Can. J. Math. Vol. 45 (6), 1993 pp. 1135–1154

DENSE SUBGROUPS OF THE AUTOMORPHISM GROUPS OF FREE ALGEBRAS

ROGER M. BRYANT AND VESSELIN DRENSKY

ABSTRACT Let F be the free metabelian Lie algebra of finite rank m over a field K of characteristic 0 The automorphism group $\mathrm{Aut}\,F$ is considered with respect to a topology called the *formal power series topology* and it is shown that the group of tame automorphisms (automorphisms induced from the free Lie algebra of rank m) is dense in $\mathrm{Aut}\,F$ for $m \geq 4$ but not dense for $m = 2$ and $m = 3$ At a more general level, we study the formal power series topology on the semigroup of all endomorphisms of an arbitrary (associative or non-associative) relatively free algebra of finite rank m and investigate certain associated modules of the general linear group $\mathrm{GL}_m(K)$

Introduction. Let K be a field of characteristic 0 and let L_m be the free Lie algebra over K of finite rank m freely generated by $x_1, \ldots, x_m$. The general linear group $\mathrm{GL}_m(K)$ acts naturally on the m-dimensional subspace of L_m spanned by $\{x_1, \ldots, x_m\}$ and we can extend this action so that $\mathrm{GL}_m(K)$ becomes a group of algebra automorphisms of L_m. If $m \geq 2$ and f belongs to the subalgebra of L_m generated by $\{x_2, \ldots, x_m\}$ then the endomorphism τ_f of L_m defined by

$$\tau_f(x_1) = x_1 + f, \quad \tau_f(x_i) = x_i \quad (i \neq 1),$$

is clearly an automorphism of L_m. By a result of Cohn [8], $\mathrm{Aut}\,L_m$ is generated by $\mathrm{GL}_m(K)$ and the automorphisms τ_f.

The main purpose of this paper is to study the automorphism group of the free metabelian Lie algebra L_m/L_m'' where L_m'' is the second derived algebra of L_m. Those automorphisms which belong to the image of the canonical homomorphism $\mathrm{Aut}\,L_m \longrightarrow \mathrm{Aut}\,L_m/L_m''$ are called *tame*. One of the questions which motivated our work was the question of whether every automorphism of L_m/L_m'' is tame.

The analogous question has been answered completely for the free metabelian groups Γ_m/Γ_m'' (where Γ_m is the free group of rank m): every automorphism of Γ_m/Γ_m'' is tame when $m \neq 3$ (see [2, 4, 12]) but Γ_3/Γ_3'' has non-tame automorphisms (see [7, 3]).

By Cohn's result, $\mathrm{Aut}\,L_2 = \mathrm{GL}_2(K)$. It follows that L_2/L_2'' has non-tame automorphisms: if v is a non-zero element of the derived algebra of L_2/L_2'' then the mapping of L_2/L_2'' defined by $u \longmapsto u + [u, v]$ for all $u \in L_2/L_2''$ is an automorphism which is clearly not induced by an element of $\mathrm{GL}_2(K)$ (see also [14, Proposition 4]). To study $\mathrm{Aut}\,L_m/L_m''$ for $m \geq 3$ we make use of a topology on $\mathrm{Aut}\,L_m/L_m''$ called the *formal power series topology* (see Section 2). We prove in Section 3 that the set of tame automorphisms is

Received by the editors April 8, 1992
AMS subject classification Primary 17B01, 17B30, 17B40, secondary 17A50, 17A36

dense in $\operatorname{Aut} L_m/L_m''$ for all $m \geq 4$ but is not dense when $m = 3$. In particular L_3/L_3'' has non-tame automorphisms.

Since the completion of our work we have been informed by Yu. A. Bahturin that he and S. Nabiyev have now proved that L_m/L_m'' has non-tame automorphisms for all $m \geq 2$ [6]. This nicely supplements our main result and shows that no exact analogue exists of the group theoretic results.

In order to study $\operatorname{Aut} L_m/L_m''$ we develop techniques which apply in a wider setting. We investigate the endomorphisms of arbitrary finitely generated relatively free algebras over K. The relevant background on relatively free algebras is described in Section 1. Our techniques are based on a combination of the methods of Anick [1] and Drensky and Gupta [9]. Anick considered the formal power series topology on the set of endomorphisms of the polynomial algebra $K[x_1, \ldots, x_m]$. He proved that the endomorphisms with invertible Jacobian matrix form a closed subset J and that the group of tame automorphisms is dense in J. Drensky and Gupta applied the representation theory of $\mathrm{GL}_m(K)$ to investigate the automorphisms of relatively free nilpotent Lie algebras. We shall develop some of these ideas further.

Let $\mathfrak{U}$ be any variety of algebras over K, let $F = F_m(\mathfrak{U})$ be the relatively free algebra of $\mathfrak{U}$ of rank m, and let $E = \operatorname{End} F$ be the semigroup of all algebra endomorphisms of F. As in the special case where $F = L_m$ we can regard $\mathrm{GL}_m(K)$ as a subgroup of $\operatorname{Aut} F$; thus $\mathrm{GL}_m(K) \subseteq E$. For $k \geq 2$ and any subsemigroup H of E, let $I_k H$ be the set of elements of H which induce the identity map on F/F^k. Thus $H \supseteq I_2 H \supseteq I_3 H \supseteq \cdots$ and each $I_k H$ is a subsemigroup of H. For $\phi, \psi \in E$ write $\phi \equiv_{k+1} \psi$ if ϕ and ψ induce the same endomorphism on F/F^{k+1}. Then it is easily verified that $\equiv_{k+1}$ is a congruence on E. We show in Section 1 that the quotient semigroup $I_k E/\equiv_{k+1}$ can be given the structure of a $K\mathrm{GL}_m(K)$-module, where the action of $\mathrm{GL}_m(K)$ comes from conjugation within E, and we determine the structure of this module. Furthermore, in Section 2 we show that the direct sum

$$\mathcal{L}(E) = \bigoplus_{k \geq 2} I_k E/\equiv_{k+1}$$

acquires the structure of a graded Lie algebra over K.

If H is any subgroup of $\operatorname{Aut} F$ then $I_k H/I_{k+1} H$ can be identified with a subgroup of $I_k E/\equiv_{k+1}$. Making this identification we show that if H is $\mathrm{GL}_m(K)$-invariant then $I_k H/I_{k+1} H$ is a $K\mathrm{GL}_m(K)$-submodule of $I_k E/\equiv_{k+1}$ and

$$\mathcal{L}(H) = \bigoplus_{k \geq 2} I_k H/I_{k+1} H$$

is a subalgebra of $\mathcal{L}(E)$. Furthermore we prove that if H_1 and H_2 are subgroups of $\operatorname{Aut} F$ such that $\mathrm{GL}_m(K) \subseteq H_1 \subseteq H_2$ then H_1 is dense in H_2 with respect to the formal power series topology if and only if $\mathcal{L}(H_1) = \mathcal{L}(H_2)$. In Section 3 we apply these ideas to the study of L_m/L_m'' by means of representation theory. We completely determine the $K\mathrm{GL}_m(K)$-modules $I_k T/I_{k+1} T$ and $I_k A/I_{k+1} A$ where T is the group of tame automorphisms of L_m/L_m'' and $A = \operatorname{Aut} L_m/L_m''$.

1. **Relatively free algebras.** Throughout this paper K will be a field of characteristic 0. By an "algebra" we shall mean a vector space R over K endowed with a multiplication which satisfies the left and right distributive laws and the law $a(r_1r_2) = (ar_1)r_2 = r_1(ar_2)$ for all $r_1, r_2 \in R$, $a \in K$. (Thus R is non-unitary and need not be commutative or associative.) Let $\mathfrak{R}$ be the class of all algebras and denote by $F(\mathfrak{R})$ the absolutely free algebra freely generated by the countable set $\{x_1, x_2, \ldots\}$. Thus the elements of $F(\mathfrak{R})$ may be regarded as polynomials without constant terms in non-commuting and non-associative variables. For each positive integer m, $F_m(\mathfrak{R})$ denotes the subalgebra of $F(\mathfrak{R})$ generated by $\{x_1, \ldots, x_m\}$.

If $f = f(x_1, \ldots, x_m) \in F(\mathfrak{R})$ we say that f is a polynomial identity of an algebra R if $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$. For a given subset W of $F(\mathfrak{R})$, the class $\mathfrak{U}$ of all algebras in which all elements of W are polynomial identities is called the *variety of algebras* defined by W. The set $T(\mathfrak{U})$ of all elements of $F(\mathfrak{R})$ which are polynomial identities of all algebras of $\mathfrak{U}$ is an ideal invariant under all endomorphisms of $F(\mathfrak{R})$. The quotient algebra $F(\mathfrak{U}) = F(\mathfrak{R})/T(\mathfrak{U})$ is the so-called relatively free algebra of $\mathfrak{U}$ of countable rank, freely generated by the set $\{y_1, y_2, \ldots\}$ where $y_i = x_1 + T(\mathfrak{U})$ for all i. Similarly $F_m(\mathfrak{U}) = F_m(\mathfrak{R})/\big(F_m(\mathfrak{R}) \cap T(\mathfrak{U})\big)$ is a relatively free algebra of $\mathfrak{U}$ of rank m. We identify it with the subalgebra of $F(\mathfrak{U})$ generated by $\{y_1, \ldots, y_m\}$, so that $F_m(\mathfrak{U})$ is freely generated by $\{y_1, \ldots, y_m\}$. If $r_1, \ldots, r_m$ are elements of any algebra R of $\mathfrak{U}$ then there is a unique homomorphism $\phi: F_m(\mathfrak{U}) \longrightarrow R$ such that $\phi(y_i) = r_i$ $(1 \leq i \leq m)$. For a fixed variety $\mathfrak{U}$ and fixed m we now write $F = F_m(\mathfrak{U})$.

We may write $F_m(\mathfrak{R}) = \oplus_{k\geq 1} F_m(\mathfrak{R})_{(k)}$ where $F_m(\mathfrak{R})_{(k)}$ is the subspace of $F_m(\mathfrak{R})$ spanned by all monomials of total degree k in $x_1, \ldots, x_m$. Since K is infinite we may see by a Vandermonde determinant argument that

$$F_m(\mathfrak{R}) \cap T(\mathfrak{U}) = \bigoplus_{k\geq 1} \big(F_m(\mathfrak{R})_{(k)} \cap T(\mathfrak{U})\big).$$

Thus we may write F as a sum of homogeneous components, $F = \oplus_{k\geq 1} F_{(k)}$, where

$$F_{(k)} \cong F_m(\mathfrak{R})_{(k)}/\big(F_m(\mathfrak{R})_{(k)} \cap T(\mathfrak{U})\big)$$

and $F_{(k)}$ is the subspace of F spanned by all monomials of total degree k in $y_1, \ldots, y_m$. Each element f of F may be written uniquely in the form $f = \sum_{k\geq 1} f_{(k)}$ with $f_{(k)} \in F_{(k)}$ for all k and $f_{(k)} = 0$ for all but finitely many k. We say that $f_{(k)}$ is the homogeneous component of f of degree k. Similarly, for any m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ of non-negative integers we write $F_\alpha = F_{(\alpha_1, \ldots, \alpha_m)}$ for the multi-homogeneous component corresponding to α; that is, the subspace of F spanned by all monomials of total degree α_i in y_i for $i = 1, \ldots, m$. Then, by similar arguments to those above, $F = \oplus_\alpha F_\alpha$ where α ranges over all m-tuples. Note that, for each positive integer k, $F^k = \oplus_{i\geq k} F_{(i)}$.

We write G for the general linear group $\mathrm{GL}_m(K)$ and let G act in the natural way on the subspace $F_m(\mathfrak{R})_{(1)}$ of $F_m(\mathfrak{R})$ spanned by $x_1, \ldots, x_m$. We extend this action so that G acts on $F_m(\mathfrak{R})$ by algebra automorphisms. Clearly the subspaces $F_m(\mathfrak{R}) \cap T(\mathfrak{U})$ and $F_m(\mathfrak{R})_{(k)}$, $k \geq 1$, are G-invariant. Thus G acts as a group of automorphisms of F such that each

$F_{(k)}$ is a KG-submodule. From now on we assume that $\mathfrak{U}$ is non-trivial, *i.e.*, $x_1 \notin T(\mathfrak{U})$. Thus $F_{(1)}$ has basis $\{y_1, \ldots, y_m\}$ and $F_{(1)}$ is the natural KG-module. In particular G acts faithfully on F and we may regard G as a subgroup of $\operatorname{Aut} F$. We write $E = \operatorname{End} F$ for the semigroup of all (algebra) endomorphisms of F.

For each integer k, $k \geq 2$, let $I_k E$ be the set of endomorphisms of F which induce the identity map on F/F^k and write $IE = I_2 E$. Thus

$$E \supseteq IE = I_2E \supseteq I_3E \supseteq \cdots$$

and each I_kE is a subsemigroup of E. For $\phi, \psi \in E$ and $k \geq 1$ we write $\phi \equiv_k \psi$ if ϕ and ψ induce the same endomorphism on F/F^k or, equivalently, $\phi(y_i) - \psi(y_i) \in F^k$ for $i = 1, \ldots, m$. It is easily verified that $\equiv_k$ is a congruence on E. For $k \geq 2$ we write $I_kE/ \equiv_{k+1}$ for the quotient semigroup of I_kE corresponding to the congruence $\equiv_{k+1}$.

For any element ϕ of I_kE let $\nu_k(\phi) = (f_1, \ldots, f_m)$ where $f_i = \big(\phi(y_i)\big)_{(k)}$ is the homogeneous component of $\phi(y_i)$ of degree k, $i = 1, \ldots, m$. Thus $\phi(y_i) \equiv y_i + f_i \pmod{F^{k+1}}$, $i = 1, \ldots, m$, and $\nu_k(\phi) \in F_{(k)}^{\oplus m}$ (the direct sum of m copies of the additive group $F_{(k)}$). It is easily verified that $\nu_k \colon I_kE \longrightarrow F_{(k)}^{\oplus m}$ is an epimorphism of semigroups. Clearly, for $\phi, \psi \in I_kE$, $\nu_k(\phi) = \nu_k(\psi)$ if and only if $\phi \equiv_{k+1} \psi$. Thus ν_k induces an isomorphism of semigroups $\tilde{\nu}_k \colon I_kE/ \equiv_{k+1} \longrightarrow F_{(k)}^{\oplus m}$. In particular, $I_kE/ \equiv_{k+1}$ is an abelian group. Furthermore, since $F_{(k)}^{\oplus m}$ is a vector space over K we can give $I_kE/ \equiv_{k+1}$ a similar structure so that $\tilde{\nu}_k$ becomes a vector space isomorphism. More explicitly, if $[\phi] \in I_kE/ \equiv_{k+1}$ is represented by $\phi \in I_kE$ and if $a \in K$ then $a[\phi]$ is represented by the endomorphism ϕ_1 defined by $\phi_1(y_i) = y_i + af_i$, for all i, where $\nu_k(\phi) = (f_1, \ldots, f_m)$.

As observed above, $F_{(1)}$ is the natural KG-module with basis $\{y_1, \ldots, y_m\}$. It will sometimes be convenient to regard elements of G as $m \times m$ matrices, corresponding to the ordered basis $\{y_1, \ldots, y_m\}$ of $F_{(1)}$. Since $G \subseteq E$ we can let G act by conjugation on E. Then it is easily verified that each I_kE is G-invariant and that if ϕ and ψ are elements of I_kE satisfying $\phi \equiv_{k+1} \psi$ then $g\phi g^{-1} \equiv_{k+1} g\psi g^{-1}$ for all $g \in G$. Thus G acts on $I_kE/ \equiv_{k+1}$. It is also easy to see that the action of G on $I_kE/ \equiv_{k+1}$ commutes with multiplication by elements of K. Thus $I_kE/ \equiv_{k+1}$ is a KG-module.

The action of G on $I_kE/ \equiv_{k+1}$ is most easily written down using the map ν_k. Let $\phi \in I_kE$, $g \in G$ and $\nu_k(\phi) = (f_1, \ldots, f_m)$. Then ν_k maps $g\phi g^{-1}$ to $\big(g(f_1), \ldots, g(f_m)\big)g^{-1}$. Here $g(f_i)$ is calculated in the G-module $F_{(k)}$, g^{-1} is regarded as an $m \times m$ matrix, and multiplication by g^{-1} is multiplication of a $1 \times m$ matrix by an $m \times m$ matrix. Let $N(1)^*$ be the vector space of $1 \times m$ row-vectors over K regarded as a left KG-module in which, for each $g \in G$, g acts as right multiplication by g^{-1} (in other words, $N(1)^*$ is the dual of the natural KG-module $N(1)$) and regard $F_{(k)} \otimes_K N(1)^*$ as a KG-module under the "diagonal" action of G. Then the map

$$\nu_k(\phi) = (f_1, \ldots, f_m) \longmapsto f_1 \otimes (1, 0, \ldots, 0) + \cdots + f_m \otimes (0, \ldots, 0, 1)$$

determines a KG-module isomorphism from $I_kE/ \equiv_{k+1}$ to $F_{(k)} \otimes_K N(1)^*$. Thus we have established the following result.

https://doi.org/10.4153/CJM-1993-063-5 Published online by Cambridge University Press

THEOREM 1.1. *Let $\mathfrak{U}$ be a non-trivial variety of algebras, let $F = F_m(\mathfrak{U})$ be the relatively free algebra of finite rank m in $\mathfrak{U}$ and let $G = \mathrm{GL}_m(K)$. Then, for $k \geq 2$, there is a KG-module isomorphism*

$$I_kE/\equiv_{k+1} \cong F_{(k)} \otimes_K N(1)^*$$

where $N(1)^*$ is the dual of the natural KG-module $N(1)$.

The proof we have given applies to any infinite field K (without need of our assumption that char $K = 0$) and is based on the proof of [9, Theorem 2.1].

Before proceeding further we need to summarise some information about KG-modules, particularly (finite dimensional) polynomial KG-modules (see [10] for basic facts and definitions). For an arbitrary integer n we write $(\det)^n$ to denote a one-dimensional KG-module which affords the representation $g \longmapsto (\det g)^n$ for all $g \in G$ (where $\det g$ is the determinant of g). Every polynomial KG-module is a direct sum of irreducible ones. The irreducible polynomial modules are indexed (up to isomorphism) by the m-tuples of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $\lambda_1 \geq \cdots \geq \lambda_m$. Such an m-tuple with $\lambda_1 + \cdots + \lambda_m = k$ is called a *partition* of k into m parts and $\mathrm{Part}(k)$ denotes the set of all such partitions. For $\lambda = (\lambda_1, \ldots, \lambda_m)$ the irreducible polynomial module corresponding to λ will be denoted by $N(\lambda)$ or $N(\lambda_1, \ldots, \lambda_m)$. The modules $N(\lambda)$ with $\lambda \in \mathrm{Part}(k)$ are precisely those irreducible polynomial modules which are homogeneous of degree k. Associated with each polynomial module W is an element of $\mathbb{Z}[X_1, \ldots, X_m]$ called the *character* of W; and the character of $N(\lambda)$ has leading term $X_1^{\lambda_1} \cdots X_m^{\lambda_m}$. When writing partitions we shall make use of standard abbreviations: thus, for example, $(2, 2, 1, 1, 1, 0)$ may be written as $(2^2, 1^3)$.

It is well known (and easy to verify by inspecting characters) that the m-dimensional natural KG-module is isomorphic to $N(1)$, and $(\det)^1 \otimes_K N(1)^* \cong N(1^{m-1})$. Thus $N(1)^* \cong (\det)^{-1} \otimes_K N(1^{m-1})$ and Theorem 1.1 may be re-stated as follows.

COROLLARY 1.2 (SEE [9, THEOREM 2.1]). *For $k \geq 2$ there is a KG-module isomorphism*

$$I_kE/\equiv_{k+1} \cong (\det)^{-1} \otimes_K N(1^{m-1}) \otimes_K F_{(k)}.$$

It is easily verified that $F_{(k)}$ is a homogeneous polynomial KG-module of degree k. Thus $F_{(k)}$ can be decomposed as a direct sum of modules each of which is isomorphic to some $N(\lambda)$ with $\lambda \in \mathrm{Part}(k)$.

We shall be particularly interested in varieties of Lie algebras (see [5]). Then, in all the above, we may replace $F(\mathfrak{N})$ by the free Lie algebra L freely generated by $\{x_1, x_2, \ldots\}$ and replace $F_m(\mathfrak{N})$ by the free Lie algebra L_m of rank m freely generated by $x_1, \ldots, x_m$. We may take polynomial identities as coming from L and take relatively free Lie algebras of rank m as quotient algebras of L_m. The following result is well known. (For a proof see, for example, [9, Lemma 3.4].)

PROPOSITION 1.3. *Let $F = L_m/L_m''$ be the free metabelian Lie algebra of finite rank $m \geq 2$ and let $G = \mathrm{GL}_m(K)$. Then the homogeneous components of F satisfy the KG-module isomorphisms $F_{(1)} \cong N(1)$ and $F_{(k)} \cong N(k-1, 1)$, $k \geq 2$.*

The tensor product of polynomial modules can be calculated by means of the Littlewood-Richardson rule. (For the rule itself see [11]. The application to $\mathrm{GL}_m(K)$ is well known and is stated in [9, Proposition 1.4].) Thus by Proposition 1.3 and Corollary 1.2 we can find the structure of the modules $I_kE/\equiv_{k+1}$ in the case where $F = L_m/L_m''$. The results are as follows (essentially as stated in [9, Lemma 3.5]).

PROPOSITION 1.4. *Let $F = L_m/L_m''$, where $m \geq 2$.*
(i) For $m = 2$, $I_2E/\equiv_3 \cong N(1)$ and $I_kE/\equiv_{k+1} \cong N(k-2, 1) \oplus N(k-1)$, $k \geq 3$.
(ii) For $m \geq 3$, $I_2E/\equiv_3 \cong \big((\det)^{-1} \otimes_K N(2^2, 1^{m-3})\big) \oplus N(1)$ and

$$I_kE/\equiv_{k+1} \cong \big((\det)^{-1} \otimes_K N(k, 2, 1^{m-3})\big) \oplus N(k-2, 1) \oplus N(k-1), \quad k \geq 3.$$

2. **Endomorphisms and automorphisms.** We now return to the general situation where $F = F_m(\mathfrak{U})$ and $\mathfrak{U}$ is a non-trivial variety of algebras. We shall continue to use all the notation of Section 1. In particular, $E = \mathrm{End}\, F$ and $G = \mathrm{GL}_m(K)$.

We consider the topology on F corresponding to the series $F \supseteq F^2 \supseteq F^3 \supseteq \cdots$; that is, the topology in which the sets $f + F^k$ ($f \in F$, $k \geq 1$) form a basis for the open sets. Since each element ϕ of E corresponds uniquely to an m-tuple $\big(\phi(y_1), \ldots, \phi(y_m)\big)$ we may give E the topology of the direct product $F \times \cdots \times F$ of m copies of F. We call this topology the *formal power series topology* on E, following Anick [1]. (This topology can be described by the metric satisfying $d(\phi, \psi) = 0$ if $\phi = \psi$ and $d(\phi, \psi) = \exp(-k)$ if $\phi \neq \psi$ and k is maximal subject to $\phi \equiv_k \psi$.)

We aim to construct a graded Lie algebra $\mathcal{L}(E)$. In order to do this it is convenient to utilise the completions of F and E. The completion $\hat{F}$ of F with respect to the series $F \supseteq F^2 \supseteq \cdots$ may be identified with the complete (unrestricted) direct sum $\widehat{\bigoplus}_{i \geq 1} F_{(i)}$. It has a natural algebra structure such that F is a subalgebra of $\hat{F}$. Each element of $\hat{F}$ may be regarded as an infinite formal sum $f = \sum_{i \geq 1} f_{(i)}$ with $f_{(i)} \in F_{(i)}$ for all i. For each $k \geq 1$ let $\hat{F}^{(k)}$ be the set of all such elements f with $f_{(i)} = 0$ for $i < k$. (In other words $\hat{F}^{(k)}$ is the completion of F^k.) Clearly the topology that $\hat{F}$ inherits from F is the same as the topology on $\hat{F}$ obtained from the series $\hat{F} \supseteq \hat{F}^{(2)} \supseteq \cdots$. It is straightforward to prove the following result.

LEMMA 2.1. *If $w_1, \ldots, w_m$ are arbitrary elements of $\hat{F}$ then there is a unique continuous endomorphism ϕ of $\hat{F}$ such that $\phi(y_i) = w_i$, $i = 1, \ldots, m$.*

Let $\hat{E}$ be the semigroup of all continuous endomorphisms of $\hat{F}$. Then Lemma 2.1 shows that each element ϕ of $\hat{E}$ corresponds uniquely to an element $\big(\phi(y_1), \ldots, \phi(y_m)\big)$ of the direct product $\hat{F} \times \cdots \times \hat{F}$ of m copies of $\hat{F}$. Clearly the set $\hat{E}$ with the topology of this direct product may be identified with the completion of E and we call this topology on $\hat{E}$ the *formal power series topology*. Note also that E is a subsemigroup of $\hat{E}$.

Because $\hat{F}^{(k)}$ is the closure of F^k, $\phi(\hat{F}^{(k)}) \subseteq \hat{F}^{(k)}$ for all $\phi \in \hat{E}$. For $k \geq 2$ we let $I_k\hat{E}$ be the set of all elements of $\hat{E}$ which induce the identity map on $\hat{F}/\hat{F}^{(k)}$. Thus $E \cap I_k\hat{E} = I_kE$ and $I_k\hat{E}$ is the completion of I_kE. We also write $I\hat{E} = I_2\hat{E}$.

LEMMA 2.2. *$I\hat{E}$ is a group.*

PROOF. Clearly $I\hat{E}$ is a subsemigroup of $\hat{E}$. Let $\phi \in I\hat{E}$. Then it is easy to see that ϕ induces the identity map on each factor $\hat{F}^{(k)}/\hat{F}^{(k+1)}$. Thus ϕ induces an automorphism of $\hat{F}/\hat{F}^{(k+1)}$. It follows that for each k there is an element ϕ_k of E such that $\phi\phi_k$ and $\phi_k\phi$ induce the identity map on $\hat{F}/\hat{F}^{(k+1)}$. The limit of the maps ϕ_k is an inverse of ϕ in $I\hat{E}$. Thus each element of $I\hat{E}$ is invertible.

It follows from Lemma 2.2 that each $I_k\hat{E}$ is a normal subgroup of $I\hat{E}$ and the topology induced on $I\hat{E}$ from $\hat{E}$ is the same as the topology associated with the series $I\hat{E} = I_2\hat{E} \supseteq I_3\hat{E} \supseteq \cdots$.

For each $k \geq 2$ we can extend the homomorphism $\nu_k \colon I_kE \longrightarrow F_{(k)}^{\oplus m}$ to a group homomorphism $\nu_k \colon I_k\hat{E} \longrightarrow F_{(k)}^{\oplus m}$ in the obvious way. Thus ν_k induces a group isomorphism $\bar{\nu}_k \colon I_k\hat{E}/I_{k+1}\hat{E} \longrightarrow F_{(k)}^{\oplus m}$. For each $k \geq 2$ we write

$$\bar{I}_kE = I_k\hat{E}/I_{k+1}\hat{E} = (I_kE)(I_{k+1}\hat{E})/I_{k+1}\hat{E}.$$

Thus $\bar{I}_kE \cong I_kE/\equiv_{k+1}$. Furthermore we can use the map $\bar{\nu}_k$ to give $\bar{I}_kE$ the structure of a vector space over K so that $\bar{\nu}_k \colon \bar{I}_kE \longrightarrow F_{(k)}^{\oplus m}$ is a vector space isomorphism. Since $G \subseteq E \subseteq \hat{E}$, G acts by conjugation on $\hat{E}$ and $\bar{I}_kE$ becomes a KG-module. Clearly $\bar{I}_kE$ and $I_kE/\equiv_{k+1}$ are isomorphic as KG-modules.

The following result is similar to several well known results and is straightforward to prove by direct calculation.

LEMMA 2.3. *Let $\phi \in I_j\hat{E}$ and $\psi \in I_k\hat{E}$ ($j,k \geq 2$). Then the group commutator $\phi^{-1}\psi^{-1}\phi\psi$ satisfies $\phi^{-1}\psi^{-1}\phi\psi \in I_{j+k-1}\hat{E}$. Furthermore, if $\nu_j(\phi) = (f_1,\ldots,f_m)$ and $\nu_k(\psi) = (g_1,\ldots,g_m)$ then $\nu_{j+k-1}(\phi^{-1}\psi^{-1}\phi\psi) = (h_1,\ldots,h_m)$ where, for $i = 1,\ldots,m$,*

$$h_i = \big(g_i(y_1+f_1,\ldots,y_m+f_m)\big)_{(j+k-1)} - \big(f_i(y_1+g_1,\ldots,y_m+g_m)\big)_{(j+k-1)}.$$

(Recall that, for $f \in F$, $f_{(j+k-1)}$ denotes the homogeneous component of f of degree $j + k - 1$.)

REMARK 2.4. In the notation of Lemma 2.3 we can write

$$\big(f_i(y_1+g_1,\ldots,y_m+g_m)\big)_{(j+k-1)} = f_i'(y_1,\ldots,y_m,g_1,\ldots,g_m),$$
$$\big(g_i(y_1+f_1,\ldots,y_m+f_m)\big)_{(j+k-1)} = g_i'(y_1,\ldots,y_m,f_1,\ldots,f_m),$$

where f_i' is linear in $g_1,\ldots,g_m$ (that is, a linear combination of monomials in $y_1,\ldots,y_m$, $g_1,\ldots,g_m$ each of which contains precisely one factor from $g_1,\ldots,g_m$) and g_i' is linear in $f_1,\ldots,f_m$.

PROPOSITION 2.5. *Let $E = \mathrm{End}\, F$ where $F = F_m(\mathfrak{U})$. Then the vector space direct sum $\mathcal{L}(E) = \bigoplus_{k\geq 2} \bar{I}_k E$ has the structure of a graded Lie algebra over K with $\bar{I}_k E$ as component of degree $k-1$ in the grading and Lie multiplication given by*

$$[\phi I_{j+1}\hat{E}, \psi I_{k+1}\hat{E}] = (\phi^{-1}\psi^{-1}\phi\psi)I_{j+k}\hat{E}$$

for all $\phi \in I_j\hat{E}$, $\psi \in I_k\hat{E}$ $(j,k \geq 2)$. Furthermore $G = \mathrm{GL}_m(K)$ acts on $\mathcal{L}(E)$ as a group of Lie algebra automorphisms.

PROOF. By Lemma 2.3 the mutual commutator groups $(I_j\hat{E}, I_k\hat{E})$ of the terms of the series $I_2\hat{E} \supseteq I_3\hat{E} \supseteq \cdots$ satisfy $(I_j\hat{E}, I_k\hat{E}) \subseteq I_{j+k-1}\hat{E}$ for all $j,k \geq 2$. Therefore the direct sum of abelian groups $\mathcal{L}(E) = \bigoplus_{k\geq 2}(I_k\hat{E}/I_{k+1}\hat{E})$ may be given the structure of a graded Lie ring in the standard way such that

$$[\phi I_{j+1}\hat{E}, \psi I_{k+1}\hat{E}] = (\phi^{-1}\psi^{-1}\phi\psi)I_{j+k}\hat{E}$$

for all $\phi \in I_j\hat{E}$, $\psi \in I_k\hat{E}, j,k \geq 2$. (See [13, Part I, Chapter II].)

We have to show that $\mathcal{L}(E)$ is a Lie algebra over K. Let $\phi \in I_j\hat{E}$, $\psi \in I_k\hat{E}$ $(j,k \geq 2)$ and let $a \in K$. In the notation of Lemma 2.3 and Remark 2.4,

$$\begin{aligned} a\Big(\big(\phi^{-1}\psi^{-1}\phi\psi(y_i)\big)_{(j+k-1)}\Big) &= ag_i'(y_1,\ldots,y_m,f_1,\ldots,f_m) - af_i'(y_1,\ldots,y_m,g_1,\ldots,g_m) \\ &= g_i'(y_1,\ldots,y_m,af_1,\ldots,af_m) - af_i'(y_1,\ldots,y_m,g_1,\ldots,g_m) \\ &= \big(g_i(y_1+af_1,\ldots,y_m+af_m)\big)_{(j+k-1)} \\ &\qquad - \big(af_i(y_1+g_1,\ldots,y_m+g_m)\big)_{(j+k-1)} \\ &= \big(\phi_1^{-1}\psi^{-1}\phi_1\psi(y_i)\big)_{(j+k-1)} \end{aligned}$$

where $\phi_1 \in I_j\hat{E}$ is defined by $\phi_1(y_i) = y_i + af_i$, $i = 1,\ldots,m$. Thus

$$a[\phi I_{j+1}\hat{E}, \psi I_{k+1}\hat{E}] = [a\phi I_{j+1}\hat{E}, \psi I_{k+1}\hat{E}],$$

and $\mathcal{L}(E)$ is a Lie algebra over K. It is easy to verify that the action of G on $\hat{E}$ by conjugation induces an action of G on $\mathcal{L}(E)$ by Lie algebra automorphisms.

Note that, for $\phi \in I_jE$, $\psi \in I_kE$, $(\phi^{-1}\psi^{-1}\phi\psi)I_{j+k}\hat{E}$ depends only on the elements $\big(\phi(y_i)\big)_{(j)}$ and $\big(\psi(y_i)\big)_{(k)}$. Thus the Lie algebra operations on $\mathcal{L}(E)$ can be defined purely in terms of E rather than $\hat{E}$.

For any subgroup H of $\mathrm{Aut}\, F$ we write $I_kH = H \cap I_k\hat{E}$, $k \geq 2$, and $IH = I_2H$. Thus I_kH is the set of elements of H which induce the identity map on F/F^k and is a normal subgroup of H. We also write $\bar{I}_kH = I_kH(I_{k+1}\hat{E})/I_{k+1}\hat{E}$. Since $I_kH \cap I_{k+1}\hat{E} = I_{k+1}H$, $\bar{I}_kH$ is naturally isomorphic to $I_kH/I_{k+1}H$. It is convenient to use $\bar{I}_kH$ rather than $I_kH/I_{k+1}H$ because of the inclusion $\bar{I}_kH \subseteq \bar{I}_kE$. Thus if H_1 and H_2 are subgroups of $\mathrm{Aut}\, F$ with $H_1 \subseteq H_2$ we have $\bar{I}_kH_1 \subseteq \bar{I}_kH_2$. The topology induced on H from E is clearly the same as the topology corresponding to the series $H \supseteq I_2H \supseteq I_3H \supseteq \cdots$.

PROPOSITION 2.6. *Let H be a subgroup of* Aut F *which is invariant under conjugation by elements of G. Then, for $k \geq 2$, $\bar{I}_k H$ is a KG-submodule of $\bar{I}_k E$.*

PROOF. It is easy to verify that $\bar{I}_k H$ is invariant under the action of G. It remains to show that it is closed under multiplication by elements of K. We repeat arguments from [1, Lemma 6] and [9, Lemma 3.1]. Let $\phi \in I_k H$ and $a \in K$. Since $\bar{\nu}_k \colon \bar{I}_k E \longrightarrow F_{(k)}^{\oplus m}$ is a vector space isomorphism, it is enough to prove that $a\nu_k(\phi) \in \nu_k(I_k H)$. Suppose first that a is rational: $a = p/q$ where p and q are integers ($q \neq 0$). Let d be the scalar matrix of G with all diagonal entries equal to $1/q$ and let $n = pq^{k-2}$. Then, by an easy calculation,

$$\nu_k\big((d\phi d^{-1})^n\big) = n\nu_k(d\phi d^{-1}) = n(1/q^{k-1})\nu_k(\phi) = a\nu_k(\phi).$$

Thus $a\nu_k(\phi) \in \nu_k(I_k H)$, as required. Now let a be a non-rational element of K. For $r = 0, 1, \ldots, k-1$, let d_r be the scalar matrix of G with all diagonal entries equal to $a+r$. Then

$$\nu_k(d_r \phi d_r^{-1}) = (a+r)^{k-1}\nu_k(\phi)$$

and so $(a+r)^{k-1}\nu_k(\phi) \in \nu_k(I_k H)$ for $r = 0, 1, \ldots, k-1$. But a can be written as a linear combination of $(a+0)^{k-1}, \ldots, \big(a + (k-1)\big)^{k-1}$ with rational coefficients. Thus $a\nu_k(\phi) \in \nu_k(I_k H)$, as required.

PROPOSITION 2.7. *Let H be a G-invariant subgroup of* Aut F. *Then $\mathcal{L}(H) = \oplus_{k\geq 2} \bar{I}_k H$ is a graded Lie algebra over K which is a G-invariant graded subalgebra of $\mathcal{L}(E)$.*

PROOF. By Proposition 2.6, $\bar{I}_k H$ is a subspace of $\bar{I}_k E$ for all k. By the definition of the Lie product in $\mathcal{L}(E)$, $[\bar{I}_j H, \bar{I}_k H] \subseteq \bar{I}_{j+k-1} H$ for all $j, k \geq 2$. The result follows.

If $\bar{I}_k H$ is identified with $I_k H / I_{k+1} H$ for each k then it is clear that $\mathcal{L}(H)$ is the same as the Lie algebra $\oplus_{k\geq 2}(I_k H / I_{k+1} H)$ obtained by means of group commutators from the series $IH = I_2 H \supseteq I_3 H \supseteq \cdots$.

PROPOSITION 2.8. *Let H_1 and H_2 be G-invariant subgroups of* Aut F *such that $H_1 \subseteq H_2$. Then IH_1 is dense in IH_2 with respect to the formal power series topology on* End F *if and only if $\mathcal{L}(H_1) = \mathcal{L}(H_2)$.*

PROOF. Suppose that IH_1 is dense in IH_2 and let $\phi \in I_k H_2$, $k \geq 2$. Then there exists $\psi \in IH_1$ such that $\psi^{-1}\phi \in I_{k+1} H_2$. Hence $\psi \in I_k H_1$ and so $I_k H_2 = (I_k H_1)(I_{k+1} H_2)$. Thus, for all k, $\bar{I}_k H_1 = \bar{I}_k H_2$ and so $\mathcal{L}(H_1) = \mathcal{L}(H_2)$. The converse is similar.

COROLLARY 2.9. *Let H_1 and H_2 be subgroups of* Aut F *such that $G \subseteq H_1 \subseteq H_2$. Then H_1 is dense in H_2 if and only if $\mathcal{L}(H_1) = \mathcal{L}(H_2)$.*

PROOF. Note that $H_i = G(IH_i)$, $i = 1, 2$. If H_1 is dense in H_2 then clearly IH_1 is dense in IH_2. Conversely, if IH_1 is dense in IH_2 then, for all $k \geq 2$, $IH_2 = (IH_1)(I_{k+1} H_2)$ and so

$$H_2 = G(IH_2) = G(IH_1)(I_{k+1} H_2) = H_1(I_{k+1} H_2),$$

which implies that H_1 is dense in H_2. The result now follows from Proposition 2.8.

3. **Automorphisms of free metabelian Lie algebras.** Let $m \geq 2$ and let L_m be the free Lie algebra of rank m freely generated by $x_1, \dots, x_m$. We shall study the free metabelian Lie algebra L_m / L_m'' of rank m freely generated by $y_1, \dots, y_m$ where $y_i = x_i + L_m''$, $i = 1, \dots, m$. From now on we write $F = L_m / L_m''$ and use all the notation previously developed for $F = F_m(\mathfrak{U})$ in the special case where $\mathfrak{U}$ is the variety of all metabelian Lie algebras. In particular, recall that $E = \operatorname{End} F$, $A = \operatorname{Aut} F$ and $G = \mathrm{GL}_m(K)$. Furthermore T will denote the group of all tame automorphisms of F. We use commutator notation for the Lie multiplication in F: thus F^k, as used previously, now denotes $[F, F, \dots, F]$ with k factors.

Let $\Omega = K[t_1, \dots, t_m]$ be the (commutative, associative, unitary) polynomial algebra over K freely generated by variables $t_1, \dots, t_m$. For $k \geq 0$ write $\Omega_{(k)}$ for the homogeneous component of Ω of degree k and $\Omega^{(k)} = \bigoplus_{i \geq k} \Omega_{(i)}$. Note that every element of the derived algebra F' of F may be written in the form

$$\sum_{1 \leq i,j \leq m} [y_i, y_j] f_{ij}(\operatorname{ad} y_1, \dots, \operatorname{ad} y_m)$$

where $f_{ij}(t_1, \dots, t_m) \in \Omega$ for all i, j. (For each $v \in F$, $\operatorname{ad} v \colon F \longrightarrow F$ is defined by $u(\operatorname{ad} v) = [u, v]$ for all $u \in F$.)

We shall use a special case of the idea of the wreath product of Lie algebras as introduced by Shmel'kin [14]. Let A_m and B_m be abelian Lie algebras (in other words vector spaces over K) with bases $\{a_1, \dots, a_m\}$ and $\{t_m, \dots, t_m\}$, respectively, and let C_m be the free right Ω-module with free generators $a_1, \dots, a_m$. Then the wreath product $A_m \operatorname{wr} B_m$ is defined to be the vector space $C_m \oplus B_m$ made into a Lie algebra over K in such a way that C_m and B_m are abelian subalgebras and

$$[a_i f(t_1, \dots, t_m), t_j] = a_i f(t_1, \dots, t_m) t_j$$

for all $f(t_1, \dots, t_m) \in \Omega$ and all $i, j \in \{1, \dots, m\}$. Thus C_m is an ideal and $A_m \operatorname{wr} B_m$ is metabelian.

As a special case of Shmel'kin's embedding theorem [14, Theorem 1], the homomorphism $\varepsilon \colon F \longrightarrow A_m \operatorname{wr} B_m$ defined by $\varepsilon(y_i) = a_i + t_i$ $(1 \leq i \leq m)$ is a Lie algebra monomorphism. If

$$f = \sum [y_i, y_j] f_{ij}(\operatorname{ad} y_1, \dots, \operatorname{ad} y_m)$$

then

$$\varepsilon(f) = \sum (a_i t_j - a_j t_i) f_{ij}(t_1, \dots, t_m).$$

LEMMA 3.1. *The element $\sum_{i=1}^m a_i f_i(t_1, \dots, t_m)$ of C_m belongs to $\varepsilon(F')$ if and only if $\sum_{i=1}^m t_i f_i(t_1, \dots, t_m) = 0$.*

PROOF. This follows from [14, Theorem 2]. It may also be proved directly as in [4, Proposition 3.1].

Our next objective is to give a matrix representation for IE which is similar to the well known representation for endomorphisms of a free metabelian group (see [4]).

https://doi.org/10.4153/CJM-1993-063-5 Published online by Cambridge University Press

Let $M = M_m(\Omega)$ be the associative algebra of all $m \times m$ matrices with entries from Ω. For $k \geq 0$ let $M_{(k)} = M_m(\Omega_{(k)})$ be the subspace of M consisting of those matrices (f_{ij}) such that $f_{ij} \in \Omega_{(k)}$ for all i, j and let $M^{(k)} = \oplus_{i \geq k} M_{(i)}$. The series $M = M^{(0)} \supseteq M^{(1)} \supseteq \cdots$ determines a topology on M with completion $\hat{M}$ where $\hat{M} = \widehat{\oplus}_{i \geq 0} M_{(i)}$ (complete direct sum). Thus $\hat{M}$ may be identified with the algebra of all $m \times m$ matrices over the formal power series algebra $K[[t_1, \ldots, t_m]]$.

Let S be the subspace of M defined by

$$S = \left\{ (f_{ij}) \in M : \sum_{i=1}^{m} t_i f_{ij} = 0, j = 1, \ldots, m \right\}$$

and, for $k \geq 1$, let $S_{(k)} = S \cap M_{(k)}$ and $S^{(k)} = S \cap M^{(k)}$. It is easily verified that $S = \oplus_{k \geq 1} S_{(k)}$ and $S^{(k)} = \oplus_{i \geq k} S_{(i)}$, $k \geq 1$. The condition $\sum_{i=1}^{m} t_i f_{ij} = 0, j = 1, \ldots, m$, may be written as $(t_1, \ldots, t_m)(f_{ij}) = (0, \ldots, 0)$, or, alternatively, $(t_1, \ldots, t_m)\big(1 + (f_{ij})\big) = (t_1, \ldots, t_m)$, where 1 denotes the identity matrix. Thus S is a right ideal of M and $1 + S$ is a multiplicative semigroup. We write $\hat{S}$ for the closure of S in $\hat{M}$ and $\hat{S}^{(k)}$ for the closure of $S^{(k)}$, $k \geq 1$. Thus $\hat{S} = \widehat{\oplus}_{k \geq 1} S_{(k)}$ and $\hat{S}^{(k)} = \widehat{\oplus}_{i \geq k} S_{(i)}$.

For $\phi \in IE$ we can write $\phi(y_j) = y_j + f_j$ with $f_j \in F'$, $j = 1, \ldots, m$. Thus, by Lemma 3.1, we can write

$$\varepsilon\big(\phi(y_j)\big) = a_j + t_j + \sum_{i=1}^{m} a_i f_{ij}, \quad j = 1, \ldots, m,$$

where the f_{ij} are elements of Ω such that $(f_{ij}) \in S$. Let $\mu(\phi)$ denote the endomorphism of the free Ω-module C_m defined by

$$\mu(\phi)(a_j) = a_j + \sum_{i=1}^{m} a_i f_{ij}, \quad j = 1, \ldots, m,$$

and identify the endomorphism algebra of C_m with M in the obvious way. Thus $\mu(\phi) \in 1 + S$ for all $\phi \in IE$.

PROPOSITION 3.2. *The mapping $\mu: IE \longrightarrow 1 + S$ is a semigroup isomorphism such that, for all $k \geq 2$, $\mu(I_k E) = 1 + S^{(k-1)}$ and $\mu(IA)$ is the set of invertible matrices of $1 + S$. Furthermore μ extends to a continuous group isomorphism $\hat{\mu}: I\hat{E} \longrightarrow 1 + \hat{S}$.*

PROOF. It is straightforward to check that μ is a semigroup monomorphism. By Lemma 3.1, for every matrix $(f_{ij}) \in S$ there exist elements $f_1, \ldots, f_m \in F'$ such that $\varepsilon(f_j) = \sum_{i=1}^{m} a_i f_{ij}, j = 1, \ldots, m$, and consequently the element ϕ of IE defined by $\phi(y_j) = y_j + f_j, j = 1, \ldots, m$, satisfies $\mu(\phi) = 1 + (f_{ij})$. Thus μ is surjective.

It may easily be verified that, for $f \in F$ and $k \geq 2$, $\varepsilon(f) \in \sum_{i=1}^{m} a_i \Omega^{(k-1)}$ if and only if $f \in F^k$. Thus

$$\mu(I_k E) = 1 + (S \cap M^{(k-1)}) = 1 + S^{(k-1)}.$$

Since $1 + S$ is the set of matrices fixing $(t_1, \ldots, t_m)$, the inverse of an invertible matrix of $1 + S$ also belongs to $1 + S$. Thus, for $\phi \in IE$, we have $\phi \in IA$ if and only if $\mu(\phi)$ is invertible.

By the above description of $\varepsilon(f)$ for $f \in F^k$, we see that, for $\phi, \psi \in IE$ and $k \geq 2$, $\phi \equiv_k \psi$ if and only if $\mu(\phi) - \mu(\psi) \in M^{(k-1)}$. Hence μ sends Cauchy sequences of IE to Cauchy sequences of M. It follows easily that μ extends to a continuous semigroup isomorphism $\hat{\mu}\colon I\hat{E} \longrightarrow 1 + \hat{S}$. Since $I\hat{E}$ is a group (by Lemma 2.2) so is $1 + \hat{S}$, and $\hat{\mu}$ is a group isomorphism.

Since S is a graded associative algebra, $S = \bigoplus_{k\geq 1} S_{(k)}$, it has the structure of a graded Lie algebra over K under the commutator operation defined by $[s_1, s_2] = s_1s_2 - s_2s_1$ for all $s_1, s_2 \in S$.

PROPOSITION 3.3. *For $k \geq 2$, $\hat{\mu}$ induces a semigroup epimorphism $\mu_k\colon I_k\hat{E} \longrightarrow S_{(k-1)}$ from $I_k\hat{E}$ to the additive group $S_{(k-1)}$. The maps μ_k induce vector space isomorphisms $\bar{\mu}_k\colon \bar{I}_kE \longrightarrow S_{(k-1)}$ and an isomorphism of graded Lie algebras from $\mathcal{L}(E)$ to S.*

PROOF. Clearly $\hat{\mu}(I_k\hat{E}) = 1 + \hat{S}^{(k-1)}$ for all $k \geq 2$. There is a group homomorphism from $1 + \hat{S}^{(k-1)}$ on to the additive group $S_{(k-1)}$ defined by $1 + u_{(k-1)} + u_{(k)} + \cdots \longmapsto u_{(k-1)}$, where $u_{(i)} \in S_{(i)}$ for all i. This induces a group isomorphism δ_k from $(1+\hat{S}^{(k-1)})/(1+\hat{S}^{(k)})$ to $S_{(k-1)}$. Thus we obtain a group epimorphism $\mu_k\colon I_k\hat{E} \longrightarrow S_{(k-1)}$ and a group isomorphism $\bar{\mu}_k\colon \bar{I}_kE \longrightarrow S_{(k-1)}$. It is easy to check that $\bar{\mu}_k$ is a vector space isomorphism. Since $\hat{\mu}\colon I\hat{E} \longrightarrow 1 + \hat{S}$ is a group isomorphism and $\hat{\mu}(I_k\hat{E}) = 1 + \hat{S}^{(k-1)}$ for all $k \geq 2$ we obtain an isomorphism from $\mathcal{L}(E)$ to the graded Lie ring

$$\mathcal{L}(1 + \hat{S}) = \bigoplus_{k\geq 2}(1 + \hat{S}^{(k-1)})/(1 + \hat{S}^{(k)}).$$

It is easy to prove that the maps δ_k give an isomorphism of graded Lie rings from $\mathcal{L}(1 + \hat{S})$ to $S = \bigoplus_{k\geq 2} S_{(k-1)}$. (One can calculate directly or use the logarithm map and the Campbell-Hausdorff formula.) Thus the maps $\bar{\mu}_k$ give an isomorphism of graded Lie rings from $\mathcal{L}(E)$ to S. Clearly this isomorphism is also an isomorphism of Lie algebras over K.

By Proposition 3.2, $IE \cong 1 + S$. We next calculate the action of G on $1 + S$ which corresponds to the action of G by conjugation on IE. Let G act in the natural way on $\Omega_{(1)}$ and extend this action so that G becomes a group of unitary algebra automorphisms of Ω. Let $\phi \in IE$ and $\mu(\phi) = 1 + (f_{ij})$. It is easy to see that, for all $g \in G$,

$$\mu(g\phi g^{-1}) = g\Big(1 + \big(g(f_{ij})\big)\Big)g^{-1}$$

where $\big(g(f_{ij})\big) \in M, g \in G \subseteq M$ and the triple product on the right hand side is the matrix product in M. We can identify M with $\Omega \otimes_K M_m(K)$. Then the action of G on $1 + S$ is the restriction to $1 + S$ of the "diagonal" action of G on $\Omega \otimes_K M_m(K)$ where G acts on Ω as described above and G acts by conjugation on $M_m(K)$. From now on when we regard M or a subspace of M as a KG-module it is always assumed that the G-action is the one just described. It is straightforward to prove the following fact.

LEMMA 3.4. *The maps $\bar{\mu}_k\colon \bar{I}_kE \longrightarrow S_{(k-1)}$, $k \geq 2$, are KG-module isomorphisms. Here $S_{(k-1)}$ is a submodule of $M_{(k-1)} = \Omega_{(k-1)} \otimes_K M_m(K)$.*

We shall now summarise some properties of (finite dimensional) rational KG-modules. For the purposes of this paper we may define a rational KG-module as one which is isomorphic to a module of the form $(\det)^{-n} \otimes_K V$ where V is a polynomial module, n is a non-negative integer, and $(\det)^{-n}$ is as defined in Section 1. Most of the properties of rational modules we need follow from elementary properties of polynomial modules as given in [10].

For $i,j \in \{1,\ldots,m\}$, let e_{ij} be the element of $M_m(K)$ or of $M = M_m(\Omega)$ which has entry 1 in the (i,j)-th position and 0 elsewhere. For $z_1,\ldots,z_m \in K \setminus \{0\}$ let

$$d(z_1,\ldots,z_m) = z_1 e_{11} + \cdots + z_m e_{mm}$$

be the corresponding diagonal element of G. If W is any rational KG-module and $\alpha = (\alpha_1,\ldots,\alpha_m)$ is any ordered m-tuple of integers, the *weight space* W^α of W is defined to be the set of those elements w of W for which $d(z_1,\ldots,z_m)(w) = z_1^{\alpha_1}\cdots z_m^{\alpha_m} w$ for all $z_1,\ldots,z_m \in K \setminus \{0\}$. The elements of W^α are called *homogeneous of weight* α. Each rational module W is the vector space direct sum of its weight spaces: $W = \oplus_\alpha W^\alpha$. If $w \in W$ and $w = \sum_\alpha w_\alpha$ with $w_\alpha \in W^\alpha$ for each α then we shall call w_α the component of w of weight α. Every rational module is a direct sum of irreducible ones. The only irreducible rational modules (up to isomorphism) are the modules $(\det)^{-n} \otimes_K N(\lambda)$, where $n \geq 0$ and $N(\lambda)$ is the irreducible polynomial module corresponding to $\lambda = (\lambda_1,\ldots,\lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ as in Section 1. The weight spaces of $(\det)^{-n} \otimes_K N(\lambda)$ and $N(\lambda)$ are related by

$$\big((\det)^{-n} \otimes_K N(\lambda)\big)^{(\alpha_1-n,\ \ldots,\alpha_m-n)} = (\det)^{-n} \otimes_K N(\lambda)^{(\alpha_1,\ \ldots,\alpha_m)}.$$

Furthermore, $N(\lambda)^\alpha \neq \{0\}$ only if $\alpha_1,\ldots,\alpha_m$ are non-negative integers satisfying $\alpha_1 + \cdots + \alpha_m = \lambda_1 + \cdots + \lambda_m$, and the dimension of $N(\lambda)^\alpha$ in this case is the number of semistandard tableaux of shape λ and content α. (In the terminology of [11], $\dim N(\lambda)^\alpha$ is the number of tableaux of shape λ and weight α: see also [9, Proposition 1.3].)

Regard $M_m(K)$ as a KG-module, as before, with G acting by conjugation. Then $M_m(K)$ is easily seen to be rational, and for $i,j \in \{1,\ldots,m\}$ the element e_{ij} is homogeneous of weight $(\varepsilon_1,\ldots,\varepsilon_m)$ where $\varepsilon_r = 0$ for $r \notin \{i,j\}$, $\varepsilon_i = \varepsilon_j = 0$ if $i = j$, and $\varepsilon_i = 1$ and $\varepsilon_j = -1$ if $i \neq j$. It follows that the module $M_{(k-1)} = \Omega_{(k-1)} \otimes_K M_m(K)$ is also rational and for all non-negative integers $\alpha_1,\ldots,\alpha_m$ with $\alpha_1 + \cdots + \alpha_m = k-1$ the element $t_1^{\alpha_1}\cdots t_m^{\alpha_m} \otimes e_{ij}$ is homogeneous of weight $(\alpha_1+\varepsilon_1,\ldots,\alpha_m+\varepsilon_m)$ where $\varepsilon_1,\ldots,\varepsilon_m$ are as above. Consequently each weight space of $M_{(k-1)}$ is spanned by those elements $t_1^{\alpha_1}\cdots t_m^{\alpha_m} \otimes e_{ij}$ which belong to it, and the weight components of any element of $M_{(k-1)}$ may be calculated by expressing it as a linear combination of elements $t_1^{\alpha_1}\cdots t_m^{\alpha_m} \otimes e_{ij}$.

We shall now begin a detailed study of $A = \operatorname{Aut} F$. For $\phi \in I_k E \subseteq I_k \hat{E}$ it will be convenient to write $\bar{\phi} = \phi I_{k+1}\hat{E}$ to denote the corresponding element of $\bar{I}_k E$.

For each element f of the subalgebra $F(y_2,\ldots,y_m)$ of F generated by $y_2,\ldots,y_m$ we define $\tau_f \in A$ by $\tau_f(y_1) = y_1 + f$ and $\tau_f(y_i) = y_i$ $(i \neq 1)$. By the description of $\operatorname{Aut} L_m$ given in the introduction, each τ_f is tame and the group T of tame automorphisms of F is generated by G together with the set of elements τ_f. If $f = f_1 + \cdots + f_n$ where $f_i \in F(y_2,\ldots,y_m)_{(i)}$, $i = 1,\ldots,n$, then $\tau_{f_1} \in G$ and $\tau_f = \tau_{f_1}\cdots\tau_{f_n}$. Thus T is generated by G together with those τ_f for which f is homogeneous of degree at least 2. Since $g\tau_f = (g\tau_f g^{-1})g$ for all $g \in G$, T can be written as a product of subgroups,

$$T = \Big\langle g\tau_f g^{-1} : g \in G, f \in \bigcup_{k\geq 2} F(y_2,\dots,y_m)_{(k)}\Big\rangle G.$$

Since $IT \cap G = \{1\}$ it follows that

$$IT = \Big\langle g\tau_f g^{-1} : g \in G, f \in \bigcup_{k\geq 2} F(y_2,\dots,y_m)_{(k)}\Big\rangle.$$

For $k \geq 2$ let P_k be the KG-submodule of $\bar{I}_k E$ generated by those elements $\bar{\tau}_f$ for which $f \in F(y_2,\dots,y_m)_{(k)}$. Note that when $m = 2$ we have $P_k = \{0\}$ for all k. For $f \in F(y_2,\dots,y_m)_{(k)} \subseteq F_{(k)}$ we can write f as a finite sum $f = \sum_\alpha f_\alpha$ where each α has the form $\alpha = (\alpha_2,\dots,\alpha_m)$ for non-negative integers $\alpha_2,\dots,\alpha_m$ with $\alpha_2+\cdots+\alpha_m = k$ and where $f_\alpha \in F_\alpha(y_2,\dots,y_m)$ is the multi-homogeneous component of f corresponding to α. Then τ_f is the product of the automorphisms τ_{f_α}. Hence P_k is generated by those $\bar{\tau}_f$ for which $f \in F(y_2,\dots,y_m)_{(k)}$ and f is multi-homogeneous.

For each $u \in F'$ define $\xi_u \in E$ by $\xi_u(y_i) = y_i + [y_i,u]$, $i = 1,\dots,m$. Since F is metabelian it follows that $\xi_u(w) = w + [w,u]$ for all $w \in F$ and so ξ_u is an automorphism with inverse ξ_{-u}. (In fact, since $[w,u,u] = 0$,

$$\xi_u(w) = w + [w,u]/1! + [w,u,u]/2! + \cdots = \exp(\operatorname{ad} u)(w)$$

and so ξ_u is an "inner" automorphism.) Let $Q_2 = \{0\} \subseteq \bar{I}_2 E$ and for $k \geq 3$ let $Q_k = \{\bar{\xi}_u : u \in F_{(k-1)}\} \subseteq \bar{I}_k E$. It is easily verified that if $k \geq 3$, $\phi \in \operatorname{Aut} F$, $u, u_1, u_2 \in F_{(k-1)}$ and $a \in K$, then $\phi\xi_u\phi^{-1} = \xi_{\phi(u)}$, $\xi_{u_1}\xi_{u_2} = \xi_{u_1+u_2}$ and $a\bar{\xi}_u = \bar{\xi}_{au}$. Hence Q_k is a KG-submodule of $\bar{I}_k E$.

Let $\Phi = K\langle s_1,\dots,s_m\rangle$ be the free associative algebra (without identity) freely generated by variables $s_1,\dots,s_m$ and let G act on Φ in the obvious way. (Thus Φ can be identified with the tensor algebra on the natural KG-module $N(1)$.) For $k \geq 1$ let $\Phi_{(k)}$ be the homogeneous component of Φ of degree k. Furthermore let $\Phi^*_{(k)}$ be the subspace of $\Phi_{(k)}$ spanned by the elements of the form $\sum_\sigma s_{i_{\sigma(1)}}\cdots s_{i_{\sigma(k)}}$ where $1 \leq i_1 \leq \cdots \leq i_k \leq m$ and σ ranges over all permutations of $\{1,\dots,k\}$. (This may be identified with the space of symmetric tensors of degree k.) It is well known and easy to prove that $\Phi^*_{(k)}$ is a KG-submodule of Φ isomorphic to $\Omega_{(k)}$ (the k-th symmetric power of $N(1)$). But $\Omega_{(k)} \cong N(k)$ (see, for example, [10, p. 54]). Thus $\Phi^*_{(k)} \cong N(k)$ as KG-module.

For each element $h(s_1,\dots,s_m)$ of $\Phi^*_{(k-1)}$, $k \geq 2$, define $\eta_h \in I_k E$ by $\eta_h(y_i) = y_i + y_i h(\operatorname{ad} y_1,\dots,\operatorname{ad} y_m)$, $i = 1,\dots,m$. For $k \geq 2$ let $R_k = \{\bar{\eta}_h : h \in \Phi^*_{(k-1)}\} \subseteq \bar{I}_k E$. It is easily verified that if $g \in G$, $h, h_1, h_2 \in \Phi^*_{(k-1)}$ and $a \in K$, then $g\eta_h g^{-1} = \eta_{g(h)}$, $\bar{\eta}_{h_1}\bar{\eta}_{h_2} = \bar{\eta}_{h_1+h_2}$ and $a\bar{\eta}_h = \bar{\eta}_{ah}$. Hence R_k is a KG-submodule of $\bar{I}_k E$.

PROPOSITION 3.5. *In the above notation let $k \geq 2$.*

(i) $\bar{I}_k E = P_k \oplus Q_k \oplus R_k$. Furthermore $P_k = \{0\}$ for $m = 2$ and $P_k \cong (\det)^{-1} \otimes_K N(k, 2, 1^{m-3})$ for $m \geq 3$, $Q_2 = \{0\}$ and $Q_k \cong N(k-2, 1)$ for $k \geq 3$, and $R_k \cong N(k-1)$.

(ii) $\bar{I}_k A = P_k \oplus Q_k$.

PROOF. (i) We apply an idea from the proof of [9, Theorem 2.2]. Assume that $k \geq 3$ and $m \geq 3$. (The other cases are treated similarly.) By Proposition 1.4, we can write $\bar{I}_k E = N_1 \oplus N_2 \oplus N_3$ where $N_1 \cong (\det)^{-1} \otimes_K N(k, 2, 1^{m-3})$, $N_2 \cong N(k-2, 1)$ and $N_3 \cong N(k-1)$. Suppose $f \in F(y_2, \ldots, y_m)_{(k)}$ where f is multi-homogeneous of multi-degree $(0, \alpha_2, \ldots, \alpha_m)$ in $y_1, y_2, \ldots, y_m$. Then it is easy to verify that $\mu_k(\tau_f)$ has the form $\sum_{\iota=2}^{m} f_{\iota 1} e_{\iota 1}$ where $f_{\iota 1} \in \Omega_{(k-1)}$, $i = 2, \ldots, m$, and where $f_{\iota 1} = 0$ if $\alpha_\iota = 0$ and $f_{\iota 1}$ has multi-degree $(0, \alpha_2, \ldots, \alpha_{\iota-1}, \alpha_\iota - 1, \alpha_{\iota+1}, \ldots, \alpha_m)$ if $\alpha_\iota > 0$. It follows that $\mu_k(\tau_f) \in S^{\beta}_{(k-1)}$ where $\beta = (-1, \alpha_2, \ldots, \alpha_m)$. Thus, by Lemma 3.4, $\bar{\tau}_f \in (\bar{I}_k E)^\beta = N_1^\beta \oplus N_2^\beta \oplus N_3^\beta$. But $N_2^\beta = N_3^\beta = \{0\}$ since the first co-ordinate of β is negative. Thus $\bar{\tau}_f \in N_1$ and so $P_k \subseteq N_1$. Since N_1 is irreducible and $P_k \neq \{0\}$, $P_k = N_1$.

The map $F_{(k-1)} \longrightarrow Q_k$ defined by $u \longmapsto \bar{\xi}_u$ is a non-zero KG-module epimorphism, and $F_{(k-1)} \cong N(k-2, 1)$ by Proposition 1.3. Thus $Q_k \cong N(k-2, 1)$. Similarly, using the map $\Phi^*_{(k-1)} \longrightarrow R_k$, $h \longmapsto \bar{\eta}_h$, we obtain $R_k \cong N(k-1)$. It follows that $\bar{I}_k E = P_k \oplus Q_k \oplus R_k$.

(ii) By Proposition 2.6, $\bar{I}_k A$ is a KG-submodule of $\bar{I}_k E$. Since the τ_f and the ξ_u are automorphisms, $P_k \oplus Q_k \subseteq \bar{I}_k A$. Let $h = s_1^{k-1} \in \Phi^*_{(k-1)}$. Thus $\eta_h(y_\iota) = y_\iota + y_\iota(\mathrm{ad}^{k-1} y_1)$ for all i and

$$\mu(\eta_h) = 1 + t_1^{k-2}\big((-t_2 e_{12} + t_1 e_{22}) + \cdots + (-t_m e_{1m} + t_1 e_{mm})\big).$$

Since $\bar{\eta}_h \in R_k$ it is enough to prove that $\bar{\eta}_h \notin \bar{I}_k A$. Suppose to get a contradiction that $\bar{\eta}_h \in \bar{I}_k A$. Then $\eta_h \equiv_{k+1} \phi$ for some $\phi \in I_k A$. Hence $\mu(\eta_h) \equiv \mu(\phi) \pmod{M^{(k)}}$ and the determinants of $\mu(\eta_h)$ and $\mu(\phi)$ are congruent modulo $\Omega^{(k)}$. But

$$\det \mu(\eta_h) = (1 + t_1^{k-1})^{m-1} \equiv 1 + (m-1)t_1^{k-1} \pmod{\Omega^{(k)}}.$$

Hence $\det \mu(\phi) \equiv 1 + (m-1)t_1^{k-1} \pmod{\Omega^{(k)}}$. On the other hand, by Proposition 3.2, $\mu(\phi)$ is invertible and so $\det \mu(\phi)$ is a unit of Ω. This is a contradiction.

By Proposition 2.6, $\bar{I}_k T$ is a KG-submodule of $\bar{I}_k A$, $k \geq 2$. Our main task now is the calculation of these submodules.

REMARK 3.6. $\bar{I}_2 T = P_2 = \bar{I}_2 A$ for all $m \geq 2$, since $Q_2 = \{0\}$. When $m = 2$, $\bar{I}_k T = \{0\}$ for all $k \geq 2$, since $IT = \{1\}$.

LEMMA 3.7. *$\bar{I}_k T = \bar{I}_k A$ for all $m \geq 3$, $k \geq 4$.*

PROOF. By Proposition 3.5, $\bar{I}_k A = P_k \oplus Q_k$ and P_k and Q_k are irreducible KG-modules. Since $P_k \subseteq \bar{I}_k T$ it suffices to show that $Q_k \cap \bar{I}_k T \neq \{0\}$. Define $\chi_1 \in I_k T$ by

$$\chi_1(y_3) = y_3 + y_2(\mathrm{ad}^{k-1} y_1), \quad \chi_1(y_\iota) = y_\iota \quad (i \neq 3).$$

Then, by an easy calculation, $\mu(\chi_1) = 1 - t_1^{k-2} t_2 e_{13} + t_1^{k-1} e_{23}$. Define $g_1 \in G$ by $g_1(y_1) = y_1 + y_3$, $g_1(y_\iota) = y_\iota$ $(i \neq 1)$. Then

$$\begin{aligned}\mu(g_1 \chi_1 g_1^{-1}) &= g_1\big(1 - g_1(t_1^{k-2} t_2) e_{13} + g_1(t_1^{k-1}) e_{23}\big) g_1^{-1} \\ &= (1 + e_{31})\big(1 - (t_1 + t_3)^{k-2} t_2 e_{13} + (t_1 + t_3)^{k-1} e_{23}\big)(1 - e_{31}) \\ &= 1 + (t_1 + t_3)^{k-2}\big(t_2(e_{11} + e_{31} - e_{13} - e_{33}) + (t_1 + t_3)(e_{23} - e_{21})\big),\end{aligned}$$

https://doi.org/10.4153/CJM-1993-063-5 Published online by Cambridge University Press

$$\mu_k(g_1\chi_1 g_1^{-1}) = (t_1+t_3)^{k-2}\big(t_2(e_{11}+e_{31}-e_{13}-e_{33})+(t_1+t_3)(e_{23}-e_{21})\big)$$

If W is any KG-submodule of $M_{(k-1)}$ and we write W as the sum of weight spaces $W = \oplus_\alpha W^\alpha$ then $W^\alpha \subseteq M^\alpha_{(k-1)}$ for all α Thus the weight components of an element w of W coincide with those obtained by regarding w as an element of $M_{(k-1)}$

The component of $\mu_k(g_1\chi_1 g_1^{-1})$ of weight $(k-2,1,0,\ldots,0)$ in $M_{(k-1)}$ is easily calculated to be

$$t_1^{k-3}\big(t_1(t_2e_{11}-t_1e_{21}+t_3e_{23}-t_2e_{33})+(k-2)t_3(-t_2e_{13}+t_1e_{23})\big)$$

Thus, since $\mu_k(g_1\chi_1 g_1^{-1}) \in \mu_k(I_kT)$, there exists $\zeta_1 \in I_kT$ such that

$$\mu_k(\zeta_1) = t_1^{k-3}\big(t_1(t_2e_{11}-t_1e_{21}+t_3e_{23}-t_2e_{33})+(k-2)t_3(-t_2e_{13}+t_1e_{23})\big)$$

Similarly define $\chi_2 \in I_kT$ by

$$\chi_2(y_3) = y_3 + y_2(\mathrm{ad}^{k-2}\, y_1)(\mathrm{ad}\, y_2), \qquad \chi_2(y_i) = y_i \quad (i \neq 3),$$

and $g_2 \in G$ by $g_2(y_2) = y_2 + y_3$, $g_2(y_i) = y_i$ $(i \neq 2)$ By considering the component of $\mu_k(g_2\chi_2 g_2^{-1})$ of weight $(k-2,1,0,\ldots,0)$, we find that there exists $\zeta_2 \in I_kT$ such that

$$\mu_k(\zeta_2) = t_1^{k-3}\big(t_2(t_2e_{12}-t_1e_{22}-t_3e_{13}+t_1e_{33})+t_3(-t_2e_{13}+t_1e_{23})\big)$$

Similarly define $\tau \in I_2T$ by $\tau(y_1) = y_1 + [y_2, y_3]$, $\tau(y_i) = y_i$ $(i \neq 1)$ and $g_3 \in G$ by $g_3(y_2) = y_1 + y_2$, $g_3(y_i) = y_i$ $(i \neq 2)$ Consideration of the component of $\mu_2(g_3\tau g_3^{-1})$ of weight $(0,0,1,0,\ldots,0)$ shows that there exists $\sigma \in I_2T$ such that $\mu_2(\sigma) = t_3e_{11} - t_1e_{31} - t_3e_{22} + t_2e_{32}$ Finally define $\zeta_3 \in I_{k-1}T$ by $\zeta_3(y_3) = y_3 + y_2(\mathrm{ad}^{k-2}\, y_1)$, $\zeta_3(y_i) = y_i$ $(i \neq 3)$ Thus $\mu_{k-1}(\zeta_3) = t_1^{k-3}(-t_2e_{13}+t_1e_{23})$

We apply Proposition 3 3 to the subalgebra $\mathcal{L}(T)$ of $\mathcal{L}(E)$ Let $\omega_1 = \sigma^{-1}\zeta_3^{-1}\sigma\zeta_3$ Then $\omega_1 \in I_kT$ and

$$\begin{aligned}\mu_k(\omega_1) &= [\mu_2(\sigma), \mu_{k-1}(\zeta_3)]\\ &= t_1^{k-3}(t_1^2e_{21} - t_1t_2e_{11} + t_2^2e_{12} - t_1t_2e_{22} - t_2t_3e_{13} - t_1t_3e_{23} + 2t_1t_2e_{33})\end{aligned}$$

By Proposition 2 6 there exist $\omega_3, \omega_0 \in I_kT$ such that

$$\mu_k(\omega_3) = \frac{1}{k-3}\big(\mu_k(\zeta_1) - \mu_k(\zeta_2) + \mu_k(\omega_1)\big) = t_1^{k-3}t_3(-t_2e_{13}+t_1e_{23}),$$

$$\mu_k(\omega_0) = -\big(\mu_k(\zeta_1) + \mu_k(\zeta_2) - k\mu_k(\omega_3)\big) = t_1^{k-3}\big(t_1(-t_2e_{11}+t_1e_{21}) + t_2(-t_2e_{12}+t_1e_{22})\big)$$

By replacing y_3 with y_p $(3 \leq p \leq m)$ in the above calculation we obtain automorphisms $\omega_p \in I_kT$ such that

$$\mu_k(\omega_p) = t_1^{k-3}t_p(-t_2e_{1p}+t_1e_{2p})$$

Let $\omega = \omega_0\omega_3\omega_4\cdots\omega_m$ Then $\omega \in I_kT$ and

$$\mu_k(\omega) = \mu_k(\omega_0) + \sum_{p=3}^{m}\mu_k(\omega_p) = \sum_{p=1}^{m} t_1^{k-3}t_p(-t_2e_{1p}+t_1e_{2p})$$

Hence $\mu_k(\omega) = \mu_k(\xi_u)$ where $u = -y_2(\mathrm{ad}^{k-2}\, y_1)$ Thus $\bar{\omega} = \bar{\xi}_u$ is a non-zero element of $Q_k \cap \bar{I}_kT$

LEMMA 3.8. *$\bar{I}_3T = \bar{I}_3A$ for all $m \geq 4$.*

PROOF. As in Lemma 3.7 it is enough to show that $Q_3 \cap \bar{I}_3T \neq \{0\}$. Define $\sigma_1, \sigma_2 \in I_2T$ by $\sigma_1(y_4) = y_4 + [y_1, y_3]$, $\sigma_1(y_i) = y_i$ $(i \neq 4)$, $\sigma_2(y_3) = y_3 + [y_2, y_4]$, $\sigma_2(y_i) = y_i$ $(i \neq 3)$. Thus

$$\mu_2(\sigma_1) = t_3e_{14} - t_1e_{34}, \quad \mu_2(\sigma_2) = t_4e_{23} - t_2e_{43}.$$

Let $\gamma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2$. Then $\gamma_1 \in I_3T$ and

$$\mu_3(\gamma_1) = [\mu_2(\sigma_1), \mu_2(\sigma_2)] = t_2(-t_3e_{13} + t_1e_{33}) + t_1(t_4e_{24} - t_2e_{44}).$$

Analogously, define $\rho_1, \rho_2 \in I_2T$ by $\rho_1(y_4) = y_4 + [y_2, y_3]$, $\rho_1(y_i) = y_i$ $(i \neq 4)$, $\rho_2(y_3) = y_3 + [y_1, y_4]$, $\rho_2(y_i) = y_i$ $(i \neq 3)$. Let $\gamma_2 = \rho_1^{-1}\rho_2^{-1}\rho_1\rho_2 \in I_3T$ and $\gamma = \gamma_1\gamma_2^{-1} \in I_3T$. Then

$$\mu_3(\gamma_2) = t_1(-t_3e_{23} + t_2e_{33}) + t_2(t_4e_{14} - t_1e_{44}),$$
$$\mu_3(\gamma) = \mu_3(\gamma_1) - \mu_3(\gamma_2) = t_3(-t_2e_{13} + t_1e_{23}) + t_4(-t_2e_{14} + t_1e_{24}).$$

Now we make use of $\zeta_1, \zeta_2 \in I_3T$ as obtained in the proof of Lemma 3.7, but with $k = 3$. Let $\psi_1 = \zeta_1\zeta_2 \in I_3T$. Then we have

$$\mu_3(\zeta_1) = t_1(t_2e_{11} - t_1e_{21} + t_3e_{23} - t_2e_{33}) + t_3(-t_2e_{13} + t_1e_{23}),$$
$$\mu_3(\zeta_2) = t_2(t_2e_{12} - t_1e_{22} - t_3e_{13} + t_1e_{33}) + t_3(-t_2e_{13} + t_1e_{23}),$$
$$\mu_3(\psi_1) = t_1(t_2e_{11} - t_1e_{21}) + t_2(t_2e_{12} - t_1e_{22}) - 3t_3(t_2e_{13} - t_1e_{23}).$$

Similarly there exists $\psi_2 \in I_3T$ such that

$$\mu_3(\psi_2) = t_1(t_2e_{11} - t_1e_{21}) + t_2(t_2e_{12} - t_1e_{22}) - 3t_4(t_2e_{14} - t_1e_{24}).$$

Thus there exist $\omega_0, \omega_3 \in I_3T$ such that

$$\mu_3(\omega_0) = \frac{1}{2}\big(-\mu_3(\psi_1) - \mu_3(\psi_2) + 3\mu_3(\gamma)\big) = t_1(-t_2e_{11} + t_1e_{21}) + t_2(-t_2e_{12} + t_1e_{22}),$$
$$\mu_3(\omega_3) = \frac{1}{3}\big(\mu_3(\omega_0) + \mu_3(\psi_1)\big) = t_3(-t_2e_{13} + t_1e_{23}).$$

The proof can now be completed as in Lemma 3.7.

LEMMA 3.9. *For $m = 3$, the Lie algebra $\mathcal{L}(A)$ satisfies $[\bar{I}_2A, \bar{I}_2A] = P_3 \underset{\neq}{\subset} \bar{I}_3A$.*

PROOF. By Remark 3.6, $\bar{I}_2A = P_2$ and, by Proposition 3.5, $\bar{I}_3A = P_3 \oplus Q_3$, where P_3 and Q_3 are non-isomorphic irreducible modules. Since $[P_2, P_2]$ is a submodule of $\bar{I}_3A$ it suffices to show that $[P_2, P_2]$ does not contain Q_3 and $[P_2, P_2] \neq \{0\}$. Since $Q_3 \cong N(1^2)$ we have $Q_3^{(1,1,0)} \neq \{0\}$. Therefore it suffices to prove that $\{0\} \neq [P_2, P_2]^{(1,1,0)} \subseteq P_3$. We shall work in $S_{(1)}$ and $S_{(2)}$ (using Proposition 3.3 and Lemma 3.4). Let $V = \bar{\mu}_2(P_2) = \mu_2(I_2A) \subseteq S_{(1)}$,

$$C = \bar{\mu}_3([P_2, P_2]) = [\bar{\mu}_2(P_2), \bar{\mu}_2(P_2)] = [V, V] \subseteq S_{(2)}$$

and $D = \bar{\mu}_3(P_3) \subseteq S_{(2)}$. We wish to prove that $\{0\} \neq C^{(1,1,0)} \subseteq D$.

Since $V \cong P_2 \cong (\det)^{-1} \otimes_K N(2^2)$, $V^\alpha \neq \{0\}$ only for

$$\alpha \in \{(-1,1,1),(1,-1,1),(1,1,-1),(1,0,0),(0,1,0),(0,0,1)\},$$

when V^α is one-dimensional. It is easy to verify that, for all α, β, $[V^\alpha, V^\beta] \subseteq [V,V]^{\alpha+\beta}$, where $\alpha+\beta$ is the componentwise sum. But $C = [\sum V^\alpha, \sum V^\beta] = \sum [V^\alpha, V^\beta]$. Thus

$$C^{(1,1,0)} = \sum_{\alpha+\beta=(1,1,0)} [V^\alpha, V^\beta] = [V^{(1,1,-1)}, V^{(0,0,1)}] + [V^{(1,0,0)}, V^{(0,1,0)}].$$

Let $\pi, \psi \in I_2A$ be defined by $\pi(y_3) = y_3 + [y_2, y_1]$, $\pi(y_i) = y_i$ $(i \neq 3)$, $\psi(y_2) = y_2 + [y_3, y_1]$, $\psi(y_i) = y_i$ $(i \neq 2)$, and let $g \in G$ be given by $g(y_3) = y_2 + y_3$, $g(y_i) = y_i$ $(i \neq 3)$. Let $\theta_1 = \pi\psi^{-1} g \psi g^{-1} \in I_2A$. Then, by easy calculations,

$$\mu_2(\pi) = -t_2e_{13} + t_1e_{23}, \quad \mu_2(\theta_1) = -t_2e_{12} + t_1e_{22} + t_3e_{13} - t_1e_{33}.$$

Similarly there exist $\theta_2, \theta_3 \in I_2A$ such that

$$\mu_2(\theta_2) = -t_3e_{23} + t_2e_{33} + t_1e_{21} - t_2e_{11},$$
$$\mu_2(\theta_3) = -t_1e_{31} + t_3e_{11} + t_2e_{32} - t_3e_{22}.$$

It follows that $V^{(1,1,-1)}$, $V^{(1,0,0)}$, $V^{(0,1,0)}$, $V^{(0,0,1)}$ are spanned by the elements $\mu_2(\pi)$, $\mu_2(\theta_1)$, $\mu_2(\theta_2)$, $\mu_2(\theta_3)$, respectively. Thus $C^{(1,1,0)}$ is spanned by $c_1 = [\mu_2(\pi), \mu_2(\theta_3)]$ and $c_2 = [\mu_2(\theta_1), \mu_2(\theta_2)]$. By direct calculation,

$$c_1 = t_1(t_2e_{11} - t_1e_{21}) + t_2(-t_2e_{12} + t_1e_{22}) + t_2t_3e_{13} + t_1t_3e_{23} - 2t_1t_2e_{33},$$
$$c_2 = t_1(-t_2e_{11} + t_1e_{21}) + t_2(-t_2e_{12} + t_1e_{22}) + 3t_3(t_2e_{13} - t_1e_{23}).$$

In particular $C^{(1,1,0)} \neq \{0\}$.

Now we use $\chi_1, \chi_2, \zeta_1, \zeta_2 \in I_3T$ as in the proof of Lemma 3.7 (but with $k = 3$). It is easy to see that $\bar{\chi}_1, \bar{\chi}_2 \in P_3$. Thus $\mu_3(\chi_1), \mu_3(\chi_2) \in D$. Because $\mu_3(\zeta_1), \mu_3(\zeta_2)$ are weight components of $\mu_3(\chi_1)$, $\mu_3(\chi_2)$ we obtain $\mu_3(\zeta_1), \mu_3(\zeta_2) \in D$. It is easy to see that $c_1 = \mu_3(\zeta_1) - \mu_3(\zeta_2)$ and $c_2 = -\mu_3(\zeta_1) - \mu_3(\zeta_2)$. Thus $C^{(1,1,0)} \subseteq D$, as required.

LEMMA 3.10. *For $m = 3$, $\bar{I}_3T = P_3 \subsetneq \bar{I}_3A$.*

PROOF. As we saw earlier, IT is generated by the automorphisms $g\tau_f g^{-1}$ where $g \in G = \mathrm{GL}_3(K)$, f is a homogeneous element of $F(y_2, y_3)'$ and τ_f is defined by $\tau_f(y_1) = y_1 + f$, $\tau_f(y_2) = y_2$, $\tau_f(y_3) = y_3$. We have $\mu(\tau_f) = 1 + f_2e_{21} + f_3e_{31}$ where $f_2, f_3 \in K[t_2, t_3] \subseteq K[t_1, t_2, t_3]$. Hence $\big(\mu(\tau_f) - 1\big)^2 = 0$. Also, for all $g \in G$,

$$\mu(g\tau_f g^{-1}) = g\big(1 + g(f_2)e_{21} + g(f_3)e_{31}\big)g^{-1}.$$

Hence $\big(\mu(g\tau_f g^{-1}) - 1\big)^2 = 0$.

Let $\phi \in I_3T$. Since $\phi \in IT$, there exist homogeneous elements $f_1, \ldots, f_n$ of $F(y_2, y_3)'$ and elements $g_1, \ldots, g_n$ of G such that $\phi = \phi_1\phi_2 \cdots \phi_n$ where $\phi_i = g_i\tau_{f_i}g_i^{-1}$, $i = 1, \ldots, n$.

(Note that $(g\tau_f g^{-1})^{-1} = g\tau_{-f}g^{-1}$.) Write $\mu(\phi_i) = 1 + u_i$, $i = 1, \ldots, n$. Thus each u_i is homogeneous of degree at least 1 and

$$\mu(\phi) = (1+u_1)\cdots(1+u_n) \equiv 1 + (u_1 + \cdots + u_n) + \sum_{i<j} u_i u_j \pmod{M^{(3)}}.$$

Let those u_i of degree 1 be $v_1, \ldots, v_p$ (taken in the same order as in $u_1, \ldots, u_n$) and let those u_i of degree 2 be $w_1, \ldots, w_q$. Then

$$\mu(\phi) \equiv 1 + (v_1 + \cdots + v_p) + (w_1 + \cdots + w_q) + \sum_{i<j} v_i v_j \pmod{M^{(3)}}.$$

Since $\phi \in I_3T$, $\mu(\phi) \equiv 1 \pmod{M^{(2)}}$. Thus $v_1 + \cdots + v_p = 0$ and

$$\mu_3(\phi) = (w_1 + \cdots + w_q) + \sum_{i<j} v_i v_j.$$

Since $\bigl(\mu(\phi_i) - 1\bigr)^2 = 0$ for all i we have $v_1^2 = \cdots = v_p^2 = 0$. Thus

$$0 = (v_1 + \cdots + v_p)^2 = \sum_{i<j}(v_i v_j + v_j v_i),$$

$$\sum_{i<j} v_i v_j = \frac{1}{2}\sum_{i<j}[v_i, v_j],$$

$$\mu_3(\phi) = (w_1 + \cdots + w_q) + \frac{1}{2}\sum_{i<j}[v_i, v_j].$$

By the definition of $w_1, \ldots, w_q$, $v_1, \ldots, v_p$ we have $w_1, \ldots, w_q \in \bar{\mu}_3(P_3)$ and $v_1, \ldots, v_p \in \bar{\mu}_2(P_2) = \bar{\mu}_2(\bar{I}_2A)$. Thus, by Lemma 3.9, $[v_i, v_j] \in \bar{\mu}_3(P_3)$ for all i,j. Hence $\mu_3(\phi) \in \bar{\mu}_3(P_3)$. This holds for all $\phi \in I_3T$ and so $\bar{I}_3T \subseteq P_3$. The result follows since $P_3 \subseteq \bar{I}_3T$ and $Q_3 \neq \{0\}$.

We now obtain the main result of this section.

THEOREM 3.11. *Let T be the group of tame automorphisms of the free metabelian Lie algebra of finite rank $m \geq 2$.*

(i) For $m \geq 4$, T is dense in $A = \operatorname{Aut} F$.

(ii) For $m = 2$ and $m = 3$, T is not dense in A and so F possesses non-tame automorphisms.

PROOF. (i) By Corollary 2.9 it suffices to show that $\mathcal{L}(T) = \mathcal{L}(A)$; that is, $\bar{I}_kT = \bar{I}_kA$ for all $k \geq 2$. This follows from Remark 3.6, Lemma 3.7 and Lemma 3.8.

(ii) It suffices to show that $\mathcal{L}(T) \neq \mathcal{L}(A)$. For $m = 2$, $\mathcal{L}(T) = \{0\}$, by Remark 3.6, and $\mathcal{L}(A) \neq \{0\}$ since $Q_3 \neq \{0\}$. For $m = 3$, $\mathcal{L}(T) \neq \mathcal{L}(A)$ by Lemma 3.10.

REFERENCES

1. D J Anıck, *Lımıts of tame automorphısms of* $k[x_1, \quad , x_N]$, J Algebra **82**(1983), 459–468

2. S Bachmuth, *Automorphısms of free metabelıan groups*, Trans Amer Math Soc **118**(1965), 93–104

3. S Bachmuth and H Y Mochızukı, *The nonfinıte generatıon of* Aut(G), *G free metabelıan of rank* 3, Trans Amer Math Soc **270**(1982), 693–700

4. ———, Aut(F) $\longrightarrow$ Aut(F/F'') *ıs surjectıve for free group F of rank* $\geq$ 4, Trans Amer Math Soc **292**(1985), 81–101

5. Yu A Bahturın, *Identıcal relatıons ın Lıe algebras*, Nauka, Moscow, 1985, Russıan, Translatıon, VNU Scıence Press, Utrecht, 1987

6. Yu A Bahturın and S Nabıyev, *Automorphısms and derıvatıons of abelıan extensıons of some Lıe algebras*, Abh Math Sem Unıv Hamburg, **62**(1992), 43–57

7. O Cheın, *IA automorphısms of free and free metabelıan groups*, Comm Pure Appl Math **21**(1968), 605–629

8. P M Cohn, *Subalgebras of free assocıatıve algebras*, Proc London Math Soc (3) **14**(1964), 618–632

9. V Drensky and C K Gupta, *Automorphısms of free nılpotent Lıe algebras*, Canad J Math **42**(1990), 259–279

10. J A Green, *Polynomıal representatıons of* GL_n, Lecture Notes ın Math **830**, Sprınger-Verlag, Berlın Heıdelberg-New York, 1980

11. I G Macdonald, *Symmetrıc functıons and Hall polynomıals*, Oxford Unıversıty Press, Oxford, 1979

12. V A Roman'kov, *Groups of automorphısms of free metabelıan groups* In Problems of relatıons between abstract and applıed algebra, Russıan, Novosıbırsk, (1985), 53–80

13. J -P Serre, *Lıe algebras and Lıe groups*, W A Benjamın Inc , New York-Amsterdam, 1965

14. A L Shmel'kın, *Wreath products of Lıe algebras and theır applıcatıon ın the theory of groups*, Trudy Moskov Mat Obshch **29**(1973), 247–260, Russıan, Translatıon, Trans Moscow Math Soc **29**(1973), 239–252

Department of Mathematıcs
Unıversıty of Manchester Instıtute of Scıence and Technology
Manchester, M60 1QD
Unıted Kıngdom

Instıtute of Mathematıcs
Bulgarıan Academy of Scıences
Sofia
Bulgarıa

https://doi.org/10.4153/CJM-1993-063-5 Published online by Cambridge University Press