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AMENABILITY AND IDEALS IN $A(G)$

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Abstract

Closed ideals in $A(G)$ with bounded approximate identities are characterized for amenable [SIN]-groups and arbitrary discrete groups. This extends a result of Liu, van Rooij and Wang for abelian groups.

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1. Introduction

Let G be a locally compact group. In [3], P. Eymard defined the Fourier algebra $A(G)$ of G to be the linear subspace of $C_0(G)$ consisting of all functions of the form $u(x) = (f * \tilde{g})^{\vee}(x)$, where $f, g \in L^2(G)$, $f^{\vee}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. If $f \in L^2(G)$ and $x \in G$, then we define $L_xF(y) = f(x^{-1}y)$ for every $y \in G$. Let $VN(G)$ denote the weak operator topology closure in $\mathscr{B}(L^2(G))$, the algebra of bounded operators on $L^2(G)$, of the linear span of $\{L_x; x \in G\}$. Then $A(G)$ can be realized as the unique predual of the von Neumann algebra $VN(G)$ [3, pp. 210–218]. $A(G)$ becomes a commutative Banach algebra under pointwise multiplication and with respect to the norm given by $\|u\|_{A(G)} = \sup\{|\langle T, u\rangle|; T \in VN(G), \|T\|_{VN(G)} \leq 1\}$. In addition, we have that the spectrum of $A(G)$ is G [3, p. 222].

Inspired by an important result of Leptin [12] that a locally compact group G is amenable if and only if $A(G)$ has a bounded approximate identity, a study of the closed ideals in $A(G)$ with bounded approximate identities was begun in [5]. There we showed the existence of such ideals to be intimately related to the amenability of G.

In this paper, we will continue from where the previous investigation left off. In particular, we are able to completely characterize those ideals with bounded approximate identities in the Fourier algebra of either an amenable [SIN]-group or an arbitrary discrete group. The first of these two results extends a theorem of Liu, van Rooij and Wang [13] for abelian locally compact groups. The latter result reaffirms the strong connection between amenability and the existence of bounded approximate identities in $A(G)$.

2. Preliminaries

Throughout this paper, G will denote a locally compact group with a fixed left Haar measure μ_G and modular function δ_G. For any subset A of G, we will denote by 1_A the characteristic function of A.

G is said to be amenable if there exists an $m \in L^\infty(G)^*$ which is such that $m \geq 0$, $m(1_G) = 1$ and $m(L_x f) = m(f)$ for every $x \in g$, $f \in L^\infty(G)$. All abelian groups and all compact groups are amenable. The free group on two generators is not amenable.

Let $B(G)$ denote the linear span of $P(G)$, the continuous positive definite functions on G. Then $B(G)$ can be identified with the dual of $C^*(G)$, the group C^*-algebra of G [3, p. 192]. When given the dual norm and pointwise multiplication, $B(G)$ becomes a commutative Banach algebra called the Fourier-Stieltjes algebra of G. The Fourier algebra $A(G)$ is a closed ideal of $B(G)$ [3, p. 208]. Further properties of $A(G)$, $VN(G)$ and $B(G)$ can be found in [3].

Let $\mathscr{A}$ be a commutative Banach algebra. A net $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ in $\mathscr{A}$ is called a bounded approximate identity if $\lim_\alpha \|u_\alpha u - u\|_{\mathscr{A}} = 0$ for every $u \in \mathscr{A}$ and if there exists an M such that $\|u_\alpha\|_{\mathscr{A}} \leq M$ for every $\alpha \in \mathscr{U}$.

Let $\Delta(\mathscr{A})$ denote the maximal ideal space of $\mathscr{A}$. By means of the Gelfand transform, $\mathscr{A}$ can be realized as a subalgebra of $C_0(\Delta(\mathscr{A}))$. Let I be an ideal in $\mathscr{A}$. Define

$$Z(I) = \{x \in \Delta(A)\,;\, u(x) = 0 \quad \text{for every } u \in I\}$$

Then $Z(I)$ is a closed subset of $\Delta(\mathscr{A})$. If E is a closed subset of $\Delta(\mathscr{A})$, define

$$I(E) = \{u \in \mathscr{A}\,;\, u(x) = 0 \quad \text{for every } x \in E\}$$

and

$$I_0(E) = \{u \in \mathscr{A}\,;\; \operatorname{supp} u \in \mathscr{F}(E)\},$$

where $\mathscr{F}(E) = \{K \subset \Delta(\mathscr{A});\ K \text{ is compact and } K \cap E = \varnothing\}$. Then $I_0(E)$ and $I(E)$ are ideals in $s\mathscr{A}$. $I(A)$ is closed. Furthermore, if I is any ideal in $\mathscr{A}$ with $Z(I) = E$, then $I_0(E) \subseteq I \subseteq I(E)$.

A closed subset E of $\Delta(\mathscr{A})$ is said to be a set of spectral synthesis, or simply an s-set, if $I(E)$ is the only closed ideal I for which $Z(I) = E$.

Let G be a locally compact group. Let A and B be closed subsets of G. Let

$$\mathscr{S}(A, B) = \{u \in B(G);\quad u(A) \equiv 1,\quad u(B) \equiv 0\},$$

$$s(A, B) = \begin{cases} \inf\{\|u\|_{B(G)};\ u \in \mathscr{S}(A, B)\} & \text{if } \mathscr{S}(A, B) \neq \varnothing, \\ \infty & \text{if } \mathscr{S}(A, B) = \varnothing, \end{cases}$$

$$\mathscr{F}(A) = \{K \subset G; K \text{ is compact and } K \cap A = \varnothing\}$$

and

$$s(A) = \sup\{s(A, K);\ K \in \mathscr{F}(A)\}.$$

If K is compact and A is a closed subset of G disjoint from K, then $s(A, K) < \infty$ by the regularity of $A(G)$.

The mappings $A(G) \to A(G)$ defined by $u \to {}_x u$ and $u \to u_x$, where ${}_x u(y) = u(xy)$ and $u_x(y) = u(yx)$, are isometric isomorphism for each $x \in G$ [3, p. 199]. It follows that the ideals $I(A)$, $I(xA)$ and $I(Ax)$ are isometrically isomorphic for every closed subset A of G and every $x \in G$. In particular, $I(A)$ has a bounded approximate identity if and only if $I(xA)$ and $I(Ax)$ have a bounded approximate identities.

3. Ideals and bounded approximate identities

We begin with the following technical lemmas.

Lemma 3.1. *Let $x, y \in G$ and let H and H_1 be subgroups of G. Then either $xH \cap yH_1 = \varnothing$ or $xH \cap yH_1 = z(H \cap H_1)$ for some $z \in G$.*

Proof. Assume that $xH \cap hH_1 \neq \varnothing$. Without loss of generality we can assume that $x = e$. Let $z \in H \cap yH_1$. Then $z \in H$ and $z = yh_1$ for some $h_1 \in H_1$.

Let $w \in z(H \cap H_1)$. Then since $z \in H$, we have $w \in H$. Since $w = zh_1'$ for some $h_1' \in H_1$, it follows that $w = yh_1h_1' \in yH_1$. Hence $z(H \cap H_1) \subseteq H \cap yH_1$.

Let $w \in H \cap yH_1$. Then $w = yh_1'$ for some $h_1' \in H_1$. Hence

$$w = yh_1(h_1^{-1}h_1') = zh_1^{-1}h_1'.$$

Since $w, z \in H$, $h_1^{-1}h_1' \in H \cap H_1$ and thus $H \cap yH_1 \subseteq z(H \cap H_1)$.

The next lemma is a generalization of a result due to Cohen [1, p. 221] for abelian groups.

LEMMA 3.2. *Let $H_1, \ldots, H_n$ be closed subgroups of G. Let $K_1, \ldots, K_n$ be compact subsets of G. If $G = \bigcup_{i=1}^{n-1} K_i H_i$, then for some $1 \le i_0 \le n$, there exists a compact subset K_0 of G with $G = K_0 H_{i_0}$.*

PROOF. Suppose that there exists a G, a collection $\{H_1, \ldots, H_n\}$ of closed subgroups of G and a collection $\{K_1, \ldots, K_n\}$ of compact subsets of G for which $G = \bigcup_{i=1}^{n} K_i H_i$ but none of the H_i's satisfies the statement of the theorem. We may assume that the collection $\{H_1, \ldots, H_n\}$ is minimal in this respect. Clearly $n > 1$. We must be able to find an $x \in G$ for which $K_n H_n \cap xH_n = \varnothing$. Thus $xH_n \subseteq \bigcup_{i=1}^{n-1} K_i H_i$. But then

$$K_n H_n = K_n x^{-1} x H_n \subseteq \bigcup_{i=1}^{n-1} K_n x^{-1} K_i H_i$$

and

$$G = \bigcup_{i=1}^{n-1} (K_i \cup K_n x^{-1} K_i) H_i$$

which is impossible by the minimality of the collection $\{H_1, \ldots, H_n\}$.

COROLLARY 3.3. *Let $G = \bigcup_{i=1}^{n} x_i H_i$, where H_i is open in G. Then for some $1 \le i_0 \le n$, $[G : H_{i_0}] < \infty$.*

DEFINITION 3.4. For any locally compact group G, let $\mathscr{R}(G)$ denote the ring of subsets of G generated by the left cosets of open subgroups of G. $\mathscr{R}(G)$ is called the coset ring of G. Define

$$\mathscr{R}_c(G) = \{A \subset G;\ A \in \mathscr{R}(G_d) \text{ and } A \text{ is closed in } G\}.$$

The following result is due to Gilbert [7] and (independently) Schreiber [17] for abelian groups. With Lemma 3.1 and Lemma 3.2 in mind, it becomes a straightforward albeit somewhat lengthy task to verify that Gilbert's proof is valid for all locally compact groups.

LEMMA 3.5. *Let $A \subset G$. Then $A \in \mathscr{R}_c(G)$ if and only if A is of the form $A = \bigcup_{i=1}^{n} x_i (H_i \backslash \Delta_i)$, where H_i is a closed subgroup of G, $\Delta_i \in \mathscr{R}(H_i)$ and $x_i \in G$.*

In [5, Proposition 3.5], we showed that if $A \subset G$ is closed and the ideal $I(A)$ has a bounded approximate identity, then $A \in \mathscr{R}_c(G)$. We now have the following.

PROPOSITION 3.6. *Let A be a closed subset of G. If $I(A)$ has a bounded approximate identity, then A has the form $A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i)$, where H_i is a closed subgroup of G, $\Delta_i \in \mathscr{R}(H_i)$ and $x_i \in G$.*

DEFINITION 3.7. A locally compact group G is called a *small invariant neighborhood group* ([SIN]-group) if every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms.

[SIN]-groups have been studied by many authors (see [15]). The class includes all discrete groups, all compact groups and all abelian locally compact groups.

For locally compact abelian groups, Liu, van Rooij and Wang [13] proved that a closed ideal I in $A(G)$ has a bounded approximate identity if and only if $I = I(A)$, where $A \in \mathscr{R}_c(G)$. We would like to extend this result to amenable [SIN]-groups.

LEMMA 3.8. *Let A and B be closed subsets of G. Suppose that $I(A)$ and $I(B)$ have bounded approximate identities. Then so does $I(A \cup B)$.*

PROOF. Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be approximate identities in $I(A)$ and $I(B)$ with bounds M_1 and M_2 respectively. Let $\{w_1, \dots, w_n\} \subseteq I(A \cup B)$ and $\varepsilon > 0$. As $I(A \cup B) \subset I(A)$, there exists i_0 such that

$$\|w_k u_{i_0} - w_k\|_{A(G)} < \varepsilon/2 \quad \text{for } k = 1, 2, \dots, n.$$

As $\{w_k u_{i_0}\}_{k=1}^{n} \subset I(B)$, there exists j_0 such that

$$\|w_k u_{i_0} - w_k u_{i_0} v_{j_0}\|_{A(G)} < \varepsilon/2 \quad \text{for } k = 1, 2, \dots, n.$$

Then $\|u_{i_0} v_{j_0}\|_{A(G)} \le M_1 M_2$, $u_{i_0} v_{j_0} \in I(A \cup B)$ and

$$\|w_k - w_k u_{i_0} v_{j_0}\|_{A(G)} \le \varepsilon \quad \text{for } k = 1, 2, \dots, n.$$

The existence of the bounded approximate identity follows easily.

LEMMA 3.9. *Let A and B be disjoint subsets of G. Assume that there exists $u \in B(G)$ with $u(A) = 1$ and $u(B) = 0$. If G is amenable, then there exists a bounded approximate identity in $I(A \cup B)$ if and only if there exists bounded approximate identities in each of $I(A)$ and $I(B)$.*

Proof. Assume that $I(A \cup B)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathscr{U}}$. By Leptin's theorem [12], $A(G)$ has a bounded approximate identity $\{v_j\}_{j \in J}$. Then

$$\{u_\alpha u - (1-u)v_j\}_{\alpha \in \mathscr{U}, j \in J}$$

is a bounded approximate identity in $I(A)$ and

$$\{(u_\alpha - u_\alpha u) + uv_j\}_{\alpha \in \mathscr{U}, j \in J}$$

is a bounded approximate identity in $I(B)$.

The converse is Lemma 3.8.

Proposition 3.10. *Let G be a* [SIN]*-group and let H be a closed subgroup of G. Then $s(H) = 1$.*

Proof. Let $K \subset G$ be compact with $K \cap H = \varnothing$. Let V be a symmetric neighborhood of e such that $\overline{V}$ is compact and

$$V^2 H \cap K^{-1} = \varnothing. \tag{1}$$

As both G and H are unimodular [15], there exists a G-invariant measure dg on the quotient space G/H [6, p. 267]. We may assume that the Haar measures μ_g and μ_H are chosen such that

$$\int_G f(g) d_{\mu_G}(g) = \int_{G/H} \left[\int_H f(gh) d_{\mu_H}(h) \right] dg$$

for every $f \in C_{00}(G)$.

By a result of Mosak [14], there exists a continuous nonnegative central function v defined on G such that $\operatorname{supp} v \subseteq V$ and

$$\begin{aligned}
1 &= \int_{G/H} \left[\int_H v(gh) d_{\mu_H}(h) \right]^2 dg \\
&= \int_{G/H} \left[\int_H v(gh') d_{\mu_H}(h') \int_H v(gh) d_{\mu_H}(h) \right] dg \\
&= \int_{G/H} \left[\int_H v(gh') d_{\mu_H}(h') \int_H v(gh'h) d_{\mu_H}(h) \right] dg \\
&= \int_G \int_H v(g) v(gh) d_{\mu_H}(h) d_{\mu_G}(g).
\end{aligned} \tag{2}$$

For every $x \in G$, define

$$u(x) = \int_G \int_H v(g) v(x^{-1} gh) d_{\mu_H}(h) d_{\mu_G}(g).$$

Following an argument of Cowling and Rodway [2, p. 95], we see that u is a coefficient function of the unitary representation obtained by inducing

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the trivial representation of H to G. Therefore, $u \in B(G)$. Furthermore, $\|u\|_{B(G)} \leq 1$.

If $x \in K$, then $v(g)v(x^{-1}gh) = 0$ for every $g \in G$ and $h \in H$ by (1). Therefore, $u(K) = 0$.

Also, if $x \in H$, then

$$\begin{aligned} u(x) &= \int_G \int_H v(g)v(x^{-1}gh)d_{\mu_H}(h)d_{\mu_G}(g) \\ &= \int_G \int_H v(g)v(gh')d_{\mu_H}(h')d_{\mu_G}(g) \quad \text{(since } v \text{ is central)} \\ &= 1 \quad \text{by (2).} \end{aligned}$$

Hence $u \in \mathscr{S}(H, K)$. Since $K \in \mathscr{F}(H)$ was arbitrary, $s(H) = 1$.

The proof of Proposition 3.10 uses an idea of Reiter which was later modified by Cowling and Rodway in [2, p. 98] to show that if G is a [SIN]-group and H is a closed subgroup of G, then for each $u \in B(H)$ there exists $v \in B(G)$ such that $v|_H = u$.

We are now able to characterize the ideals with bounded approximate identities in the Fourier algebra of an amenable [SIN]-group.

Theorem 3.11. *Let G be an amenable* [SIN]*-group. Let I be an ideal in $A(G)$. Then I has a bounded approximate identity if and only if $Z(I) \in \mathscr{R}_c(G)$. Moreover, each $A \in \mathscr{R}_c(G)$ is a set of spectral synthesis, so if I is closed, $I = I(Z(I))$. In any case, the bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ can be chosen such that*

(i) $u_\alpha \in A(G) \cap C_{00}(G)$,

(ii) *if $K \in \mathscr{F}(A)$, then there exists a sequence $\{u_{k_n}\} \subseteq \{u_\alpha\}_{\alpha\in\mathscr{U}}$ such that if $v \in I$ and* supp $v \subseteq K$, *then*

$$\|u_{k_n}v - v\|_{A(G)} \leq 1/n.$$

Proof. Let $A \in \mathscr{R}_c(G)$. By Lemma 3.5,

$$A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i),$$

where $x_i \in G$, H_i is a closed subgroup of G and $\Delta_i \in \mathscr{R}(H_i)$.

It follows from Proposition 3.10 that $s(H_i) = 1$ for each i. Hence $I(H_i)$ has a bounded approximate identity for each i [5, Proposition 3.2].

As $\Delta_i \in \mathscr{R}(H_i)$, $1_{\Delta_i} \in B(H_i)$ by Host's noncommutative version of Cohen's idempotent theorem [10]. By [2, Theorem 2], 1_{Δ_i} extends to a $u \in B(G)$. Then $u \in \mathscr{S}(\Delta_i, H_i \backslash \Delta_i)$. Since G is amenable, Lemma 3.9 implies

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that $I(H_i \backslash \Delta_i)$ has a bounded approximate identity. Clearly $I(x_i(H_i \backslash \Delta_i))$ does as well. It now follows immediately from Lemma 3.8 that $I(A) = I(\bigcup_{i=1}^n x_i(H_i \backslash \Delta_i))$ has a bounded approximate identity.

A careful examination of the respective proofs of Lemma 3.8, Lemma 3.9 and [5, Proposition 3.2] shows that the bounded approximate identity for $I(\bigcup_{i=1}^n x_i(H_i \backslash \Delta_i))$ can be constructed so as to satisfy conditions (i) and (ii) above.

By (i), the approximate identity for $I(A)$ lies in $A(G) \cap C_{00}(G)$. Therefore A is an s-set. Moreover, if I is any ideal with $Z(I) = A$, then $I_0(A) \subseteq I \subseteq I(A)$ [9, p. 93], so that the bounded approximate identity for $I(A)$ is also a bounded approximate identity for I.

Conversely, assume that I has a bounded approximate identity. Then $Z(I) \in \mathscr{R}_c(G)$ exactly as was shown in the proofs of [5, Lemma 3.3] and [5, Proposition 3.5].

Corollary 3.12. *Let G be abelian, compact or amenable and discrete. Then $I(A)$ has a bounded approximate identity if and only if $A \in \mathscr{R}_c(G)$. Furthermore, each such set A is an s-set.*

For an arbitrary locally compact group, it is not necessarily true that every $u \in B(H)$ extends to some $v \in B(G)$. Therefore, it is not clear that $\mathscr{S}(\Delta, H \backslash \Delta) \neq \varnothing$ for every $\Delta \in \mathscr{R}(H)$. It may well be that our technique will fail for groups which are far from [SIN]-groups.

In [5], we showed the existence of bounded approximate identities in various ideals of $A(G)$ to be intimately related to the amenability of G. In particular, we were able to prove the following result.

Theorem 3.13. *Let $G \neq \{e\}$ be a locally compact group. Then the following are equivalent:*

(i) *G is amenable;*

(ii) *$I(H)$ has a bounded approximate identity for some amenable closed subgroup H of G.*

We can now strengthen this result considerably. We begin as follows.

Lemma 3.14. *Let G be a locally compact group. Let H be a proper closed subgroup of G. If $I(H)$ has a bounded approximate identity, then G is amenable.*

Proof. Since H is proper, there exists $x \in G$ with $x \notin H$. Furthermore $I(xH)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathscr{U}}$.

Assume that $v \in C_{00}(H) \cap A(H)$ with $\operatorname{supp} v = K \subset H$. We can find a neighborhood V of K in G and a $u \in B(G)$ such that $u(K) = 1$, $\operatorname{supp} u \subseteq V$ and $V \cap xH = \varnothing$. By [8, Theorem 1a], we can find a $u_1 \in A(G)$ such that $u_{1|_H} = v$. Let $w = u_1 u$. Then $w \in I(xH)$ and $w_{|H} = v$.

Let $v_\alpha = u_{\alpha|_H}$. Then $v_\alpha \in A(H)$ and $\|v_\alpha\|_{A(H)} \leq \|u_\alpha\|_{A(G)}$ [8, Theorem 1a]. Furthermore, if $v \in A(H) \cap C_{00}(H)$ and $w \in I(xH)$ with $w|_H = v$, then

$$0 \leq \lim_\alpha \|v_\alpha v - v\|_{A(H)} \leq \lim_\alpha \|u_\alpha w - w\|_{A(G)} = 0.$$

Since $A(H) \cap C_{00}(H)$ is dense in $A(H)$, $\{v_\alpha\}_{\alpha \in \mathscr{U}}$ is a bounded approximate identity in $A(H)$. By Leptin's theorem [12], H is amenable. It follows from the previous theorem that G is amenable.

COROLLARY 3.15. *Let G be a locally compact group. Then the following are equivalent:*

(i) *G is amenable;*

(ii) *$I(H)$ has a bounded approximate identity for some amenable closed subgroup H of G;*

(iii) *$I(H)$ has a bounded approximate identity for some closed proper subgroup H of G.*

PROPOSITION 3.16. *Let G be a locally compact group. Let*

$$A = \bigcup_{i=1}^{n} x_i(H_i \backslash \Delta_i) \in \mathscr{R}_c(G).$$

If $I(A)$ has a bounded approximate identity, then either G is amenable or for some $1 \leq i_0 \leq n$ there exists a compact subset K of G such that $G = KH_{i_0}$.

PROOF. Assume that for each $1 \leq i \leq n$ that there does not exist a compact K_0 for which $K_0 H_i = G$. Let $K \subset G$ be an arbitrary compact subset. Assume also that for every $x \in G$ that $Ax \cap K \neq \varnothing$. Then $G = K^{-1}A = \bigcup_{i=1}^n K^{-1} x_i H_i$, which is impossible by Lemma 3.2.

Therefore, we can find an $x_K \in G$ such that $K \cap Ax_K = \varnothing$. Since $I(A)$ has a bounded approximate identity, so does $I(Ax)$. By proceeding exactly as in the proof of [5, Proposition 3.22], we can conclude that G is amenable.

PROPOSITION 3.17. *Let G be a locally compact group. Let $A = \bigcup_{i=1}^n x_i H_i$, where $x_i \in G$ and H_i is a closed subgroup of G. If I is a closed ideal in $A(G)$ with $Z(I) = A$ and I has a bounded approximate identity, then either $A = G$ or there exists a closed amenable subgroup H_0 of G and a compact subset K of G such that $G = KH_0$.*

PROOF. We proceed by induction on n. Assume that $A = xH$. If I is a closed ideal in $A(G)$ with $Z(I) = xH$, then $I = I(A)$ since A is an s-set [8, Theorem 2]. If $H \neq G$, then since $I(H)$ also has a bounded approximate identity, G is amenable by Lemma 3.14.

Assume that the statement of the theorem holds true for any G and any union of at most $n-1$ cosets. Let G be a locally compact group containing a subset $A = \bigcup_{i=1}^{n} x_i H_i$ for which there exists a closed ideal I in $A(G)$ with $Z(I) = A$ and is such that I has a bounded approximate identity. If $A \neq G$, then we may assume that G is nonamenable and hence, by Proposition 3.16, that there exists a compact subset K_0 of G such that $K_0 H_{i_0} = G$ for some $1 \leq i_0 \leq n$. By translating if necessary, we can also assume that $e \notin A$.

Let $J = I|H_{i_0}$. Since $A(G)|H_{i_0} = A(H_{i_0})$, J is an ideal in $A(H_{i_0})$. If $x \in H_{i_0} \backslash (H_{i_0} \cap A)$, there exists a $u \in A(G) \cap C_{00}(G)$ with $u(x) = 1$ and $\operatorname{supp} u \cap A = \varnothing$. Because $u \in A(G) \cap C_{00}(G)$, we have that $u \in I$ [9, Theorem 39.18]. Therefore, $Z(J) = H_{i_0} \cap A$. Furthermore, if $\{u_\alpha\}_{\alpha \in \mathscr{U}}$ is a bounded approximate identity in I, then $\{u_\alpha | H_{i_0}\}_{\alpha \in \mathscr{U}}$ is a bounded approximate identity in J and thus also in J^-, the closure of J in $A(H_{i_0})$.

If $H_{i_0} \cap A = \varnothing$, then $J^- = A(H_{i_0})$ since $\varnothing$ is an s-set of H_{i_0}. By Leptin's theorem, H_{i_0} is amenable and we are done. Otherwise, $H_{i_0} \cap A = \bigcup_{i=1, i \neq i_0}^{n} (x_i H_i) \cap H_{i_0} = \bigcup_{j=1}^{m} z_i (H_i \cap H_{i_0})$ for some $m < n$. By the inductive hypothesis, we can find an amenable closed subgroup H_0 of H_{i_0} and a compact subset K_1 of H_{i_0} for which $H_{i_0} = K_1 H_0$. But then $G = (K_0 K_1) H_0$ and the statement of the theorem follows.

COROLLARY 3.18. *Let B be a locally compact group for which $\delta_G | H = \delta_H$ for every closed subgroup H of G. Let $A = \bigcup_{i=1}^{n} x_i H_i$ where $x_i \in G$ and H_i is a closed subgroup of G. If I is a closed ideal in $A(G)$ with $Z(I) = A$ and I has a bounded approximate identity, then either $A = G$ or G is amenable.*

PROOF. If $A \neq G$, then by Proposition 3.17, G has an amenable closed subgroup H_0 for which the coset space G/H_0 is compact. Since $\delta_G | H = \delta_H$, G acts amenably on G/H_0 [4, p. 17]. By [4, p. 16], G is also amenable.

A locally compact group G is called an [IN]-group (*invariant neighborhood group*) if G has a compact neighborhood of e invariant under all inner automorphisms. Clearly [SIN] $\subset$ [IN]. All discrete groups are [IN]-groups.

G is called a [MAP]-group (*maximally almost periodic*) if the finite dimensional irreducible unitary representations of G separate points.

It is well known that both [IN]-groups and [MAP] -groups are unimodular (see [15]). Furthermore, both classes are stable with respect to the taking of closed subgroups [15]. Therefore, we have:

COROLLARY 3.19. *Let G be either in* [IN]*-group or a* [MAP]*-group. Let $A = \bigcup_{i=1}^n x_i H_i$, where $x_i \in G$ and H_i is a closed subgroup of G. If I is a closed ideal in $A(G)$ with $Z(I) = A$ and if I has a bounded approximate identity, then either $A = G$ or G is amenable.*

For discrete groups, we can now completely characterize the closed ideals in $A(G)$ with bounded approximate identities.

THEOREM 3.20. *Let G be a discrete group. Let I be a closed ideal in $A(G)$ with a bounded approximate identity. Let $A = Z(I)$. Then $I = I(A)$, A is an s-set, and either $A = G$ or $G\backslash A = \bigcup_{i=1}^n x_i(H_i\backslash\Delta_i)$, where $x_i \in G$ and each H_i is an amenable subgroup of G with either $\Delta_i = \varnothing$ or $\Delta_i = \bigcup_{j=1}^n x_{i_j} H_{i_j}$ where $x_{i_j} \in H_i$ and each H_{i_j} is a subgroup of H_i.*

Conversely, every such closed ideal I has a bounded approximate identity.

PROOF. If I has a bounded approximate identity, then $Z(I) = A \in \mathscr{R}(G)$ [5, Lemma 3.3]. It follows that $1_A \in B(G)$ and that A is an s-set. Therefore $I = I(A)$.

Since $A \in \mathscr{R}(G)$, $G\backslash A \in \mathscr{R}(G)$ as well. Given Corollary 3.3, we can apply an argument of Cohen [1, p. 221] to conclude that either $G\backslash A = \varnothing$ or $G\backslash A = \bigcup_{i=1}^n x_i(H_i\backslash\Delta_i)$ where $x_i \in G$, H_i is a subgroup of G and either $\Delta_i = \varnothing$ or $\Delta_i = \bigcup_{j=1}^{m_i} x_{i_j} H_{i_j}$, $x_{i_j} \in H_i$ and H_{i_j} is a subgroup of H_i.

We may assume that $A \neq G$ and that $\delta_i \neq H_i$. By translating if necessary, we may also assume that $x_1 = e$. Observe that $1_{H_1\backslash\Delta_1} \in B(G)$. If $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ is a bounded approximate identity for $I(A)$, then $\{u_{\alpha|_{H_1}}\}_{\alpha\in\mathscr{U}}$ is a bounded approximate identity for $1_{H_1\backslash\Delta_1}\cdot I(A)|_{H_1} = I_{H_1}(\Delta_i)$. Since $\Delta_i \neq H_1$, it follows from Corollary 3.18 that H_1 is amenable.

We can show that $H_2, \dots, H_n$ are amenable in the same way.

Conversely, let A be as in the statement of the theorem. Let I be a closed ideal with $Z(I) = A$. As $1_A \in B(G)$, A is an s-set and $I = I(A)$. If $A = G$, then $I = \{0\}$ and we are done. Otherwise, $G\backslash A = \bigcup_{i=1}^n x_i(H_i\backslash\Delta_i)$, where H_i is an amenable subgroup of G and $\Delta_i \in \mathscr{R}(H_i)$. By Theorem 3.11, $I_{H_i}(\Delta_i)$ has a bounded approximate identity $\{v_{\alpha_i}\}_{\alpha_i\in\mathscr{U}_i}$. We may assume that each $v_{\alpha_i} \in A(G)$ by abuse of notation [8, Proposition 5].

By taking all possible intersections of the sets $x_i(H_i\backslash\Delta_i)$, we can write $G\backslash A = \bigcup_{j=1}^m \Gamma_j$ where $\Gamma_j \in \mathscr{R}(G)$, $\Gamma_{j_1} \cap \Gamma_{j_2} = \varnothing$ if $j_1 \neq j_2$ and $\Gamma_j \subseteq$

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$x_{i_j}(H_{i_j}\backslash\Delta_{i_j})$ for some i_j. As $1_{\Gamma_j} \in B(A)$, we have that $w_{\alpha_j} = 1_{\Gamma_j} v_{\alpha_{i_j}} \in A(G)$. For every

$$\alpha = (\alpha_1, \alpha_1, \dots, \alpha_m) \in \mathscr{U}_{i_1} \times \cdots \times \mathscr{U}_{i_m},$$

let $w_\alpha = w_{\alpha_1} + w_{\alpha_2} + \cdots + w_{\alpha_m}$. Then $\{w_\alpha\}_\alpha \in \mathscr{U}_{i-1} \times \cdots \times \mathscr{U}_{i_m}$ is a bounded approximate identity for $I(A)$.

Given a Banach algebra $\mathscr{A}$, both $\mathscr{A}^*$ and $\mathscr{A}^{**}$ becomes $\mathscr{A}$-bimodules in a natural way. In particular, $\mathscr{A}^{**}$ becomes a Banach algebra with respect to either Arens product [9]. We close with the following application.

THEOREM 3.21. *Let G be an amenable* [SIN]*-group. Let $A \in \mathscr{R}_c(G)$ and let $I = I(A)$. Then there exists a bounded linear map $\mathscr{M}$ from* $\mathrm{Hom}^I(I, I)$ *into $(I \cdot I^*)^*$. Furthermore, $\mathscr{M}$ is onto if and only if G is discrete.*

PROOF. If G is an amenable [SIN]-group, then by Theorem 3.11, $I(A)$ has a bounded approximate identity. The existence of $\mathscr{M}$ follows immediately from [5, Proposition 6.6].

If G is discrete, then $A(G) \cdot A(G)^{**} \subseteq A(G)$ [11, Theorem 3.7]. Cohen's factorization theorem [9, p. 268] implies that $I^2 = I$. Therefore

$$I \cdot I^{**} = I \cdot ((I^\perp)^\perp) \subseteq (I \cdot I) \cdot A(G)^{**} = I \cdot (I \cdot A(G))^{**} \subseteq I.$$

By [5, Proposition 6.6], $\mathscr{M}$ is onto.

Conversely, assume that $\mathscr{M}$ is onto. By translating A if necessary, we may assume that $e \in G\backslash A$. Let $u_0 \in P(G) \cap I$ with $u(e) = q$. Following an idea of Lau's [11, Theorem 3.7], let

$$K = \{u_0 \cdot \Gamma; \Gamma \in A(G)^{**},\ \Gamma \geq 0,\ \|\Gamma\| = 1\}.$$

Since the map $\Gamma \to u_0\Gamma$ is weak-$*$ to weak-$*$ continuous, K is weak-$*$ compact. By [5, Proposition 6.6], K is a weakly compact subset of $A(G)$. Let $\Lambda = \{u \in P(G) \cap I;\ u(e) = 1\}$. For each $u \in \Lambda$, define $\Lambda_u(v) = uv$ for every $v \in K$. Then $\{\Lambda_u; u \in \Lambda\}$ is a commutative semigroup of continuous maps from $\{K,\ \text{weak}\}$ into $\{K,\ \text{weak}\}$. By the Markov-Kakutani fixed-point theorem, there exists some $v_0 \in X$ such that

$$\Lambda_u(v_0) = v_0 \quad \text{for every} \quad u \in \Lambda.$$

If $x \neq e$, there exists $u \in \Lambda$ such that $u(x) \neq 1$. Therefore $v_0 = 1_{\{e\}}$ and G is discrete.

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