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RESEARCH ARTICLE

Sparse analytic systems

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Abstract
Erdős [7] proved that the Continuum Hypothesis (CH) is equivalent to the existence of an uncountable family $\mathcal{F}$ of (real or complex) analytic functions, such that $\{f(x) : f \in \mathcal{F}\}$ is countable for every x. We strengthen Erdős' result by proving that CH is equivalent to the existence of what we call *sparse analytic systems* of functions. We use such systems to construct, assuming CH, an equivalence relation $\sim$ on $\mathbb{R}$ such that any 'analytic-anonymous' attempt to predict the map $x \mapsto [x]_\sim$ must fail almost everywhere. This provides a consistently negative answer to a question of Bajpai-Velleman [2].

1. Introduction

In the early 1960s, John Wetzel posed the following problem.

> **Wetzel's Problem:** If $\mathcal{F}$ is a family of analytic functions (on some common domain) such that $\{f(x) : f \in \mathcal{F}\}$ is countable for every x, must $\mathcal{F}$ be a countable family?

A few years later, Erdős proved that an affirmative answer to Wetzel's Problem is equivalent to the negation of Cantor's *Continuum Hypothesis (CH)*. Combined with Paul Cohen's proof of the independence of CH, this showed that Wetzel's Problem is independent of the standard axioms of mathematics (ZFC). Upon learning of Erdős' theorem, Wetzel remarked to his dissertation advisor (Halsey Royden) that '. . . once again a natural analysis question has grown horns!' This quote, and other interesting history surrounding Wetzel's Problem, appears in Garcia-Shoemaker [10]. Erdős' proof even made it into Aigner-Ziegler's 'Proofs from the Book' ([1]). It will be more convenient for us to state and refer to Erdős' equivalence in the negated form.

Theorem 1 (Erdős [7]). *The following are equivalent:*

(1) *CH;*
(2) *There exists an uncountable family $\mathcal{F}$ of analytic functions on some fixed open domain D of either*

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$\mathbb{R}$ or $\mathbb{C}$, such that for every $x \in D$,

$$\{f(x) \;:\; f \in \mathcal{F}\}$$

is countable.

Motivated by connections to work of Hardin-Taylor ([11], [12]) and Bajpai-Velleman [2] described below, we strengthen Theorem 1 as follows. If $P \in \mathbb{R}^2$, we denote the first coordinate of P by x_P and the second coordinate by y_P. Define a **sparse (real) analytic system** to mean a collection

$$\{f_P \;:\; P \in \mathbb{R}^2\}$$

such that:

(1) for all $P \in \mathbb{R}^2$, f_P is an increasing, analytic bijection from $\mathbb{R} \to \mathbb{R}$ that passes through the point P; and
(2) For all $z \in \mathbb{R}$, the sets

$$\{f_P(z) \;:\; P \in \mathbb{R}^2 \text{ and } z \neq x_P\}$$

and

$$\{f_P^{-1}(z) \;:\; P \in \mathbb{R}^2 \text{ and } z \neq y_P\}$$

are both countable.

We prove the following strengthening of Erdős' Theorem 1.

Theorem 2. *The following are equivalent:*

(1) *CH*
(2) *There exists a sparse real analytic system.*

We use Theorem 2 to answer a question of Bajpai and Velleman, assuming CH. Given a nonempty set S, let ${}^{\mathbb{R}}\boldsymbol{S}$ denote the collection of total functions from $\mathbb{R}$ to S, and let $\overset{\mathbb{R}}{\smile}\boldsymbol{S}$ denote the collection of all S-valued functions f such that $\mathrm{dom}(f) = (-\infty, t_f)$ for some $t_f \in \mathbb{R}$. An ***S*-predictor** will refer to any function $\mathcal{P}$ with domain and codomain as follows:

$$\mathcal{P} : \overset{\mathbb{R}}{\smile}S \to S. \tag{1}$$

An S-predictor $\mathcal{P}$ will be called **good** if for all $F \in {}^{\mathbb{R}}S$, the set

$$\Big\{t \in \mathbb{R} \;:\; F(t) = \mathcal{P}\big(F \restriction (-\infty, t)\big)\Big\}$$

has full measure in $\mathbb{R}$. So $\mathcal{P}$ is good if for any total $F : \mathbb{R} \to S$, $\mathcal{P}$ 'almost always' correctly predicts $F(t)$ based only on $F \restriction (-\infty, t)$.[1] Hardin-Taylor [11] proved that for any set S, there exists a good S-predictor, and in [12], they raised the question of whether these good predictors could also be arranged to be 'Γ-anonymous' with respect to certain classes $\Gamma \subseteq \mathrm{Homeo}^+(\mathbb{R})$;[2] an S-predictor $\mathcal{P}$ is Γ-anonymous if for every $\varphi \in \Gamma$ and every $f \in \overset{\mathbb{R}}{\smile}S$,

$$\mathcal{P}(f) = \mathcal{P}(f \circ \varphi),$$

where $f \circ \varphi$ is the member of $\overset{\mathbb{R}}{\smile}S$ whose domain is understood to be $\big(-\infty, \varphi^{-1}(t_f)\big)$. Bajpai and Velleman [2] gave a positive and a negative result:

[1] Note that F is allowed to be highly discontinuous; otherwise, the problem trivializes since one could simply predict $F(t)$ by considering $\lim_{x \nearrow t} F(x)$, which only depends on $F \restriction (-\infty, t)$.
[2] $\mathrm{Homeo}^+(\mathbb{R})$ denotes the set of increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

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- ◦ For every set S, there exists a good S-predictor that is anonymous with respect to the class of affine functions on the reals. This strengthened a previous theorem of Hardin-Taylor [12], who had gotten the same result for the smaller class of affine functions of slope 1 (i.e, shifts).
- ◦ There is an equivalence relation $\sim$ on $\mathbb{R}$ such that, letting $S := \mathbb{R}/\sim$, there is **no** good S-predictor that is anonymous with respect to the class of increasing C^∞ bijections on $\mathbb{R}$.

They asked about classes intermediate between the affine functions and the C^∞ functions.

Question 3 (Bajpai-Velleman [2], page 788). Does there exist (for every set S) a good S-predictor that is anonymous with respect to the analytic members of $\mathrm{Homeo}^+(\mathbb{R})$?

We use Theorem 2, together with an argument from Bajpai-Velleman [2], to prove:

Theorem 4. *Assuming CH, the answer to Question 3 is negative.*

Section 2 provides an interpolation theorem that will be used in the proof of Theorem 2, Section 3 proves Theorem 2, Section 4 proves Theorem 4, and Section 5 has concluding remarks and open questions.

2. An interpolation theorem

A key part of the proof of Theorem 2 is the (ZFC) Theorem 5 below. One of the referees pointed out that Theorem 5 follows from known results; in particular, it follows from the much more powerful Theorem 3.2 of Burke [4] or, with modifications in the proofs, either Theorem 2 of Barth-Schneider [3] or Corollary 1.9 of Burke [5]. Since deriving Theorem 5 from those more powerful theorems is not trivial, we choose to present our original direct proof of Theorem 5.

Recall that Cantor proved that any two countable dense subsets of $\mathbb{R}$ are order-isomorphic and that this order-isomorphism easily extends uniquely to a homeomorphism of $\mathbb{R}$. Franklin [9] considered the question of how nice this homeomorphism could be arranged to be, and showed that if D and E are countable dense subsets of $\mathbb{R}$, then there is an order-isomorphism of D with E that extends to a real analytic function. A series of papers improved this result, culminating in Barth-Schneider [3], who proved that there is an order-isomorphism of D with E that extends to an entire function $f : \mathbb{C} \to \mathbb{C}$, answering (one interpretation of) Question 24 of Erdős [8].[3] Subsequent work of Burke, mentioned above, further strengthened those results. The variant we will need for the proof of Theorem 2 follows.

Theorem 5. *Suppose $\mathcal{D}$ is a partition of $\mathbb{R}$ into dense subsets of $\mathbb{R}$; for each $z \in \mathbb{R}$, let D_z denote the unique $D \in \mathcal{D}$ such that $z \in D$.*

Then for any $P = (x_P, y_P) \in \mathbb{R}^2$ and any countable set W of reals, there is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that:

(1) *$f \restriction \mathbb{R}$ is real-valued (hence analytic, since $f : \mathbb{C} \to \mathbb{C}$ is entire);*
(2) *$f \restriction \mathbb{R}$ is a bijection with strictly positive derivative;*
(3) *$f(x_P) = y_P$; and*
(4) *for each $w \in W$,*
   (a) *if $w \neq x_P$, then $f(w) \in D_w$;*
   (b) *if $w \neq y_P$, then $f^{-1}(w) \in D_w$.*

Let us give a brief outline of the following proof of Theorem 5, which is inspired by the proof of Nienhuys-Thiemann [14]. We will inductively define a sequence of functions $\langle f_n : n \in \mathbb{N}\rangle$ whose limit will be the desired function f. Each function f_n will satisfy a version of Theorem 5(4) for finitely many points in W. When we define the next function f_{n+1}, we will want it to be equal to f_n on these finitely many points in W that have already been taken care of, and we will want f_{n+1} to satisfy Theorem 5(4a) or Theorem 5(4b), depending on whether n is even or odd, for an additional point in W. We will write A_n to denote the set of finitely many points of W that have already been taken care of at stage n with

[3]See also Maurer [13], Nienhuys-Thiemann [14] and Sato-Rankin [15] for related results. Burke [5] provides a nice historical overview of this literature on this topic.

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regard to Theorem 5(4a), and we will write B_n to denote the set of finitely many points of W that have been taken care of in regard to Theorem 5(4b).

Suppose $\mathcal{D}$ is a partition of $\mathbb{R}$ into dense sets, W is a countable set of real numbers, and $P = (x_P, y_P)$ is a point in $\mathbb{R}^2$. Fix a 1-1 enumeration $\{w_n : n \in \mathbb{N}\}$ of W, and for each n, let D_n be the unique member of $\mathcal{D}$ containing w_n. Since $\mathcal{D}$ is a partition, we have

$$\forall k, n \in \mathbb{N} \ \left(w_k \in D_n \iff D_k = D_n \iff w_n \in D_k\right). \tag{*}$$

Suppose $p : \mathbb{R} \to \mathbb{R}$ is a continuous positive function such that

$$\forall n \in \mathbb{N} \ \lim_{t\to\infty} \frac{p(t)}{t^n} = \infty. \tag{2}$$

We will inductively define sequences $\langle f_n : n \in \mathbb{N}\rangle$, $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle B_n : n \in \mathbb{N}\rangle$ such that $A_0 = \emptyset$ and $B_0 = \emptyset$ and for all $n \in \mathbb{N}$, we have

- (I)$_n$ $f_n : \mathbb{C} \to \mathbb{C}$ is entire and $f_n \restriction \mathbb{R}$ is real-valued;
- (II)$_n$ $f_n(x_P) = y_P$;
- (III)$_n$ $\forall x \in \mathbb{R}\ f_n'(x) \geq \frac{1}{2} + \frac{1}{2^n}$, and thus $f_n \restriction \mathbb{R}$ is a bijection;
- (IV)$_n$ if $n > 0$, then $\forall z \in \mathbb{C}\ |f_n(z) - f_{n-1}(z)| < \frac{1}{2^n} p(|z|)$;
- (V)$_n$ if $n = 2k+1$ is odd, then $A_n = A_{n-1} \cup \{w_k\}$, $B_n = B_{n-1}$ and we have $w_k \neq x_P \implies f_n(w_k) \in D_k$;
- (VI)$_n$ if $n = 2k+2$ is even, then $A_n = A_{n-1}$, $B_n = B_{n-1} \cup \{w_k\}$ and we have $w_k \neq y_P \implies f_n^{-1}(w_k) \in D_k$; and
- (VII)$_n$ if $n > 0$, then $f_n \restriction A_{n-1} = f_{n-1} \restriction A_{n-1}$ and $f_n^{-1} \restriction B_{n-1} = f_{n-1}^{-1} \restriction B_{n-1}$.

First, let us show that, assuming we have sequences $\langle f_n : n \in \mathbb{N}\rangle$ and $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle B_n : n \in \mathbb{N}\rangle$ satisfying (I)$_n$-(VII)$_n$ for all n, the pointwise limit defined by $f(z) = \lim_{n\to\infty} f_n(z)$ has all of the desired properties. Suppose D is any compact subset of $\mathbb{C}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and since $p(|z|)$ is bounded on D, the fact that (IV)$_n$ holds for all n ensures that the sequence $\langle f_n : n \in \mathbb{N}\rangle$ is uniformly Cauchy on D. Hence, we can define a function $f : \mathbb{C} \to \mathbb{C}$ by letting $f(z) = \lim_{n\to\infty} f_n(z)$. Since the sequence $\langle f_n : n \in \mathbb{N}\rangle$ is uniformly Cauchy on any compact set, it follows that the convergence of $\langle f_n : n \in \mathbb{N}\rangle$ to f is uniform on any compact set, and hence, f is an entire function.

Now let us verify that Theorem 5(1)–(4) hold for f. By (I)$_n$ and closure of $\mathbb{R}$ in $\mathbb{C}$, we see that $f \restriction \mathbb{R}$ is real valued, and since (III)$_n$ holds for all n, we have $f'(x) \geq \frac{1}{2}$ for all $x \in \mathbb{R}$. Thus, Theorem 5(1) and Theorem 5(2) hold. Theorem 5(3) holds since the sequence $\langle f_n(x_P) : n \in \mathbb{N}\rangle$ is constantly equal to y_P. To show that Theorem 5(4) holds, let us prove that for all $i \in \mathbb{N}$, if $w_i \neq x_P$, then $f(w_i) \in D_i$, and if $w_i \neq y_P$, then $f^{-1}(w_i) \in D_i$. Fix $i \in \mathbb{N}$. We have $w_i \in A_{2i+1}$ and $w_i \in B_{2i+2}$, and furthermore, by (V)$_{2i+1}$ and (VI)$_{2i+2}$, $w_i \neq x_P$ implies $f_{2i+1}(w_i) \in D_i$ and $w_i \neq y_P$ implies $f_{2i+2}^{-1}(w_i) \in D_i$. Since (VII)$_n$ holds for all n, we see that both of the sequences $\langle f_n(w_i) : n \in \mathbb{N}\rangle$ and $\langle f_n^{-1}(w_i) : n \in \mathbb{N}\rangle$ are eventually constant, and indeed, for $n \geq 2i+2$, we have $f_n(w_i) = f_{2i+1}(w_i)$ and $f_n^{-1}(w_i) = f_{2i+2}^{-1}(w_i)$. Therefore, $f(w_i) = f_{2i+1}(w_i)$ and $f^{-1}(w_i) = f_{2i+2}^{-1}(w_i)$, so (4) holds.

It remains to show that we can inductively define sequences $\langle f_n : n \in \mathbb{N}\rangle$, $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle B_n : n \in \mathbb{N}\rangle$ that satisfy (I)$_n$–(VII)$_n$ for all $n \in \mathbb{N}$.

Let $f_0 : \mathbb{C} \to \mathbb{C}$ be $f_0(z) = \frac{3}{2}(z - x_P) + y_P$, $A_0 = \emptyset$ and $B_0 = \emptyset$. One may easily verify that (I)$_0$–(VII)$_0$ hold. For $n > 0$, Section 2.1 shows how f_n is constructed when n is odd, and Section 2.2 shows how f_n is constructed when n is even.

2.1. When n is odd

Suppose $n = 2k+1 > 0$ is odd and that f_i, A_i and B_i satisfying (I)$_i$–(VII)$_i$ have already been defined for $i \leq 2k$. If $k = 0$, we have $A_0 = \emptyset$ and $B_0 = \emptyset$, whereas if $k > 0$, we have

$$A_{n-1} = A_{2k} = A_{2(k-1)+2} = \{w_0, \ldots, w_{k-1}\}$$

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and

$$B_{n-1} = B_{2k} = \{w_0, \ldots, w_{k-1}\}.$$

In any case, we let $A_n = A_{n-1} \cup \{w_k\}$ and $B_n = B_{n-1}$. We define $f_n = f_{2k+1}$ in two cases as follows.

Case 2.1. A: $w_k \notin \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$. Let us argue that there is an entire function g_n such that

(i) $(\forall z \in \mathbb{C})\ g_n(z) = 0 \iff z \in \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$,
(ii) $(\forall z \in \mathbb{C})\ |g_n(z)| \leq \frac{1}{2^n} p(|z|)$ and
(iii) $(\forall x \in \mathbb{R})\ g_n'(x) \geq -\frac{1}{2^n}$.

Take

$$h_n(z) = (z - x_P)^{\beta_n}(z - w_0)\cdots(z - w_{k-1})(z - f_{n-1}^{-1}(w_0))\cdots(z - f_{n-1}^{-1}(w_{k-1})),$$

where $\beta_n \in \{1, 2\}$ is such that the degree of h_n is odd. We will show that for small enough positive $\alpha_n \in \mathbb{R}$, the function $g_n(z) = \alpha_n h_n(z)$ satisfies (i)–(iii). Clearly, h_n satisfies (i), so any such function $g_n(z)$ satisfies (i). For (ii), choose $m \in \mathbb{N}$ and some positive $c \in \mathbb{R}$ such that $|h_n(z)| \leq |z|^m + c$ for all $z \in \mathbb{C}$. By our assumption on p, we have $\lim_{|z|\to\infty} \frac{p(|z|)}{|z|^m + c} = \infty$, and thus we can let $D \subseteq \mathbb{C}$ be a large enough closed disk centered at the origin such that $z \in \mathbb{C} \setminus D$ implies $1 \leq \frac{p(|z|)}{|z|^m + c}$. Since p is a continuous positive function, we can choose a positive $\alpha_n \in \mathbb{R}$ such that $\alpha_n \leq \frac{1}{2^n}$ and $\alpha_n \leq \frac{p(|z|)}{2^n(|z|^m + c)}$ for all $z \in D$. Then it follows that for every $z \in \mathbb{C}$, we have

$$|\alpha_n h_n(z)| \leq \alpha_n(|z|^m + c) \leq \frac{1}{2^n} p(|z|).$$

Let us verify that (iii) holds for small enough α_n. Since h_n is odd and has a positive leading coefficient, the derivative of $h_n \restriction \mathbb{R}$ is bounded below. So we may let $d = \inf\{h_n'(x) : x \in \mathbb{R}\} \in \mathbb{R}$. Thus, we may choose a small enough positive $\alpha_n \in \mathbb{R}$ such that $\alpha_n d \geq -\frac{1}{2^n}$, and then it follows that for all $x \in \mathbb{R}$, we have $\alpha_n h_n'(x) \geq \alpha_n d \geq -\frac{1}{2^n}$.

Using the case assumption that $w_k \notin \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$, we see that $g_n(w_k) \neq 0$, and hence it follows that the set

$$\{f_{n-1}(w_k) + M g_n(w_k) : M \in [0, 1]\}$$

is a nontrivial interval of real numbers. Thus, since D_k is dense in $\mathbb{R}$, it follows that there is some $M_n \in [0, 1]$ such that $f_{n-1}(w_k) + M_n g_n(w_k) \in D_k$. We define

$$f_n(z) = f_{n-1}(z) + M_n g_n(z).$$

Let us show that $(\mathrm{I})_n$–$(\mathrm{VII})_n$ hold. It is trivial to see that $(\mathrm{I})_n$ and $(\mathrm{II})_n$ are true. For $(\mathrm{III})_n$, notice that because $M_n \in [0, 1]$, and since (iii) and $(\mathrm{III})_{n-1}$ both hold, we have for all $x \in \mathbb{R}$,

$$f_n'(x) = f_{n-1}'(x) + M_n g_n'(x) \geq \frac{1}{2} + \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^n},$$

and thus $f_n : \mathbb{R} \to \mathbb{R}$ is a bijection. For $(\mathrm{IV})_n$, we have for all $z \in \mathbb{C}$,

$$|f_n(z) - f_{n-1}(z)| = M_n |g_n(z)| \leq \frac{1}{2^n} p(|z|),$$

where the last inequality follows since $M_n \in [0, 1]$ and (ii) holds. Let us verify that $(\mathrm{V})_n$ holds. From the definition of $f_n = f_{2k+1}$ and the way we chose M_n, it follows that $f_n(w_k) \in D_k$ (notice that $w_k \neq x_P$ by our case assumption). Thus, $(\mathrm{V})_n$ holds. $(\mathrm{VI})_n$ holds trivially since n is odd. To see that $(\mathrm{VII})_n$ holds,

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note that since $g_n(z) = 0$ if $z \in \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$, it follows directly from the definition of f_n that $f_n \restriction A_{n-1} = f_n \restriction A_{n-1}$ and $f_n^{-1} \restriction B_n = f_{n-1}^{-1} \restriction B_{n-1}$.

Case 2.1. B: $\boldsymbol{w_k \in \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})}$. Then we let $f_n = f_{n-1}$, $A_n = A_{n-1} \cup \{w_k\}$ and $B_n = B_{n-1}$. Let us argue that this definition of f_n satisfies $(\mathrm{V})_n$; the rest of $(\mathrm{I})_n$–$(\mathrm{VII})_n$ are easily seen to hold by the inductive hypothesis. Suppose $w_k \neq x_P$. Since the enumeration of W is one-to-one, we have $w_k \neq w_j$ for all $j \leq k-1$. Thus, for some $j \leq k-1$, we have $w_k = f_{n-1}^{-1}(w_j)$, and because $f_{n-1}(x_P) = y_P$, f_{n-1} is injective and $w_k \neq x_P$, it follows that $w_j \neq y_P$. Since $2j+2 \leq n-1$ and since it follows by our inductive assumptions $(\mathrm{VII})_\ell$ for $\ell \leq n-1$, that $f_{n-1} \restriction A_{2j+2} = f_{2j+2} \restriction A_{2j+2}$, we see that $w_k = f_{n-1}^{-1}(w_j) = f_{2j+2}^{-1}(w_j) \in D_j$. Then $D_j = D_k$ by (*) from page 4. So, $f_n(w_k) = f_{n-1}(w_k) = w_j \in D_j = D_k$, and hence, $(\mathrm{V})_n$ holds.

2.2. When n is even

Now suppose $n = 2k+2$ is even, where $k > 0$, and that f_i, A_i and B_i satisfying $(\mathrm{I})_i$–$(\mathrm{VI})_i$ have already been defined for $i \leq 2k+1$. We have

$$A_{2k+1} = \{w_0, \ldots, w_k\}$$

and

$$B_{2k+1} = \{w_0, \ldots, w_{k-1}\}.$$

We will define f_n, A_n and B_n in two cases as follows.

Case 2.2. A: $\boldsymbol{f_{n-1}^{-1}(w_k) \notin \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})}$. Then we let g_n be an entire function such that

(i) $(\forall z \in \mathbb{C})\ g_n(z) = 0 \iff z \in \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$,
(ii) $(\forall z \in \mathbb{C})\ |g_n(z)| \leq \frac{1}{2^n} p(|z|)$ and
(iii) $(\forall x \in \mathbb{R})\ g_n'(x) \geq -\frac{1}{2^n}$.

For example, as in the case above where n was odd, we could take

$$g_n(z) = \alpha_n (z - x_P)^{\beta_n} (z - w_0) \cdots (z - w_k)(z - f_{n-1}^{-1}(w_0)) \cdots (z - f_{n-1}^{-1}(w_{k-1}))$$

satisfying (i)–(iii) by choosing α_n small enough and $\beta_n \in \{1, 2\}$ so that the degree of g_n is odd. By our inductive assumption about f_{n-1} and by (iii), it follows that for any $M \in [0,1]$ and any $x \in \mathbb{R}$, we have

$$f_{n-1}'(x) + M g_n'(x) \geq \frac{1}{2} + \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^n} > 0.$$

Thus, the function $f_{n-1} + M g_n : \mathbb{R} \to \mathbb{R}$ is a bijection. Let us argue that the set

$$\{(f_{n-1} + M g_n)^{-1}(w_k) : M \in [0,1]\}$$

is a nontrivial interval of real numbers. It will suffice to show that $(f_{n-1} + g_n)^{-1}(w_k) \neq f_{n-1}^{-1}(w_k)$. Suppose $(f_{n-1} + g_n)^{-1}(w_k) = f_{n-1}^{-1}(w_k)$. Then $f_{n-1}(f_{n-1}^{-1}(w_k)) = w_k$ and $(f_{n-1} + g_n)(f_{n-1}^{-1}(w_k)) = w_k$. This implies that the functions f_{n-1} and $f_{n-1} + g_n$ are equal at the point $f_{n-1}^{-1}(w_k)$, and hence, $g_n(f_{n-1}^{-1}(w_k)) = 0$, which contradicts (i) by our case assumption that $f_{n-1}^{-1}(w_k) \notin \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$.

Thus, since D_k is dense in $\mathbb{R}$, it follows that there is some $M_n \in [0,1]$ such that $(f_{n-1} + M_n g_n)^{-1}(w_k) \in D_k$. We fix such an M_n and define

$$f_n(z) = f_{n-1}(z) + M_n g_n(z).$$

We also let $A_n = A_{n-1}$ and $B_n = B_{n-1} \cup \{w_k\}$. The verification that $(\mathrm{I})_n$–$(\mathrm{VII})_n$ hold is straightforward and similar to the above; it is therefore left to the reader.

Case 2.2. B: $f_{n-1}^{-1}(w_k) \in \{x_P\} \cup A_{n-1} \cup f_{n-1}^{-1}(B_{n-1})$, or equivalently, $w_k \in \{y_P\} \cup f_{n-1}(A_{n-1}) \cup B_{n-1}$. Then we define $f_n = f_{n-1}$. As in the odd case above, this definition of f_n is easily seen to satisfy $(\mathrm{I})_n$–$(\mathrm{V})_n$ and $(\mathrm{VII})_n$. Let us check $(\mathrm{VI})_n$. Suppose $w_k \neq y_P$. Since the enumeration of W is one-to-one, we have $w_k \neq w_j$ for all $j \leq k-1$, and hence, $w_k = f_{n-1}(w_j)$ for some $j \leq k$, where $w_j \neq x_P$. Since $2j+1 \leq n-1$, it follows by our inductive assumptions $(\mathrm{V})_\ell$ for $\ell \leq n-1$ that $f_{n-1} \restriction A_{2j+1} = f_{2j+1} \restriction A_{2j+1}$ and $w_k = f_{n-1}(w_j) = f_{2j+1}(w_j) \in D_j$. Then by (*) from page 4, $D_j = D_k$. So $f_n^{-1}(w_k) = f_{n-1}^{-1}(w_k) = w_j \in D_j = D_k$.

This concludes the proof of Theorem 5.

3. Proof of Theorem 2

To prove the $\Leftarrow$ direction of Theorem 2, assume that $\left\{f_P \ : \ P \in \mathbb{R}^2\right\}$ is a sparse analytic system and consider the subcollection $\left\{f_{(0,y)} \ : \ y \in \mathbb{R}\right\}$. Since $f_{(0,y)}$ passes through the point $(0,y)$ and each $f_{(0,y)}$ is analytic, and hence continuous, it follows that for $y \neq y'$, $f_{(0,y)} \restriction (-\infty,0) \neq f_{(0,y')} \restriction (-\infty,0)$. So

$$\mathcal{F} := \left\{f_{(0,y)} \restriction (-\infty,0) \ : \ y \in \mathbb{R}\right\}$$

is a continuum-sized collection of analytic functions on the common domain $D := (-\infty,0)$. Furthermore, given any $z \in D$, since $z \neq 0$ and the f_P's formed a sparse analytic system, it follows that

$$\{f_{(0,y)}(z) \ : \ y \in \mathbb{R}\}$$

is countable. So $\mathcal{F}$ is a collection of analytic functions as in clause (2) of Erdős' Theorem 1. So by that theorem, CH must hold.

To prove the $\Rightarrow$ direction of Theorem 2 – which is heavily inspired by Erdős' proof of Theorem 1 – assume CH and fix an enumeration $\langle w_\alpha \ : \ \alpha < \omega_1\rangle$ of $\mathbb{R}$. Fix any partition $\mathcal{D}$ of the reals into countable dense subsets of $\mathbb{R}$.[4] For each $\alpha < \omega_1$, let D_α be the unique member of $\mathcal{D}$ containing w_α. Also fix an ω_1-enumeration $\langle P_\alpha = (a_\alpha, b_\alpha) \ : \ \alpha < \omega_1\rangle$ of $\mathbb{R}^2$.

Fix an $\alpha < \omega_1$. By Theorem 5, there exists an entire $f_\alpha : \mathbb{C} \to \mathbb{C}$ such that:

(1) $f_\alpha \restriction \mathbb{R}$ is a real analytic bijection with strictly positive derivative;
(2) $f_\alpha(a_\alpha) = b_\alpha$ (i.e., $f_\alpha \restriction \mathbb{R}$ passes through the point P_α);
(3) For each w_ξ in the countable set $W_\alpha := \{w_\xi \ : \ \xi < \alpha\}$,
   (a) if $w_\xi \neq a_\alpha$, then $f_\alpha(w_\xi) \in D_\xi$; and
   (b) if $w_\xi \neq b_\alpha$, then $f_\alpha^{-1}(w_\xi) \in D_\xi$.

We claim that $\{f_\alpha \restriction \mathbb{R} \ : \ \alpha < \omega_1\}$ is a sparse analytic system, and the only nontrivial requirement to verify is that if $w \in \mathbb{R}$, then both

$$A_w := \left\{f_\alpha(w) \ : \ \alpha < \omega_1 \text{ and } w \neq a_\alpha\right\}$$

and

$$B_w := \left\{f_\alpha^{-1}(w) \ : \ \alpha < \omega_1 \text{ and } w \neq b_\alpha\right\}$$

are countable. Say $w = w_\xi$; then,

$$A_w = A_{w_\xi} \subseteq \underbrace{\{f_\alpha(w_\xi) \ : \ \xi < \alpha < \omega_1 \text{ and } w_\xi \neq a_\alpha\}}_{\subseteq D_\xi, \text{ by 3a}} \cup \underbrace{\{f_\alpha(w_\xi) \ : \ \alpha \leq \xi\}}_{\text{countable because } \xi < \omega_1},$$

[4]For example, define an equivalence relation $\sim$ on $\mathbb{R}$ by: $x \sim y$ iff $y = rx$ for some nonzero $r \in \mathbb{Q}$. Then the set of equivalence classes constitutes a partition of $\mathbb{R}$ into countable dense subsets of $\mathbb{R}$. We thank Alex Misiats for pointing out this example (since our original draft used CH to get such a partition).

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and hence, A_w is countable. Similarly,

$$B_w = B_{w_\xi} \subseteq \underbrace{\{f_\alpha^{-1}(w_\xi) \ : \ \xi < \alpha < \omega_1 \text{ and } w_\xi \neq b_\alpha\}}_{\subseteq D_\xi,\text{ by 3b}} \cup \underbrace{\{f_\alpha^{-1}(w_\xi) \ : \ \alpha \leq \xi\}}_{\text{countable because } \xi<\omega_1},$$

and hence, B_w is countable.

4. Proof of Theorem 4

The next lemma is the key connection between sparse analytic systems and predictors.

Lemma 6. *Suppose $\mathcal{F} = \langle f_P \ : \ P \in \mathbb{R}^2\rangle$ is a sparse analytic system. Let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by the set*

$$X := \Big\{(u,v) \in \mathbb{R}^2 \ : \ \exists P \in \mathbb{R}^2 \ \big(u \neq x_P \ \wedge \ v \neq y_P \ \wedge \ f_P(u) = v\big)\Big\}.$$

Then,

(1) *Each $\sim$-equivalence class is countable.*
(2) *For every $P = (x_P, y_P) \in \mathbb{R}^2$ and every $z \in \mathbb{R}$: if $z \neq x_P$, then $z \sim f_P(z)$.*

Before proving Lemma 6, we say how the proof of Theorem 4 is finished: assuming CH, Theorem 2 yields the existence of a sparse analytic system. Let $\sim$ be the equivalence relation on $\mathbb{R}$ induced by the sparse analytic system via Lemma 6. The properties of $\sim$ listed in the conclusion of Lemma 6 satisfy the assumptions of Lemma 20 of Cox-Elpers [6], and that lemma tells us that if $S := \mathbb{R}/\sim$ and

$$\mathcal{P} : {}^{\mathbb{R}}_{\smile}S \to S$$

is any analytic-anonymous S-predictor,[5] then $\mathcal{P}$ fails to predict the function $x \mapsto [x]_\sim$ for almost every $x \in \mathbb{R}$.[6] In particular, there is no good analytic-anonymous S-predictor.

(Proof of Lemma 6). Part (2) holds because, by the definition of sparse analytic system, f_P is injective and $f_P(x_P) = y_P$. So if $z \neq x_P$, then $f_P(z) \neq y_P$; so not only is $z \sim f_P(z)$, but the pair $\big(z, f_P(z)\big)$ is an element of X.

To prove part (1), since X generates $\sim$, it suffices to prove that for every $z \in \mathbb{R}$, both

$$z^\uparrow := \{v \in \mathbb{R} \ : \ (z,v) \in X\} = \{v \in \mathbb{R} \ : \ \exists P \in \mathbb{R}^2 \ (z \neq x_P \wedge v \neq y_P \wedge f_P(z) = v)\}$$

and

$$z_\downarrow := \{u \in \mathbb{R} \ : \ (u,z) \in X\} = \{u \in \mathbb{R} \ : \ \exists P \in \mathbb{R}^2 \ (u \neq x_P \wedge z \neq y_P \wedge f_P(u) = z)\}$$

are countable. But

$$z^\uparrow \subseteq \{f_P(z) \ : \ z \neq x_P\}$$

and

$$z_\downarrow \subseteq \{f_P^{-1}(z) \ : \ z \neq y_P\},$$

which are both countable by definition of sparse analytic system. □

[5] Recall these notions were defined in Section 1.
[6] Strictly speaking, the *statement* of [6, Lemma 20] only implies that an analytic-anonymous predictor fails to predict $x \mapsto [x]_\sim$ on a *positive*-measure set. This is good enough to answer Question 3, since such a predictor would not be good. But the *proof* of [6, Lemma 20] – which was due essentially to Bajpai-Velleman [2] – shows that an analytic-anonymous predictor can successfully predict $x \mapsto [x]_\sim$ only for those x lying in some fixed equivalence class, which, in the context of Lemma 6, is countable. So analytic-anonymous predictors fail to predict $x \mapsto [x]_\sim$ almost everywhere in this situation.

5. Concluding Remarks

The notion of a sparse analytic system obviously generalizes to a sparse Γ-system for any $\Gamma \subseteq \mathrm{Homeo}^+(\mathbb{R})$, and Lemma 6 easily generalizes to such systems. In fact, Section 4 of Bajpai-Velleman [2] and Section 5 of Cox-Elpers [6] can both be viewed as constructions, in ZFC alone, of sparse Γ-systems (with Γ = 'increasing C^∞ bijections' in [2] and Γ = 'increasing smooth diffeomorphisms' in [6]).

We have shown that CH implies a negative answer to Bajpai-Velleman's Question 3, but it is open whether ZFC alone implies a negative solution.

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