\langle E_t \rangle$ could be increased by deleting $[x_j, y_j]$ and adding $[w_1, x_j]$ and $[w_2, y_j]$. Therefore the set S of vertices adjacent to w_2 in $\langle E_t \rangle$ is contained in $V - \{y_1, \ldots, y_\lambda\}$ which has $2k-\lambda$ vertices. But $|S| \geq \lambda$. Hence $\lambda \leq 2k-\lambda$ from which we deduce $\lambda \leq k$. Therefore

$$2s-1-\sum \leq k$$

or

$$(s-k)+(s-1-\sum) = 0.$$

But $s-k>0$ and $(s-1-\sum) \geq 0$ and we have the required contradiction showing that K_{2s} has no acceptable t-colouring.

THEOREM 3. *Let* $\sum \geq s$.

(i) $M = \sum + s$ *if* $\sum$ *is even and some* m_i *is even.*
(ii) $M = \sum + s + 1$ *otherwise.*

Proof. By definition there exists a $(t-1)$-colouring $E_1, \ldots, E_{t-1}$ of the complete graph on $r(K_{1,m_1}, \ldots, K_{1,m_{t-1}})-1$ vertices such that $\langle E_i \rangle \not\supseteq K_{1,m_i}$ for $i=1, \ldots, t-1$. Take the join (see [5] page 21) of this graph with a distinct K_{s-1} and let E_t be the set of edges of the K_{s-1} together with all joining edges. $E_1, \ldots, E_t$ is an acceptable t-colouring of the complete graph on $r(K_{1,m_1}, \ldots, K_{1,m_{t-1}})-1+(s-1)$ vertices. Hence

$$M > r(K_{1,m_1}, \ldots, K_{1,m_{t-1}})+s-2.$$

We now apply theorem 1 and establish that M is greater than or equal to the numbers asserted in this theorem.

The proof of part (ii) of the theorem may now be completed by assuming an acceptable t-colouring of $K_{\Sigma+s+1}$ and obtaining a contradiction by reasoning identical to that used in the proof of theorem 2. We omit the details.

Part (i) seems to be more difficult. Suppose there is an acceptable t-colouring $E_1, \ldots, E_t$ of $K_{\Sigma+s}$. Let $V=\{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ be the set of $2k$ vertices incident with the k edges $\{[x_i, y_i], i=1, \ldots, k\}$ in a maximal matching in $\langle E_t\rangle$ where $k \le s-1$ and let W be the set of vertices not in V. We note $|W|=\sum+s-2k\ge 2$. The degree of each vertex $\bigcup_{i=1}^{t-1}\langle E_i\rangle$ is no more than $\sum$, hence the degree of each vertex in $\langle E_t\rangle$ is at least $(\sum+s-1)-\sum=s-1$.

Let w_1, w_2 be in W. Since there are no edges in E_t which are incident with two vertices in W there are at least $2(s-1)$ edges in E_t from $\{w_1, w_2\}$ to V. The reasoning used in theorem 2 establishes (we omit the details):

(a) $k=s-1$.

(b) For each $w \in W$, the degree of w in $\langle E_t\rangle$ is exactly $s-1$.

(c) For each $i=1, \ldots, s-1$ there are precisely two edges in E_t which join $\{w_1, w_2\}$ to $\{x_i, y_i\}$, and the subgraph of $\langle E_t\rangle$ induced by $\{w_1, w_2, x_i, y_i\}$ is of type A or B depicted in Fig. 2.

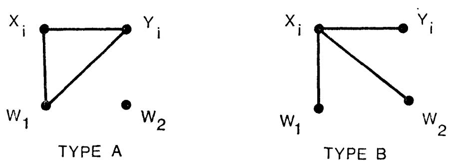

FIGURE 2

Suppose that for all $w_1, w_2 \in W$ and all i, the induced subgraph of $\{w_1, w_2, x_i, y_i\}$ of $\langle E_t\rangle$ is of Type B. Then for each i, one of the vertices x_i, y_i, say x_i is adjacent in $\langle E_t\rangle$ to every vertex in W while y_i is adjacent to no vertex of W. Further, no edge $[y_\alpha, y_\beta]$ is in E_t for otherwise the maximal matching in $\langle E_t\rangle$ could be increased by deletion of $[x_\alpha, y_\alpha]$, $[x_\beta, x_\beta]$ and the addition of $[y_\alpha, y_\beta]$, $[x_\alpha, w_1]$, $[x_\beta, w_2]$. Therefore the graph induced by $\{W \cup \{y_1, \ldots, y_{s-1}\}\}$ is a complete graph on $\sum+1$ vertices whose edges are in $\bigcup_{i=1}^{t-1} E_i$, i.e. we have constructed a partition $F_1, \ldots, F_{t-1}$ of $E(K_{\Sigma+1})$ such that for each i, $\langle F_i\rangle \not\supseteq K_{1,m_1}$. But this is impossible by Theorem 1.

Suppose for some $w_1, w_2 \in W$ and some $i \in \{1, \ldots, s-1\}$ the subgraph induced by $\{w_1, w_2, x_i, y_i\}$ in $\langle E_t\rangle$ is type A. If w_3 is a third point of W then of the subgraphs of $\langle E_t\rangle$ induced by $\{w_1, w_3, x_i, y_i\}$, $\{w_2, w_3, x_i, y_i\}$, one is type B, since otherwise the maximal matching in $\langle E_t\rangle$ could be increased. On the other hand, if $|W|=2$, then $\sum=s$. In this case since $s-1$ is odd, for some $j \in \{1, \ldots, s-1\}$ the induced subgraph of $\{w_1, w_2, x_j, y_j\}$ in $\langle E_t\rangle$ is type B. Thus, in either case, for some w_1, $w_2 \in W$ we may re-index the edges in the maximal matching in $\langle E_t\rangle$ so that the subgraph of $\langle E_t\rangle$ induced by $\{w_1, w_2, x_j, y_j\}$ is type A for $j=1, \ldots, \lambda$ and type B for $j=\lambda+1, \ldots, s-1$, where $1 \le \lambda \le s-2$. Suppose the vertices are labelled so that y_j is adjacent to neither w_1 nor w_2 in $\langle E_t\rangle$ for $j=\lambda+1, \ldots, s-1$. Reasoning as in the preceding paragraph shows that no $[y_\alpha, y_\beta]$ where $\alpha, \beta \in \{\lambda+1, \ldots, s-1\}$ is in E_t. Moreover for each $j \in \{1, \ldots, \lambda\}$, neither $[x_j, y_{s-1}]$ nor $[y_j, y_{s-1}]$ is in E_t.

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For suppose $[x_j, y_{s-1}] \in E_t$ where $[x_j, w_1]$, $[y_j, w_1]$ are also in E_t. Then the maximal matching in $\langle E_t\rangle$ may be increased by deletion of $[x_j, y_j]$, $[x_{\lambda-1}, y_{s-1}]$ and addition of $[x_j, y_{s-1}]$, $[y_j, w_1]$ and $[x_{s-1}, w_2]$. Hence the edges $[x_j, y_{s-1}]$, $[y_j, y_{s-1}]$ for $j=1,\ldots,\lambda$, $[y_\alpha, y_{s-1}]$ for $\alpha=\lambda+1,\ldots,s-2$ and $[w, y_{s-1}]$ for each $w \in W$ are in $\bigcup_{j=1}^{t-1} E_i$ and the degree of y_{s-1} in $\bigcup_{i=1}^{t-1} \langle E_i\rangle$ is at least

$$2\lambda+\{(s-2)-(\lambda+1)+1\}+(\sum -s+2) = \sum+\lambda \geq \sum+1.$$

Hence for some $i \in \{1,\ldots,t-1\}$, y_{s-1} has degree $>m_i-1$ in $\langle E_i\rangle$, i.e., $\langle E_i\rangle > K_{1,m_i}$.

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