J. Aust. Math. Soc. **82** (2007), 183–207

PERFECT EFFECT ALGEBRAS ARE CATEGORICALLY EQUIVALENT WITH ABELIAN INTERPOLATION PO-GROUPS

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(Received 2 February 2005; revised 7 March 2006)

Communicated by M. Jackson

Abstract

We introduce perfect effect algebras and we show that every perfect algebra is an interval in the lexicographical product of the group of all integers with an Abelian directed interpolation po-group. To show this we introduce prime ideals of effect algebras with the Riesz decomposition property (RDP). We show that the category of perfect effect algebras is categorically equivalent to the category of Abelian directed interpolation po-groups. Moreover, we prove that any perfect effect algebra is a subdirect product of antilattice effect algebras with the RDP.

2000 *Mathematics subject classification*: primary 06F20; secondary 03G12, 03B50.
Keywords and phrases: effect algebra, the Riesz decomposition property, radical, perfect effect algebra, interpolation po-group, unital po-group, categorical equivalence.

1. Introduction

Effect algebras are partial algebras with a partially defined addition $+$. They were introduced in 1994 by Foulis and Bennett [6] as a common basis for the system $\mathcal{E}(H)$ of all Hermitian operators A on a Hilbert space H such that $O \leq A \leq I$, where O and I are the zero and identity operators on H. Such operators are of great importance for modelling events of an unsharp nature in quantum mechanics. More about effect algebras and D-posets (which are equivalent structures introduced in [10]) is in [4].

One very important class of effect algebras is the class of effect algebras with the Riesz decomposition property, which means, roughly speaking, that we can do a refinement of decompositions of unity 1. Such algebras are always intervals in unital interpolation po-groups, see [12]. In addition, every MV-algebra (introduced by Chang [1] in 1957 to model many-valued reasoning) can be understood as a lattice

https://doi.org/10.1017/S1446788700016025 Published online by Cambridge University Press

effect algebra with the Riesz decomposition property. Such algebras are always intervals in unital ℓ-groups.

Motivated by MV-algebras, we introduce perfect effect algebras. In such algebras, every element is either in its radical or in its logical complement. Such algebras are always intervals in the lexicographical product of the group of all integers with an Abelian directed interpolation po-group. Moreover, we show that we have a categorical equivalence of the category of perfect effect algebras and the category of Abelian interpolation directed po-groups. This generalizes an analogous result of Di Nola and Lettieri [2] for perfect effect algebras.

The paper is organized as follows. Effect algebras are presented in Section 2 with the main emphasis on the Riesz decomposition property. Ideals of effect algebras (maximal ideals, prime ideals and values) are studied in Section 3. In Section 4, we introduce infinitesimals of effect algebras and the radical, the most important part of every effect algebra. We show that in contrast to MV-algebras, not every radical consists of all its infinitesimals. Therefore, we introduce effect algebras with the Rad-property. Perfect effect algebras are introduced in Section 5, where we show the categorical equivalence of the category of perfect effect algebras with the category of Abelian directed interpolation po-groups. In Section 6, we study quotient effect algebras, and in Section 7 we prove that every perfect effect algebra is a subdirect product of antilattice effect algebras with the RDP.

2. Effect algebras with the Riesz decomposition property

An *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a+b$ is defined in E if and only if $b+a$ is defined, and in this case $a+b = b+a$;

(ii) $a + b$, $(a + b) + c$ are defined if and only if $b + c$ and $a + (b + c)$ are defined, and in this case $(a + b) + c = a + (b + c)$;

(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;

(iv) if $a + 1$ is defined in E then $a = 0$.

The relation $\leq$ on E is defined for $a, b \in E$ by $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$. This relation is a partial ordering and we write $c = b - a$.

For example, if (G, u) is an Abelian unital po-group with a strong unit[1] u and if $\Gamma(G, u)$ is defined to be $\{g \in G : 0 \leq g \leq u\}$ and $+$ is the restriction to $\Gamma(G, u)$ of the group addition in G, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

[1]An element $u \in G^+$ is said to be a *strong unit* for a po-group G if, given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

Let E and F be two effect algebras. A mapping $h : E \to F$ is said to be a *homomorphism* if (i) $h(a+b) = h(a) + h(b)$ whenever $a + b$ is defined in E, and (ii) $h(1) = 1$. A bijective homomorphism h such that h^{-1} is homomorphism is said to be an *isomorphism* of E and F.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property* (RIP) if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j; (ii) the *Riesz decomposition property* (RDP) if $x \leq y_1 + y_2$ implies that there exist two elements $x_1, x_2 \in E$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice then E trivially has the RIP but the converse is not true, as we see below; (2) E has the RDP if and only if, $x_1 + x_2 = y_1 + y_2$ implies that there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$ and $y_2 = c_{12} + c_{22}$, [4, Lemma 1.7.5]; (3) the RDP implies the RIP but the converse is not true (for example, if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without the RDP). On the other hand, every finite poset with the RIP is a lattice.

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have an infimum or a supremum. It is clear that any linearly ordered poset is an antilattice and every finite effect algebra with the RIP is a lattice.

The following example shows that there exists an effect algebra with the RIP which is not a lattice.

EXAMPLE 2.1. Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$. Then $u = (1, 1)$ is a strong unit for G. The effect algebra $E = \Gamma(G, u)$ is an antilattice having both the RIP and the RDP, but E is not a lattice.

A partially ordered Abelian group $(G; +, 0)$ is said to satisfy the *Riesz decomposition property* provided, given x, y_1, y_2 in G^+ such that $x \leq y_1 + y_2$, there exist x_1, x_2 in G^+ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for each j. This condition is equivalent by [9, Proposition 2.1] to the following two equivalent conditions:

(a) Given x_1, x_2, y_1, y_2 in G such that $x_i \leq y_j$ for all i, j, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j.

(b) Given x_1, x_2, y_1, y_2 in G^+ such that $x_1 + x_2 = y_1 + y_2$, there exist $z_{11}, z_{12}, z_{21}, z_{22}$ in G^+ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

According to [9], a group G with the Riesz decomposition property is said to be an *interpolation group*.

It is clear that if (G, u) is a unital group interpolation group, then $E = \Gamma(G, u)$ has the RDP.

We recall that by a *universal group* for an effect algebra E we mean a pair (G, γ) consisting of an additive Abelian group G and a G-valued measure $\gamma : E \to G^+$ (so

$\gamma(a+b) = \gamma(a) + \gamma(b)$ whenever $a+b$ is defined in E) such that the following conditions hold: (i) $\gamma(E)$ generates G. (ii) If H is an additive Abelian group and $\phi : E \to H$ is an H-valued measure, then there is a group homomorphism $\phi^* : G \to H$ such that $\phi = \phi^* \circ \gamma$. According to [6], every effect algebra possesses a universal group.

Ravindran [12] ([4, Theorem 1.17.17]) proved the following important result.

THEOREM 2.2. *Let E be an effect algebra with the Riesz decomposition property. Then there exists a unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic with E, and there is a G-valued injective measure γ such that (G, γ) is a universal group for E.*

We recall that all finite meets and joins from E are preserved in (G, u), see [5, Proposition 6.3].

Let G be a directed Abelian po-group and define the lexicographical product

$$\mathbb{Z}(G) = \mathbb{Z} \times_{lex} G, \tag{2.1}$$

where $\mathbb{Z}$ is the group of all integers. Then the element $(1, 0)$ is a strong unit in the po-group $\mathbb{Z}(G)$ and if we define

$$E(G) = \Gamma(\mathbb{Z}(G), (1, 0)), \tag{2.2}$$

then $E(G)$ is an effect algebra. Every element $a \in E(G)$ is of the form either $a = (1, -g)$ or $a = (0, g)$, where $g \in G^+$. In addition, if G is a directed interpolation group then $\mathbb{Z}(G)$ is an interpolation group by [9, Corollary 2.12] and $E(G)$ satisfies the Riesz interpolation property.

In what follows, we introduce perfect effect algebras and show that every perfect effect algebra is isomorphic with $E(G)$ for some interpolation Abelian po-group G.

3. Ideals of effect algebras

In the present section we introduce ideals, prime and maximal ideals, and values.

Let a be any element of an effect algebra E and n an integer with $n \geq 0$. We define recurrently

$$0a = 0, \quad 1a = a, \quad (n+1)a = na + a, \quad n \geq 1,$$

supposing that na and $na + a$ are defined in E. By the *isotropic index* $i(a)$ of an element $a \in E$ we mean $i(a) = \sup\{n \geq 1 : na \in E\}$.

An *ideal* of an effect algebra E is a non-empty subset I of E such that (i) $x \in E$, $y \in I$, $x \leq y$ imply $x \in I$, and (ii) if $x, y \in I$ and $x + y$ is defined in E then $x + y \in I$.

An ideal I is said to be a *Riesz ideal* if, for all $x \in I$ and $a, b \in E$ with $x \leq a + b$, there exist $a_1, b_1 \in I$ such that $x \leq a_1 + b_1$ and $a_1 \leq a$ and $b_1 \leq b$.

For example, if E has the RDP, then every ideal of E is a Riesz ideal.

If we denote by $\mathcal{I}(E)$ the set of all ideals of E then $\{0\}, E \in \mathcal{I}(E)$. We recall that if E is linearly ordered then $\mathcal{I}(E)$ is linearly ordered with respect to the set-theoretical inclusion. Indeed, let I_1, I_2 be two ideals of E and assume $a \in I_1 \setminus I_2$ and $b \in I_2 \setminus I_1$. We can assume that, for example, $a \leq b$ which implies $a \in I_2$, a contradiction.

PROPOSITION 3.1. *Let A be a subset of an effect algebra E satisfying the RDP. Then the ideal $I_0(A)$ of E generated by A is the set*

$$I_0(A) = \left\{ x \in E \,\middle|\, \begin{array}{l} \exists\, n \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_n^0 \in E,\ \exists\, a_1, \ldots, a_n \in A, \\ a_i^0 \leq a_i,\ i = 1, \ldots, n,\ x = a_1^0 + \cdots + a_n^0 \end{array} \right\}.$$

PROOF. If $A = \emptyset$, then $I_0(A) = \{0\}$. Let $A \neq \emptyset$. Then $A \subseteq I_0(A)$. Assume $x \in E$, $y \in I_0(A)$ and $x \leq y$. Then $y = a_1^0 + \cdots + a_n^0$, where $a_i^0 \in E$, $a_i^0 \leq a_i$ for some $a_1, \ldots a_n \in A$. The the RDP implies $x \in I_0(A)$.

Suppose $a, b \in I_0(A)$ and $a + b \in E$. Then $a = a_1^0 + \cdots + a_n^0$, $b = b_1^0 + \cdots + b_m^0$, where $E \ni a_i^0 \leq a_i \in A$, $E \ni b_j^0 \leq b_j \in A$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, which gives $a + b \in I_0(A)$.

Since any ideal containing A must contains $I_0(A)$, we have the assertion in question. □

PROPOSITION 3.2. *Let J be an ideal of an effect algebra E with the RDP and $a \in E$. Then the ideal $I_0(J, a)$ of E generated by J and a is given by*

$$I_0(J, a) = \left\{ x \in E \,\middle|\, \begin{array}{l} \exists\, y \in J,\ \exists\, k \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_k^0 \in E, \\ a_i^0 \leq a,\ x = y + a_1^0 + \cdots + a_k^0 \end{array} \right\}.$$

PROOF. It follows from Proposition 3.1. □

In particular, the ideal $I_0(a)$ of E generated by an element a is, by Proposition 3.2,

$$I_0(a) = \left\{ x \in E : \exists\, k \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_k^0 \in E,\ a_i^0 \leq a,\ x = a_1^0 + \cdots + a_k^0 \right\}. \tag{3.1}$$

Let a be a non-zero element of E. By Zorn's lemma, there exists an ideal V which does not contain a and is maximal with respect to this property. Such an ideal is said to be a *value* of a in E. We denote by $\Gamma(a)$ the set of all values of the element a. We say that an ideal J *covers* I (or J is a *cover* of I) if $I \subset J$ and, for any ideal K of E, $I \subseteq K \subseteq J$ implies either $I = K$ or $J = K$.

PROPOSITION 3.3. *Every value in an effect algebra E has a cover.*

PROOF. Suppose that $a > 0$ and V is a value of a in E. Let $V(a)$ be the intersection of all ideals of E that contain V and a. Then $V(a)$ covers V because if J is an ideal of E and $J \supset V$ then $a \in J$ by the maximality of V, hence $V(a) \subseteq J$. □

PROPOSITION 3.4. *For any effect algebra E satisfying the RDP, $\mathcal{I}(E)$ is a distributive lattice. Moreover, for all $I, I_s \in \mathcal{I}(E)$ $(s \in S)$ we have $I \cap \bigvee_s I_s = \bigvee_s (I \cap I_s)$. In addition, $I \vee (I_1 \cap I_2) = (I \vee I_1) \cap (I \vee I_2)$.*

PROOF. Let $\{I_s\}_{s \in S}$ be a family of ideals of E and let I be the ideal of E generated by $\bigcup_s I_s$. Clearly $\bigvee_s (I \cap I_s) \subseteq I \cap \bigvee_s I_s$. If $x \in I \cap \bigvee_s I_s$ then by Proposition 3.1, $x = x_1 + \cdots + x_n$ with $x_i \in I_{s_i}$, $i = 1, \dots, n$, and if simultaneously $x \in I$ then we have $x_i \in I$ for $i = 1, \dots, n$, which gives $x \in \bigvee_s (I \cap I_s)$.

For the rest, it is clear that $I \vee (I_1 \cap I_2) \subseteq (I \vee I_1) \cap (I \vee I_2)$. Assume $x \in (I \vee I_1) \cap (I \vee I_2)$. Then by Proposition 3.2, $x = x_1 + x_2 = y_1 + y_2$, where $x_1, y_1 \in I$ and $x_2 \in I_1$, $y_2 \in I_2$. By the RDP, there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$ and $y_2 = c_{12} + c_{22}$. Hence $c_{11}, c_{12}, c_{21} \in I$ and $c_{22} \in I_1 \cap I_2$ and $x = (c_{11} + c_{12} + c_{21}) + c_{22} \in I \vee (I_1 \cap I_2)$. □

We say that an ideal P of an effect algebra E with the RDP is *prime* if, for all ideals I and J of E, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(E)$ the set of all prime ideals of E.

PROPOSITION 3.5. *If an effect algebra E satisfies the RDP, then every value is a prime ideal of E.*

PROOF. Let V be a value of an element $a \in E \setminus \{0\}$. Assume $I \cap J \subseteq V$. By Proposition 3.4, $V = V \vee (I \cap J) = (V \vee I) \cap (V \vee J)$. Since $a \notin V$, we have that either $a \notin V \vee I$ or $a \notin V \vee J$. The maximality of V gives $V = V \vee I$ or $V = V \vee J$. □

PROPOSITION 3.6. *Let E be an effect algebra with the RDP. The following statements are equivalent*:

(i) *P is a prime ideal.*
(ii) *$I \cap J = P$ implies $P = I$ or $P = J$.*
(iii) *$I_0(a) \cap I_0(b) \subseteq P$, $a, b \in E$, implies $a \in P$ or $b \in P$.*

If the set $\{I \in \mathcal{I}(E) : I \supseteq P\}$ is an antilattice with respect to the set-theoretical inclusion, then all of conditions (i)–(iii) *hold.*

PROOF. (i) $\Rightarrow$ (ii). Let $I \cap J = P$. By (i), $I \subseteq P$ or $J \subseteq P$. Assume $I \subseteq P$, then $I \subseteq P = I \cap J \subseteq I$, which proves $I = P$.

(ii) ⇒ (i). Let $I \cap J \subseteq P$. By Proposition 3.4, $P = P \vee (I \cap J) = (P \vee J) \cap (P \vee J)$ which gives $P = P \vee I$ or $P = P \vee J$, so $P \supseteq I$ or $P \supseteq J$.

(i) ⇒ (iii). Let $I_0(a) \cap I_0(b) \subseteq P$. Then $I_0(a) \subseteq P$ or $I_0(b) \subseteq P$, that is, $a \in P$ or $b \in P$.

(iii) ⇒ (i). Let $I \cap J \subseteq P$. Then $I = \bigvee_{a \in I} I_0(a)$ and $J = \bigvee_{b \in J} I_0(b)$. By Proposition 3.4, $I \cap J = \bigvee_{a,b}(I_0(a) \cap I_0(b)) \subseteq P$. If we could find some $a \in I \setminus P$ and some $b \in J \setminus P$ then we would have $I_0(a) \cap I_0(b) \subseteq P$, which gives $a \in P$ or $b \in P$, a contradiction.

(iv) ⇒ (ii). Let $P = I \cap J$. Then $P \subseteq I$ and $P \subseteq J$, so that $I \subseteq J$ or $J \subseteq I$, which proves $P = I$ or $P = J$. □

It is interesting to recall that (iv) in Proposition 3.6 is not equivalent to (i).

We now use a notion of prime ideals to present a criterion when an effect algebra is a lattice with the RDP.

PROPOSITION 3.7. *Let an effect algebra E satisfy the condition: For any $a_1, a_2, b_1, b_2 \in E$ with $a_1 + a_2 = b_1 + b_2$ there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{12} + c_{22}$, and such that, for any prime ideal P of E, $c_{12} \in P$ or $c_{21} \in P$. Then $c_{12} \wedge c_{21} = 0$ and E is a lattice with the RDP.*

PROOF. It is evident that E satisfies the RDP. Assume now $x \leq c_{12}, c_{21}$ and let $x \neq 0$. Take a value P of x. By Proposition 3.5, P is a prime ideal. Hence, $c_{12} \in P$ or $c_{21} \in P$, which gives $x \in P$, a contradiction. Hence $x = 0$.

CLAIM. *If E is an effect algebra with the RDP and $a \wedge b = 0$ then $a + b$ and $a \vee b$ are defined in E and $a \vee b = a + b$.*

Indeed, if x is an upper bound of a and b then $x = c + a$ for some $c \in E$. Hence $b = b_1 + b_2$ where $b_1 \leq c$ and $b_2 \leq a$, which yields $b_2 = 0$ and therefore $b = b_1$ and $b + a$ is defined in E. It is clear now that $a + b = a \vee b$.

To finish the proof, we see that $a + a' = 1 = b' + b$, which gives four elements $x_{11}, x_{12}, x_{21}, x_{22} \in E$ such that $a = x_{11} + x_{12}$, $a' = x_{21} + x_{22}$, $b = x_{11} + x_{21}$ and $b' = x_{21} + x_{22}$. We can assume that $x_{12} \wedge x_{21} = 0$. Hence, by the Claim above, we have $x_{11} + (x_{12} \vee x_{21}) = x_{11} + x_{12} + x_{21} \in E$, so that $x_{11} + (x_{12} \vee x_{21}) = (x_{11} + x_{12}) \vee (x_{11} + x_{21}) = a \vee b \in E$. □

EXAMPLE 3.8. Let (G, u) be the unital po-group with the RDP from Example 2.1. Then $E = \Gamma(G, u)$ has only one maximal ideal I, namely $I = \{(0, 0)\}$. In addition, E is an antilattice and $\mathcal{I}(E) = \{\{0\}, E\}$.

EXAMPLE 3.9. Let (G, u) be the unital po-group, where $G = \mathbb{R}^2$ and $u = (1, 1)$, and $G^+ = \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 : x > 0,\ y \geq 0\}$. Then (G, u) has the RDP and if $I = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x < 1\}$ then I is a unique maximal ideal of $E = \Gamma(G, u)$. In addition, E is an antilattice and $\mathcal{I}(E) = \{\{0\}, I, E\}$.

From [8, Lemma 3.3.3], we have that if G is an ℓ-group, then an ideal P of G is prime if and only if, for all $f, g \in G$ there is a $c \in P$ such that $f \leq c+g$ or $g \leq c+f$. We recall that an analogous assertion is not valid for all effect algebras. Indeed, take E from Example 3.9, and set $f = (0.4, 0.5)$ and $g = (0.4, 0.6)$. Then there is no $c = (x, 0) \in I$ such that $f \leq c + g$ or $g \leq c + f$.

4. Infinitesimal elements of effect algebras

In the present section, we define infinitesimal elements of an effect algebra E, and we show that they are elements of the radical of E. Radicals, as intersection of all maximal ideals, are important parts of effect algebras, containing much information about the effect algebras. In contrast to lattice effect algebras with the RDP, it can happen that not every element of the radical is an infinitesimal.

It is possible to prove that if $a \wedge b$ is defined in an effect algebra E then

$$(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0. \tag{4.1}$$

An element a is said to be *infinitesimal* if $i(a) = \infty$, and we denote by $\mathrm{Infinit}(E)$ the set of all infinitesimals of E. Then (i) $0 \in \mathrm{Infinit}(E)$, (ii) if $b \in E$, $a \in \mathrm{Infinit}(E)$ and $b \leq a$ then $b \in \mathrm{Infinit}(E)$, and (iii) $1 \notin \mathrm{Infinit}(E)$.

We say that an effect algebra E is *Archimedean* if $\mathrm{Infinit}(E) = \{0\}$.

A proper ideal I of an effect algebra E is said to be *maximal* if is not a proper subset of another proper ideal of E (or equivalently, if I is a value of 1). By Zorn's lemma, E possesses at least one maximal ideal. Let $\mathcal{M}(E)$ be the set of all maximal ideals of E. We define the *radical* of E, $\mathrm{Rad}(E)$, by $\mathrm{Rad}(E) = \bigcap\{I : I \in \mathcal{M}(E)\}$.

PROPOSITION 4.1. *Let E be an effect algebra satisfying the RDP. Then*

$$\mathrm{Infinit}(E) \subseteq \mathrm{Rad}(E). \tag{4.2}$$

PROOF. Suppose that $a \notin \mathrm{Rad}(E)$. Then there exists a maximal ideal I of E such that $a \notin I$. Hence the ideal $I_0(I, a)$ generated by I and a must coincide with E. Therefore there exist elements $z \in I$, $a_1^0, \dots, a_k^0 \in E$ such that $a_i^0 \leq a$ $(i = 1, \dots, k)$ and $1 = z + a_1^0 + \cdots + a_k^0$. If $a \in \mathrm{Infinit}(E)$ then $ka \leq a'$ and one would have $z + a_1^0 + \cdots + a_k^0 = a + a'$, so that $z = a + (a' - (a_1^0 + \cdots + a_k^0)) \geq a$, whence $a \in I$, a contradiction. Hence $a \notin \mathrm{Infinit}(E)$ and $\mathrm{Infinit}(E) \subseteq \mathrm{Rad}(E)$. ☐

PROPOSITION 4.2. *Let E be a lattice effect algebra with the RDP. If $a_1+a_2 = b_1+b_2$ for some $a_1, a_2, b_1, b_2 \in E$, then*

$$a_1 - (a_1 \wedge b_1) = b_2 - (a_2 \wedge b_2),$$
$$a_2 - (a_2 \wedge b_2) = b_1 - (a_1 \wedge b_1).$$

PROOF. Due to the RDP, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_1 = c_{11}+c_{12}$, $a_2 = c_{21}+c_{22}$, $b_1 = c_{11}+c_{21}$ and $b_2 = c_{12}+c_{22}$. Due to [4, Section 1.10], we can assume that $c_{12}\wedge c_{21} = 0$. Then $c_{11} = c_{11}+0 = (c_{11}+c_{12})\wedge(c_{11}+c_{21}) = a_1\wedge b_1$, and in a similar way $c_{22} = a_2 \wedge b_2$, which proves the assertion in question. □

It is valuable to recall that it is possible to have inequality in (4.2). For example, in Example 3.9, $\mathrm{Infinit}(E) = \{(0,0)\}$ and $\mathrm{Rad}(E) = \{(x,0) : 0 \le x < 1\}$, while in Example 3.8, $\mathrm{Infinit}(E) = \{(0,0)\} = \mathrm{Rad}(E)$. Inspired by this, we say that an effect algebra E has the Rad-*property* if

$$\mathrm{Infinit}(E) = \mathrm{Rad}(E). \tag{4.3}$$

An effect algebra E is said to be *simple* if it has only trivial ideals. For example, Example 3.8 gives a simple effect algebra.

THEOREM 4.3. *Assume that E is an effect algebra with the RDP of one of the following kinds*:

(1) *E is a lattice.*
(2) *E is a simple effect algebra.*
(3) *$E = E(G) = \Gamma(\mathbb{Z}(G), (1,0))$, where $\mathbb{Z}(G)$ is the lexicographical product of $\mathbb{Z}$ with a directed Abelian po-group G with interpolation.*
(4) *For every $x \in E$, $x \le 1-x$ or $1-x \le x$.*

Then E has the Rad-*property. In addition, in* (3), $\mathrm{Infinit}(E)$ *is a unique maximal ideal of E.*

PROOF. (1) Suppose that $a > 0$ is not infinitesimal. Then there exists a natural number $m \ge 1$ such that $ma \in E$ and $ma \not\le a'$, and, by Proposition 4.2, we have $a - ((ma)' \wedge a) = (ma) - ((ma) \wedge a') = c$, say, and $c > 0$. Let P be a value of c; by Proposition 3.5, P is a prime ideal of E. Assume that M is any maximal ideal of E containing P. Since $((ma) - ((ma) \wedge a')) \wedge (a' - ((ma) \wedge a')) = 0$ by (4.1), we have by Proposition 3.6, $a' - ((ma) \wedge a') \in P \subseteq M$. The maximality of M entails $(ma) \wedge a' + a = (a' - ((ma) \wedge a'))' \notin M$. Hence $a \notin M$ which proves $\mathrm{Rad}(E) \subseteq \mathrm{Infinit}(E)$.

(2) It is evident.

(3) It is clear that $\text{Infinit}(E(G)) = \{(0, g) : g \in G^+\}$, and $\text{Infinit}(E(G))$ is an ideal of $E(G)$. We assert that it is a (unique) maximal ideal of $E(G)$. Indeed, take $(1, -g)$, where $g \in G^+$. Then $(1, -g)' = (0, g)$, so that $(1, 0)$ is an element of the ideal generated by $\text{Infinit}(E(G))$ and the element $(1, -g)$.

(4) Assume $x \in \text{Rad}(E)$. Then only $x \leq 1 - x$, otherwise $1 \in \text{Rad}(E)$, which is impossible. Hence, $x + x$ is defined in E. Since $\text{Rad}(E)$ is an ideal of E, we have $2x \in \text{Rad}(E)$. Repeating this process with $2x$, we see that $2x + 2x \in \text{Rad}(E)$, and so on. Therefore, nx is defined in E for any integer $n \geq 1$, so $x \in \text{Infinit}(E)$. □

5. Perfect effect algebras and Abelian po-groups with interpolation

In this section we give the main result on perfect effect algebras, showing that every perfect algebra E is of the form (2.2) for some interpolation directed po-group G. In addition, we show that the category of perfect effect algebras is categorically isomorphic with the category of Abelian directed interpolation po-groups, which generalizes an analogous result of Di Nola and Lettieri [2] who proved this for MV-algebras (which are equivalent to lattice effect algebras with the RDP).

If E is an effect algebra and $A \subseteq E$, we define $A' = \{a' : a \in A\}$.

PROPOSITION 5.1. *Let E be an effect algebra with the RDP satisfying the* Rad-*property. Then* $\text{Rad}(E)$ *is a commutative semigroup with respect to* $+$ *such that*

(i) 0 *is a neutral element,*
(ii) *the cancellation law holds in* $\text{Rad}(E)$,
(iii) *if* $a + b = 0$ *for* $a, b \in \text{Rad}(E)$ *then* $a = b = 0$,
(iv) $\text{Rad}(E) \cap \text{Rad}'(E) = \emptyset$,
(v) *if* $a, b \in \text{Rad}'(E)$ *then* $a + b$ *is not defined in* E,
(vi) *if* $a \in \text{Rad}(E)$ *and* $b \in \text{Rad}'(E)$ *then* $a \leq b$.

PROOF. Assume $a, b \in \text{Rad}(E)$. Then $a = a_1 + c$ and $b = b_1 + c$ for some $a_1, b_1, c \in E$ such that $a_1 + b_1 + c \in E$, and then $a_1, b_1, c \in \text{Rad}(E)$. Due to the Rad-property, we have $a_1 + b_1 + c \in \text{Infinit}(E) = \text{Rad}(E)$, therefore $(a_1 + b_1 + c) + (a_1 + b_1 + c) \in E$, which implies $a + b$ is defined in E and consequently $a + b \in \text{Rad}(E)$. The conditions (i)–(iii) are now evident.

(iv) Assume $a \in \text{Rad}(E) \cap \text{Rad}'(E)$. Then $a \in \text{Rad}'(E)$, so that $a' \in \text{Rad}(E)$, which implies $1 = a + a' \in \text{Rad}(E)$, a contradiction.

(v) Assume $a + b \in E$. Then $a \leq b' \in \text{Rad}(E)$, which implies $a \in \text{Rad}(E)$, a contradiction.

(vi) If $b \in \text{Rad}'(E)$ then $b' \in \text{Rad}(E)$, which implies $a + b'$ is defined in $\text{Rad}(E)$. Hence $a \leq (b')' = b$. □

https://doi.org/10.1017/S1446788700016025 Published online by Cambridge University Press

We say that an effect algebra E with the RDP and the Rad-property is *perfect* if, for any element $a \in E$, either $a \in \mathrm{Rad}(E)$ or $a' \in \mathrm{Rad}(E)$. For example, if $E(G)$ is defined by (2.2) for a directed po-group G with interpolation then $E(G)$ is a perfect algebra. In what follows, we show that every perfect effect algebra is of this nature.

Let $\mathcal{PEA}$ be the category of all perfect effect algebras where the objects are perfect effect algebras and morphisms are homomorphisms of effect algebras. Let $\mathcal{GI}$ be the category of Abelian po-groups, where the objects are directed Abelian po-groups with interpolation and the morphisms are positive homomorphisms of po-groups.

For each object G in $\mathcal{GI}$ let

$$\mathcal{E}(G) = E(G) = \Gamma(\mathbb{Z} \times_{\mathrm{lex}} G, (1, 0)), \tag{5.1}$$

and for each morphism $h : G \to G'$ in $\mathcal{GI}$ let $\mathcal{E}(h)$ be defined, for some $g \in G^+$, by

$$\mathcal{E}(h)(x) = \begin{cases} (0, h(g)) & \text{if } x = (0, g), \\ (1, -h(g)) & \text{if } x = (1, -g). \end{cases} \tag{5.2}$$

Then $\mathcal{E}$ is a functor from $\mathcal{GI}$ to $\mathcal{PEA}$.

PROPOSITION 5.2. *$\mathcal{E}$ is a faithful and full functor from the category $\mathcal{GI}$ of directed Abelian po-groups with interpolation into the category $\mathcal{PEA}$ of perfect effect algebras.*

PROOF. Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for any $g \in G^+$, consequently $h_1 = h_2$.

To prove that $\mathcal{E}$ is a full functor, suppose that $f : E(G) \to E(G')$ is a morphism of effect algebras. Then $f(0, g) = (0, g')$ for a unique $g' \in G'^+$. Define a mapping $h : G^+ \to G'^+$ by $h(g) = g'$ if and only if $f(0, g) = (0, g')$. Then $h(g_1 + g_2) = h(g_1) + h(g_2)$ if $g_1, g_2 \in G^+$. Assume now that $g \in G$ is arbitrary. If $g = g_1 - g_2 = g_1' - g_2'$, where $g_1, g_2, g_1', g_2' \in G^+$, then $g_1 + g_2' = g_1' + g_2$, which implies $h(g_1) - h(g_2) = h(g_1') - h(g_2')$ and hence that letting $h(g) = h(g_1) - h(g_2)$ establishes a well-defined extension of h from G^+ onto G. In addition, h is a homomorphism of po-groups. Therefore, $\mathcal{E}(h) = f$ as desired. □

PROPOSITION 5.3. *Let E be a perfect effect algebra. Then there is a unique (up to isomorphism) directed Abelian po-group G with interpolation such that $E \cong E(G)$.*

PROOF. Let E be a perfect effect algebra. By Proposition 5.1, $\mathrm{Rad}(E)$ is a cancellative semigroup satisfying conditions of Birkhoff [7, Theorem II.4] which guarantee that $\mathrm{Rad}(E)$ is a positive cone of a unique (up to isomorphism) po-group G. Since E has the RDP, G is a directed Abelian po-group with interpolation.

Without loss of generality, we can assume that any element $g \in G$ can be written in the form $g = g_1 - g_2$ with $g_1, g_2 \in \mathrm{Rad}(E)$. The mapping $\gamma : \mathrm{Rad}(E) \to G$ given by

$$\gamma(a) = a, \quad a \in \mathrm{Rad}(E), \tag{5.3}$$

is injective and preserves $+$ in $\mathrm{Rad}(E)$.

Define $E(G)$ by (2.2). By Theorem 4.3, $E(G)$ is a perfect effect algebra and we can define a mapping $f : E \to E(G)$ by $f(a) = (0, a)$ and $f(a') = (1, -a)$ for $a \in \mathrm{Rad}(E)$. Then f is an isomorphism of the effect algebras E and $E(G)$. □

Extend the mapping $\gamma : \mathrm{Rad}(E) \to G$ defined by (5.3) to a mapping $\hat{\gamma} : E \to G$ by

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & \text{if } a \in \mathrm{Rad}(E), \\ -\gamma(a') & \text{if } a \notin \mathrm{Rad}(E). \end{cases} \tag{5.4}$$

THEOREM 5.4. *Let E be a perfect effect algebra. Then the pair $(G, \hat{\gamma})$, where G is from Proposition* 5.3 *and $\hat{\gamma}$ is defined in* (5.4), *is a universal group for E.*

PROOF. Due to the Rad-property, $\hat{\gamma}$ is a mapping from E into G that is injective and preserves $+$, and $\hat{\gamma}(E)$ generates G. Assume that H is an additive Abelian group and $\phi : E \to H$ is an H-valued measure. Define a mapping $\phi^* : G^+ \to H$ by $\phi^*(g) = \phi(\hat{\gamma}(a))$, where a is a unique element in $\mathrm{Rad}(E)$ such that $\hat{\gamma}(a) = g$. Then ϕ^* preserves addition in G^+, so ϕ^* can be uniquely extended to a group isomorphism (denoted again as ϕ^*) on all of G by $\phi^*(g) = \phi^*(g_1) - \phi^*(g_2)$ whenever $g = g_1 - g_2$ $(g_1, g_2 \in G^+)$. Hence we have $\phi(a) = \phi^*(\hat{\gamma}(a))$ for any $a \in E$, which proves that $(G, \hat{\gamma})$ is a universal group for E. □

PROPOSITION 5.5. *The functor $\mathcal{E}$ from the category $\mathcal{GI}$ into the category $\mathcal{PEA}$ is a right-adjoint.*

PROOF. We show that given a perfect effect algebra E there is a universal arrow (G, f), which means that f is a homomorphism from E into $\mathcal{E}(G)$ such that if G' is an object from $\mathcal{GI}$ and f' is a homomorphism from E into $\mathcal{E}(G')$ then there is a unique homomorphism $f^* : G \to G'$ such that $\mathcal{E}(f^*) \circ f = f'$.

Let $(G, \hat{\gamma})$ be a universal group for E guaranteed by Theorem 5.4. Then it is straightforward to verify that $(G, \hat{\gamma})$ is a universal arrow for E. □

Define a morphism $\mathcal{P} : \mathcal{PEA} \to \mathcal{GI}$ by $\mathcal{P}(E) = G$ whenever $(G, \hat{\gamma})$ is a universal group for E. It is clear that if f is a morphism from E into F then f can be uniquely extended to a homomorphism $\mathcal{P}(f)$ from G into G_1, where $(G_1, \hat{\gamma}_1)$ is a universal group for the perfect effect algebra F.

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PROPOSITION 5.6. *$\mathcal{P}$ is a functor from the category $\mathcal{PEA}$ into the category $\mathcal{GI}$ which is left-invariant to $\mathcal{E}$.*

PROOF. It follows from the construction of universal groups. □

We present now the main result on categorical equivalence of the category of perfect effect algebras and the category of directed Abelian po-groups with interpolation.

THEOREM 5.7. *$\mathcal{E}$ is a categorical equivalence of the category $\mathcal{GI}$ of directed Abelian po-groups with interpolation and the category $\mathcal{PEA}$ of perfect effect algebras.*

PROOF. According to [11, Theorem IV.4.1], it is necessary to show that, for a perfect effect algebra E, there is an object G in $\mathcal{GI}$ such that $\mathcal{E}(G)$ is isomorphic to E. To show that, we take a universal group $(G, \hat{\gamma})$ for E from Theorem 5.4. Then $\mathcal{E}(G)$ and E are isomorphic. □

To complete our categorical equivalences of appropriate categories of effect algebras and interpolation po-groups, let $\mathcal{EA}_{\rm RDP}$ be the category whose objects are effect algebras with the RDP and morphisms are homomorphisms of effect algebras, and let $\mathcal{UIG}$ be the category whose objects are unital interpolation po-groups (G, u) with a fixed strong unit u and morphisms are homomorphisms of unital po-groups, that is, positive homomorphisms of unital po-groups which preserve fixed strong units. Using Theorem 2.2 and methods used for the categories of perfect effect algebras, we can prove the following important result.

THEOREM 5.8. *The mapping $\Gamma : \mathcal{UIG} \to \mathcal{EA}_{\rm RDP}$ defines the categorical equivalence of the category $\mathcal{UIG}$ of unital interpolation po-groups and the category of effect algebras with the RDP.*

In addition, suppose that $h : \Gamma(G, u) \to \Gamma(H, v)$ is a homomorphism of effect algebras with the RDP. Then there is a unique homomorphism $f : (G, u) \to (H, v)$ of unital po-groups such that $h = \Gamma(f)$ and

(i) *if h is surjective, so is f;*
(ii) *if h is an isomorphism, so is f.*

6. Ideals and quotients of effect algebras

In the present section we establish a one-to-one bijection between the set of all ideals of an effect algebra with the RDP and the ideals of the corresponding representation unital po-groups. In addition, we study quotient effect algebras and antilattice effect algebras.

Let P be a (proper) ideal of an effect algebra E with the RDP. We define a relation $\sim_P$ on E by $a \sim_P b$ if and only if $a - e = b - f$ for some $e, f \in P$. According to [4, Section 3.1.2], we have that $\sim_P$ is an equivalence such that (i) $a + b \in E$, $a_1 + b_1 \in E$, $a \sim_P a_1$, $b \sim_P b_1$ imply $(a + b) \sim_P (a_1 + b_1)$, (ii) $a \sim_P b$ implies $a' \sim_P b'$, (iii) $a+b \in E, c \sim_P a$ imply there exists an element $d \in E$ such that $d \sim_P b$ and $d + c \in E$, (iv) $a + b, a_1 + b_1 \in E, a_1 \sim_P a$, $(a_1 + b_1) \sim_P (a + b)$ imply $b_1 \sim_P b$. If we define $a/P = [a] = [a]_P = \{b \in E : b \sim_P a\}$ and $E/P = \{[a]_P : a \in E\}$, then E/P is an effect algebra, where $[a] + [b] = [c]$ if and only if there exist $a_1 \in [a]$, $b_1 \in [b]$, $c_1 \in [c]$ such that $a_1 + b_1 = c_1$. As the constant elements in E/P we take $[0]$ and $[1]$. We recall that

$$(6.1) \qquad [a]_P \le [b]_P \text{ in } E/P \quad \Longleftrightarrow \quad \text{there exists } a_1 \in [a]_P \text{ such that } a_1 \le b.$$

Indeed, if $[a]_P \le [b]_P$ then there exists a $c \in E$ such that $[a]_P + [c]_P = [b]_P$. Then there are elements $u, d, e, f \in P$ and $a_0 \in [a]_P, b_1 \in [b]_P$ such that $a_0 + c = b_1$, $a_0 - u = a - v$ and $b_1 - e = b - f$. Since $a_0 = (a - v) + u$ and $b_1 = (b - f) + e$, we have $(a - v) + u + c = (b - f) + e$. Due to the RDP, there are $e_1, e_2, e_3 \in P$ with $e_1 \le a - v$, $e_2 \le u$, $e_3 \le c$ and $e = e_1 + e_2 + e_3$, which yields

$$(a - (v_1 + e_1)) + (u - e_2) + (c - e_3) = b - f,$$
$$(a - (v_1 + e_1)) + (u - e_2) + (c - e_3) + f = b.$$

If we set $a_1 = a - (v + e_1)$ then $a_1 \le b$.

PROPOSITION 6.1. *Let P be a proper ideal of an effect algebra E with the RDP. Then E/P is an effect algebra with the RDP.*

PROOF. Assume that $[a_1] + [a_2] = [b_1] + [b_2]$. Without loss of generality we can assume that $a_1 + a_2$ and $b_1 + b_2$ are defined in E. There are $e, f \in P$ such that $(a_1 + a_2) - e = (b_1 + b_2) - f$, so that $a_1 + a_2 = ((b_1 + b_2) - f) + e$. By the the RDP holding in E, there are $c_{11}, c_{12}, c_{21}, c_{22}$ in E such that

$$a_1 = c_{11} + c_{12}, \qquad (b_1 + b_2) - f = c_{11} + c_{21},$$
$$a_2 = c_{21} + c_{22}, \qquad e = c_{12} + c_{22}.$$

This gives $c_{11} + c_{21} + f = b_1 + b_2$. Again due to the RDP, there are $d_{11}, d_{12}, d_{21}, d_{22}, d_{31}, d_{31} \in E$ such that

$$c_{11} = d_{11} + d_{12}, \qquad b_1 = d_{11} + d_{21} + d_{31}, \qquad f = d_{31} + d_{32},$$
$$c_{21} = d_{21} + d_{22}, \qquad b_2 = d_{12} + d_{22} + d_{32}.$$

It is clear that $c_{12}, c_{22}, d_{31}, d_{32} \in P$, which gives $[a_1] = [c_{11}] = [d_{11}] + [d_{12}]$, $[a_2] = [c_{21}] = [d_{21}] + [d_{22}]$, $[b_1] = [d_{11}] + [d_{21}]$, $[b_2] = [d_{12}] + [d_{22}]$, which proves that E/P has the RDP. $\square$

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We say that an effect algebra E is *finitely subdirectly irreducible* if, for any two ideals I and J of E with $I \cap J = \{0\}$, we have $I = \{0\}$ or $J = \{0\}$.

For $\emptyset \neq A \subseteq E$, we set $A^\perp = \{x \in E : x \wedge a = 0 \text{ for any } a \in A\}$, and for $a \in E$ we define $a^\perp = \{a\}^\perp$. Then

$$a^\perp \cap a^{\perp\perp} = \{0\}, \quad a \in E. \tag{6.2}$$

PROPOSITION 6.2. *Let E be an effect algebra with the RDP. If $\emptyset \neq A \subseteq E$, then $A^\perp$ is an ideal of E.*

PROOF. $0 \in A^\perp$. If $x, y \in E$ and $x \leq y \in A^\perp$, then $x \in A^\perp$. Assume now $x, y \in A^\perp$ and $x + y \in E$. Fix $a \in A$. If $z \leq x + y$ and $z \leq a$, then $z = x_1 + y_1$, where $x_1 \leq x$ and $y_1 \leq y$ and $x_1, y_1 \in a^\perp$. Since $x_1, y_1 \leq a$, we have $x_1 = x_1 \wedge a = 0 = y_1 \wedge a = y_1$, which proves $z = 0$. □

PROPOSITION 6.3. *An effect algebra E with the RDP is finitely subdirectly irreducible if and only if E is an antilattice.*

PROOF. If an effect algebra E with the RDP is not finitely subdirectly irreducible then there exist two non-zero ideals I and J such that $I \cap J = \{0\}$. Hence, if $a \in I$ and $b \in J$ are non-zero elements then $a \wedge b = 0$, whence E cannot be an antilattice.

Conversely, assume that E is finitely subdirectly irreducible and let there be $a, b \in E \setminus \{0\}$ with $a \wedge b = 0$. Then $a \in b^\perp$ and $b \in a^\perp$. In view of the Claim from the proof of Proposition 3.7, we have $0 \neq a + b = a \vee b \in E$, so that $a^\perp \cap b^\perp = (a+b)^\perp$. Since $(a+b)^\perp \cap (a+b)^{\perp\perp} = \{0\}$ and $a + b \in (a+b)^{\perp\perp}$, the irreducibility implies $(a+b)^\perp = \{0\}$, so $a^\perp \cap b^\perp = \{0\}$, which gives $b \in a^\perp = \{0\}$ or $a \in b^\perp = \{0\}$, so $b = 0$ or $a = 0$, a contradiction. □

If $A \subseteq E$ and P is an ideal of E, then we set $A/P = \{a/P : a \in A\}$.

PROPOSITION 6.4. *Let E be an effect algebra with the RDP and let P be a proper ideal of E.*

(i) *If I is an ideal of E then so is I/P in E/P. Moreover, if I is a proper ideal of E containing P, then I/P is a proper ideal of E/P.*

(ii) *If M is an ideal of E/P and we let $\kappa(M) = \{x \in E : x/P \in M\}$ then $\kappa(M)$ is an ideal of E and $\kappa(M)/P = M$. If M is a proper ideal of E then so is $\kappa(M)$ in E.*

PROOF. (i) $0/P \in I/P$. If $x/P \leq y/P$, where $y \in I$ then there exists $x_1 \in [x]_P$ such that $x_1 \leq y$, which gives $x_1 \in I$ and $x_1/P = x/P \leq y/P$. If $x/P + y/P$ is defined in E/P for some $x, y \in I$ then there are $x_1 \in [x]_P$, $y_1 \in [y]_P$ and $e, f, u, v \in P$ such that $x_1 - e = x - f \in I$ and $y_1 - u = y - v \in I$ and $x_1 + y_1 \in E$.

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Then $x/P + y/P = x_1/P + y_1/P = (x_1 + y_1)/P = ((x - f) + e + (y - v) + u)/P = ((x - f) + (y - v))/P$ and $(x - f) + (y - v) \in I$.

Let now $I \supseteq P$ and $1/P = x/P$, where $x \in I$. Then there are $e, f \in P$ such that $1 - e = x - f$, so $1 - x = e - f \in P \subseteq I$, which gives a contradiction.

(ii) We have $\kappa(M) \supseteq P$. If $x \leq y \in \kappa(M)$ then $x/P \leq y/P \in M$, so that $x \in \kappa(M)$. Now let $x, y \in \kappa(M)$ and $x + y \in E$. Then $(x + y)/P = x/P + y/P \in M$, so $x + y \in \kappa(M)$.

Finally, assume M is a proper ideal of E/P. Then $1/P \notin M$, hence $1 \notin \kappa(M)$. □

It is worth recalling that it can happen that $M/P = E/P$ even for a maximal ideal M. Indeed, take a Boolean algebra E as an effect algebra, let M be a maximal ideal and let a be an element of E such that $a \notin M$ and $0 \neq a \neq 1$. Then for the ideal P generated by a we have $1/P = a'/P \in M/P = E/P$.

PROPOSITION 6.5. *A proper ideal P of an E with the RDP is prime if and only if E/P is an antilattice.*

PROOF. Let P be a prime ideal of E and I and J be two ideals of E/P with $I \cap J = \{0\}$. We set $I_0 = \{x \in E : x/P \in I\}$ and $J_0 = \{x \in E : x/P \in J\}$. Then I_0 and J_0 are ideals of E such that $I_0 \cap J_0 = P$ which, by (ii) of Proposition 3.6, gives $I_0 = P$ or $J_0 = P$, so $I = \{0\}$ or $J = \{0\}$. Applying Proposition 6.3, we conclude that E/P is an antilattice.

Conversely, let P/I be an antilattice and let I and J be two ideals of E such that $I \cap J = P$. Then $I/P \cap J/P = \{0\}$ which, by Proposition 6.3, yields $I/P = \{0\}$ or $J/P = \{0\}$, so $I = P$ or $J = P$, and therefore, by Proposition 3.6, P is a prime ideal of E. □

COROLLARY 6.6. *An effect algebra with the RDP is an antilattice if and only if $\{0\}$ is a prime ideal of E.*

PROOF. Since $E \cong E/\{0\}$, the assertion follows from Proposition 6.5. □

PROPOSITION 6.7. *An ideal M of an effect algebra E with the RDP is maximal if and only if E/M is a simple effect algebra.*

PROOF. Let M be a maximal ideal of E, and let J be a proper ideal of E/M. Define $J_0 = \{x \in E : x/M \in J\}$. Then J_0 is an ideal of E containing M and the maximality of M entails $J = \{0/M\}$.

Conversely, let E/M be simple and let I_1 be a proper ideal of E containing M. The set $\{a/M : a \in I_1\}$ is an ideal of E/I. Therefore, by Proposition 6.4, this set coincides with the zero-ideal $\{0/M\}$ of E/M, so $I_1 = M$, which proves that M is maximal. □

COROLLARY 6.8. *Every simple effect algebra with the RDP is an antilattice.*

PROOF. Let M be a maximal ideal of E. Then $M = \{0\}$ and M is a prime ideal. Since $E \cong E/M$, Proposition 6.5 mplies that E is an antilattice. □

PROPOSITION 6.9. *Let E be an effect algebra with the RDP. Then $a \wedge b \in E$ if and only if $a \vee b \in E$. In addition, if I is an ideal of E and $a, b \in I$ and $a \vee b$ is defined in E, then $a \vee b \in I$.*

PROOF. Let $a \wedge b \in E$. Due to the RDP, there are $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c \in E$. Since $c \leq a \wedge b$, we have $a_1 \geq a - (a \wedge b)$ and $b_1 \geq b - (a \wedge b)$, which yield $x_0 = a \wedge b + (a - (a \wedge b)) + (b - (a \wedge b)) \in E$. According to (4.1) and the Claim from the proof of Proposition 3.7, we have $a_0 + b_0 = a_0 \vee b_0$, where $a_0 = a - (a \wedge b)$ and $b_0 = b - (a \wedge b)$. We state $x_0 = a \vee b$. Assume $x \geq a, b$. Then $x \geq a_0 + (a \wedge b)$ and $x \geq b_0 + (a \wedge b)$, which gives $x - (a \wedge b) \geq a_0 \vee b_0$, so $x \geq x_0 = a_0 \vee b_0 + (a \wedge b) = a_0 + b_0 + (a \wedge b) = a \vee b$.

Assume now $a \vee b \in E$. Then $(a \vee b)' = a' \wedge b' \in E$ and, by the claim above, this implies $a' \vee b' \in E$. Hence $a \wedge b = (a' \vee b')' \in E$.

Let I be an ideal of E. Then, by the first part of the present proof, $a \wedge b \in E$ and $a \vee b = a + (b - (a \wedge b))$, which proves $a \vee b \in I$. □

PROPOSITION 6.10. *Let E be an effect algebra with the RDP and let P be a proper ideal of E. If $a \wedge b \in E$ then $[a \wedge b]_P = [a]_P \wedge [b]_P$ and $[a \vee b]_P = [a]_P \vee [b]_P$. In addition, if E is a lattice then so is E/P.*

PROOF. It is clear that $[a \wedge b]_P \leq [a]_P, [b]_P$. Assume $[x]_P \leq [a]_P$ and $[x]_P \leq [b]_P$. According to (6.1), there are $x_1, x_2 \in [x]_P$ such that $x_1 \leq a$ and $x_2 \leq b$. Since $x_1 \sim_P x_2$, there are $e, f \in P$ with $x_1 - e = x_2 - f$. Hence, for $x_0 = x_1 - e$, we have $x_0 \in [x]_P$ and $x_0 \leq x_1$, $x_0 \leq x_2$. Consequently, $x_0 \leq a, b$ which yields $x_0 \leq a \wedge b$, so $[x]_P = [x_0]_P \leq [a \wedge b]_P$.

The second equality follows from the first part of the present proof and of Proposition 6.9. □

We recall that an *o-ideal* of a po-group G is any directed convex subgroup of G. An o-ideal I of a po-group G is said to be (i) *maximal* if it is a proper subset of G and it is not contained in any proper o-ideal of G, (ii) *prime* if, for all o-ideals P and Q of G with $P \cap Q \subseteq I$, we have $P \subseteq I$ or $J \subseteq I$, and (iii) a *value* of a non-zero element g if $g \notin I$ and I is maximal with respect to this property. Let (G, u) be a unital Abelian po-group; by $\mathcal{I}(G, u)$, $\mathcal{M}(G, u)$ and $\mathcal{P}(G, u)$ we denote the set of all o-ideals, maximal o-ideals and prime o-ideals, respectively, of (G, u).

THEOREM 6.11. *Let (G, u) be a unital interpolation po-group and let $E = \Gamma(G, u)$. For any ideal I of E, we let*

$$\phi(I) = \{x \in G : \exists\, x_i, y_j \in I,\ x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{6.3}$$

Then $\phi(I)$ is an o-ideal of (G, u). The mapping ϕ defines a one-to-one mapping preserving the set-theoretical inclusion. The inverse mapping ψ is given by

$$\psi(K) = K \cap [0, u], \quad K \in \mathcal{I}(G, u). \tag{6.4}$$

The restriction of ϕ to $\mathcal{P}(E)$ or $\mathcal{M}(E)$ gives a bijection between $\mathcal{M}(E)$ and $\mathcal{M}(G, u)$, and $\mathcal{P}(E)$ and $\mathcal{P}(G, u)$, respectively, which preserves the set-theoretical inclusion.

PROOF. Let I be an ideal of E. Then $I \subseteq \phi(I)$. Let $x, y \in \phi(I)$, then $x + y \in \phi(I)$ and $-x \in \phi(I)$. Assume $0 \le x \le y \in \phi(I)$ for $x \in G$. Then $y = y_1 + \cdots + y_n$ with $y_i \in I$ for $i = 1, \ldots, n$. The interpolation in G implies that $x = x_1 + \cdots + x_n$ for $x_i \le y_i \in I$, which proves $x \in \phi(I)$. It is clear that $\phi(I)$ is directed.

Assume that I and J are two different ideals of E. Then there exists $x \in I \setminus J$ or $x \in J \setminus I$. In the first case we assert $x \in \phi(I) \setminus \phi(J)$. Indeed, it is clear that $x \in \phi(I)$. If also $x \in \phi(J)$, then $x = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, where $x_i, y_j \in J$. Hence, $x + y_1 + \cdots + y_m = x_1 + \cdots + x_n$ and $0 \le x \le x_1 + \cdots + x_n$. The interpolation in (G, u) gives $x = x_1^0 + \cdots + x_n^0$ with $x_i^0 \le x_i \in J$, which implies that $x \in J$, a contradiction.

Assume now K is an o-ideal of (G, u). Then $\psi(K) \subseteq E$ and $0 \in \psi(K)$. If $x \le y$ for $x \in E$ and $y \in \psi(K)$, then $x \in K$, so $x \in \psi(K)$. If now $a, b \in \psi(K)$ and if $a + b$ is defined in E, then $a + b \in K$, that is $a + b \in \psi(K)$.

It is clear that $\phi(\psi(K)) = K$ for any $K \in \mathcal{I}(G, u)$ and $\psi(\phi(I)) = I$ for any $I \in \mathcal{I}(E)$.

Since ϕ is bijective, so are its restriction to $\mathcal{M}(E)$ and $\mathcal{P}(E)$.

Assume now that P is a prime o-ideal of E and $K_1 \cap K_2 \subseteq \phi(P)$ where K_1, K_2 are o-ideals of (G, u). From (6.4) we have $\psi(K_1) \cap \psi(K_2) = \psi(K_1 \cap K_2)$, that is $\psi(K_1) \cap \psi(K_2) \subseteq P$ which gives $\psi(K_1) \subseteq P$ or $\psi(K_2) \subseteq P$. Hence, $K_1 \subseteq \phi(P)$ or $K_2 \subseteq \phi(K_2)$.

In a similar manner we prove that if P is a prime o-ideal of (G, u) then so is $\psi(P)$ in E. □

We can prove that if a is a non-zero element of E then an o-ideal V is a value of a if and only if $\phi(V)$ is a value of a in (G, u).

Let I be an o-ideal of (G, u). For $f, g \in G$, we define $f \sim_I g$ if and only if $f - g$ and $g - f \in I$. Then $\sim_I$ is a congruence on G, and $(G/I, u/I)$ is an interpolation group, see [9, Proposition 2.3].

THEOREM 6.12. *Let I be a proper ideal of an effect algebra E with the RDP and let $E \cong \Gamma(G, u)$. Then $E/I \cong \Gamma(G/\phi(I), u/\phi(I))$.*

PROOF. Let $f : G \to G/\phi(I)$ be the canonical mapping defined by $f(g) = g/\phi(I)$, $g \in G$, and let $h : E \to E/I$ be given by $h(a) = [a]_I$, $a \in E$.

We can prove that, for $a, b \in E$, $[a]_I = [b]_I$ if and only if $a/\phi(I) = b/\phi(I)$. Indeed, if $[a]_I = [b]_I$, then there are $e, f \in I$ such that $a - e = b - f$, which proves that $a/\phi(I) = b/\phi(I)$.

Conversely, if $a/\phi(I) = b/\phi(I)$ then there is $x \in \phi(I)$ such that $a - b = x$. Since $x = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, where $x_i, y_j \in I$, we have $a + y_1 + \cdots + y_m = b + x_1 + \cdots + x_n$. Due to the RDP holding in G, we find elements $c_{ij} \in E$ such that $a = c_{00} + c_{01} + \cdots + c_{0n}$, $y_j = c_{j0} + c_{j1} + \cdots + c_{jn}$ for $j = 1, \ldots, m$, and $b = c_{00} + c_{10} + \cdots + c_{m0}$, $x_i = c_{0i} + c_{1i} + \cdots + c_{mi}$ for $i = 1, \ldots, n$. Then $c_{00} + c_{01} + \cdots + c_{0n} + c_{10} + \cdots + c_{m0} = a + b - c_{00}$. Therefore $a - (a - c_{00}) = b - (b - c_{00})$ and if we set $e =: a - c_{00} = c_{01} + \cdots + c_{0n} \in I$ and $f = b - c_{00} = c_{10} + \cdots + c_{m0} \in I$ then $a - e = b - f$, which proves $[a]_I = [b]_I$.

Using Theorem 5.7, we have the following assertion. □

PROPOSITION 6.13. *Let $E = \Gamma(G, u)$, where (G, u) is an Abelian unital po-group satisfying the RDP. If K is a prime o-ideal of (G, u) and we let $\gamma(K) = K \cap E$ then $\gamma(K)$ is a prime ideal of E. If P is a prime ideal of E and ϕ is defined as in Theorem* 6.11 *then $\phi(P)$ is a prime o-ideal of (G, u). In addition, the mappings γ and ϕ are mutually inverse and preserve the set-theoretical inclusion.*

PROOF. It follows from Theorem 6.11. □

For $a \in G$, let $G_0(a)$ be the o-ideal of G generated by a.

PROPOSITION 6.14. *For an o-ideal P of an Abelian po-group G the following statements are equivalent*:

(i) *P is prime.*

(ii) *For $a, b \in G$, $G_0(a) \cap G_0(b) \subseteq P$ implies $a \in P$ or $b \in P$.*

(iii) *For $a, b \in G^+$, $G_0(a) \cap G_0(b) \subseteq P$ implies $a \in P$ or $b \in P$.*

PROOF. (i) $\Rightarrow$ (ii). It is evident.

(ii) $\Rightarrow$ (i). Assume $G_0(a) \cap G_0(b) \subseteq P$ implies $a \in P$ or $b \in P$. Let I and J be two directed convex subgroups of G with $I \cap J \subseteq P$. If $I \not\subseteq P$ and $J \not\subseteq P$ then there are two elements $a \in I \setminus P$ and $b \in J \setminus P$. Then $G_0(a) \cap G_0(b) \subseteq I \cap J \subseteq P$ which gives $a \in P$ or $b \in P$, a contradiction.

(ii) $\Rightarrow$ (iii). It is evident.

(iii) $\Rightarrow$ (ii). As in the implication (ii) $\Rightarrow$ (i) we can find two elements $a \in I \setminus P$ and $b \in J \setminus P$. There are two elements $a_1, a_2 \in G_0(a)$ such that $a = a_1 - a_2$. Since $-a_2 \leq a \leq a_1$, we have $a_1 \notin P$ or $a_2 \notin P$. In a similar way, $b = b_1 - b_2$, where $b_1, b_2 \in G_0(b)$, and $b_1 \notin P$ or $b_2 \notin P$. Say $a_1 \notin P$ and $b_1 \notin P$. Then $G_0(a_1) \cap G_0(b_1) \subseteq I \cap J \subseteq P$ which gives $a_1 \in P$ or $b_1 \in P$, a contradiction. ☐

We recall that a directed po-group G is an antilattice if and only if $a \wedge b = 0$ implies $a = 0$ or $b = 0$.

PROPOSITION 6.15. *An Abelian directed po-group G with the RDP is an antilattice if and only if the null ideal $\{0\}$ is prime.*

PROOF. Suppose $\{0\}$ is a prime o-ideal and $a \wedge b = 0$. Then $G_0(a) \cap G_0(b) = \{0\}$ which proves $a = 0$ or $b = 0$, so G is an antilattice.

Conversely, suppose that G is an antilattice, and $G_0(a) \cap G_0(b) = \{0\}$. Then $a \wedge b = 0$ which yields $a = 0$ or $b = 0$, so $\{0\}$ is prime. ☐

PROPOSITION 6.16. *A proper o-ideal P of a unital Abelian po-group (G, u) satisfying the RDP is prime if and only if G/P is an antilattice.*

PROOF. Let P be prime and assume $a/P \wedge b/P = 0/P$. Without loss of generality, we can assume $a, b \geq 0$ (indeed, there are $e, f \in P$ such that $0 \leq a+e$ and $0 \leq b+f$ and $e = e_1 - e_2$, $f = f_1 - f_2$, where $e_1, e_2, f_1 f_2 \in P^+$, which gives $e_2 \leq a + e_1$ and $f_2 \leq b + f_1$).

We claim that $G_0(a) \cap G_0(b) \subseteq P$. There are two cases.

(i) Suppose $x \in G_0(a) \cap G_0(b)$ and $x \geq 0$. Then $x = a_1 + \cdots + a_m = b_1 + \cdots + b_n$, where $a_i \leq a$ and $b_j \leq b$ for all i and all j. The the RDP implies that there is a system $\{c_{ij}\}$ of elements of E such that $a_i = \sum_j c_{ij}$ and $b_j = \sum_i c_{ij}$. Since $c_{ij} \leq a, b$, we have $c_{ij}/P = 0/P$, so $c_{ij} \in P$, which yields $a_i \in P$ and $x \in P$.

(ii) Suppose that $x \in G_0(a) \cap G_0(b)$. Then $x = x_1 - x_2 = -y_1 + y_2$, where $x_1, x_2 \in G_0(a)^+$ and $y_1, y_2 \in G_0(b)^+$. Due to the RDP holding in (G, u), there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $y_1 = c_{11} + c_{12}$, $x_1 = c_{21} + c_{22}$, $y_2 = c_{11} + c_{21}$ and $x_2 = c_{12} + c_{22}$. By (i), $c_{12}, c_{21} \in P$. Therefore, $x = x_1 - x_2 = c_{21} + c_{22} - c_{22} - c_{12} = c_{21} - c_{12} = -c_{12} - c_{11} + c_{11} + c_{12} \in P$.

Combining (i) and (ii), we have $G_0(a) \cap G_0(b) \subseteq P$ which gives $G_0(a) \subseteq P$ or $G_0(b) \subseteq P$, so $a \in P$ or $b \in P$, and $a/P = 0/P$ or $b/P = 0/P$ which proves that G/P is an antilattice.

Conversely, suppose that G/P is an antilattice. According to Proposition 6.14, for $a, b \in G^+$ we have $G_0(a) \cap G_0(b) \subseteq P$. We claim that $G_0(a/P) \cap G_0(b/P) = \{0/P\}$.

Indeed, let $0/P \leq x/P = a_1/P + \cdots + a_m/P = b_1/P + \cdots + b_n/P$, where $a_i/P \leq a/P$ and $b_j/P \leq b/P$. There are $a_i' \in a_i/P$ and $b_j' \in b_j/P$ such that

https://doi.org/10.1017/S1446788700016025 Published online by Cambridge University Press

$a'_i \le a$ and $b'_j \le b$. Hence, there is a system $\{c_{ij}\}$ of elements in G^+ such that $a'_i/P = \sum_j c_{ij}/P$ and $b'_j/P = \sum_i c_{ij}/P$. For each i and each j, there is an element $c'_{ij} \in c_{ij}/P$ such that $c'_{ij} \le a'_i, b'_j$, so $c'_{ij} \le a_i, b_j$. Then $c'_{ij} \in P$, which proves $x \in P$.

If now $x/P \in G_0(a/P) \cap G_0(b/P)$, then $x/P = x_1/P - x_2/P = -y_1/P + y_2/P$, where $x_1/P, x_2/P \in G_0(a/P)^+$ and $y_1/P, y_2/P \in G_0(b/P)^+$. As in (ii), we can prove $G_0(a/P) \cap G_0(b/P) = \{0/P\}$. Since G/P is an antilattice, by Proposition 6.15, $\{0/P\}$ is a prime o-ideal of G/P, consequently $a/P = 0/P$ or $b/P = 0/P$, so $a \in P$ or $b \in P$, which proves that P is prime. □

THEOREM 6.17. *Let $E = \Gamma(G, u)$, where (G, u) is an Abelian unital po-group with the RDP. Then E is an antilattice if and only if G is an antilattice.*

PROOF. It follows from Proposition 6.5, Theorem 6.11, Proposition 6.14, Proposition 6.16, and Theorem 6.12. □

7. Subdirect representation and perfect effect algebras

We give a representation of perfect effect algebras as a subdirect product of antilattice perfect effect algebras with the RDP and we introduce the class of $\mathcal{BP}_0$ effect algebras and give their subdirect representation.

We start with a new proof of Ravindran's representation, [12, Theorem 2.18], of effect algebras with the RDP as subdirect products of antilattice effect algebras with the RDP.

Let $\{E_i\}_{i\in I}$ be an indexed system of effect algebras. The Cartesian product $\prod_{i\in I} E_i$ can be organized into an effect algebra with the partial addition defined by coordinates. Each E_i has the RDP if and only if the same is true of $\prod_i E_i$. We say that an effect algebra E is a *subdirect product* of effect algebras $\{E_i\}_{i\in I}$ if there is an injective homomorphism $f : E \to \prod_{i\in I} E_i$ such that $f(a) \le f(b)$ if and only if $a \le b$ ($a, b \in E$), and for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from E onto E_j, where π_j is the projection of $\prod_i E_i$ onto E_j.

We first give another characterization of the finite irreducibility of effect algebras, or equivalently of antilattice effect algebras with the RDP.

PROPOSITION 7.1. *An effect algebra E with the RDP is finitely subdirectly irreducible if and only if E is a subdirect product of E_1 and E_2, and if f is an injective homomorphism from E into $E_1 \times E_2$ such that $f(x) \le f(y)$, whenever $x \le y$, and such that $\pi_1 \circ f$ and $\pi_2 \circ f$ are surjective, then $\pi_1 \circ f$ or $\pi_2 \circ f$ is an isomorphism.*

PROOF. Suppose that E is not finitely subdirectly irreducible, that is, there are two non-zero ideals A and B of E such that $A \cap B = \{0\}$. The mapping f :

$E \to E/A \times E/B$ given by $f(a) = ([a]_A, [a]_B)$, $a \in E$, is a homomorphism of effect algebras. If $f(a) = f(b)$ then there are $e, f_1 \in A$ and $u_1, v \in B$ such that $a - e = b - f_1$ and $a - u_1 = b - v$. If we now take the addition and subtraction in the corresponding unital interpolation group (G, u) such that $E = \Gamma(G, u)$, then $a - b = e - f_1 \in \phi(A)$ and $a - b = u_1 - f_1 \in \phi(B)$, so $a - b = 0$ and f is an injective homomorphism.

Assume $f(x) \leq f(y)$ for some $x, y \in E$, so $x/A \leq y/A$ and $x/B \leq y/B$. There are two elements $a \in A$ and $b \in B$ with $a, b \leq x$ such that $x - a \leq y$ and $x - b \leq y$. Since $a \wedge b = 0$, we have $x = x - (a \wedge b) = (x - a) \vee (x - b)$ (while all existing meets in E are preserved in the corresponding representation group (G, u)), which gives $x \leq y$.

Hence E is a subdirect product of E/A and E/B, and $a \mapsto a/A$ and $a \mapsto a/B$ are not isomorphisms.

Conversely, suppose E is a subdirect product of E_1 and E_2 and let $f : E \to E_1 \times E_2$ be an injective homomorphism with $f(x) \leq f(y)$ if and only if $x \leq y$ and such that every $\pi_i \circ f : E \to E_i$, $i = 1, 2$, is not an isomorphism. Define $A_i = \{a \in E : \pi_i \circ f(a) = 0\}$, $i = 1, 2$. Then A_1 and A_2 are non-zero ideals of E. Assume that $x \in A_1 \cap A_2$. Then $f(x) = (0, 0)$ and the injectivity of f gives $x = 0$, which proves $A_1 \cap A_2 = \{0\}$, so E is not finitely subdirectly irreducible. □

The following result is the Ravindran representation mentioned above.

THEOREM 7.2. *Every effect algebra E with the RDP is a subdirect product of antilattice effect algebras with the RDP, and all existing meets and joins in E are preserved in the subdirect product.*

PROOF. (1) Assume $E = \Gamma(G, u)$. Choose $g \in G$, $g \not\leq 0$, and set $U(g) = \{h \in G : h \geq g\}$. We denote by $A(g)$ an ideal of E which is maximal with respect to the property $U(g) \cap A(g) = \emptyset$. Since $0 \notin U(g)$, $A(g)$ exists due to the Zorn lemma. We assert $A(g)$ is a prime ideal of E. Let $I \cap J = A(g)$, where I and J are ideals of E. To obtain a contradiction, assume $A(g)$ is a proper subset of I as well as of J. Take $a \in I \cap U(g)$ and $b \in J \cap U(g)$. We have $0, g \leq a, b$. By the RIP holding in (G, u), there is an element $c \in G$ such that $0, g \leq c \leq a, b$, Since $0 \leq c \leq a$, we have $c \in E$ and $g \leq c \in I \cap J = A(g)$, which gives $c \in U(g) \cap A(g)$, a contradiction.

Moreover, $A(g)$ is a proper ideal of E whenever $g \leq u$.

In particular, if a is a nonzero element of E then $A(a)$ is a value of E.

(2) There exists a system $\mathcal{P}$ of proper prime ideals of E such that $\bigcap\{P : P \in \mathcal{P}\} = \{0\}$. For that, let $\mathcal{P}$ be the system of all proper prime ideals of E. Then

$$\bigcap\{P : P \in \mathcal{P}\} = \{0\}.$$

Indeed, take a non-zero element $a \in \bigcap\{P : P \in \mathcal{P}\}$. Then any value V of a is a prime ideal by Proposition 3.5. Fix a value of a, say $V(a)$, then $a \notin V(a)$.

The mapping $f : E \to \prod_{P\in\mathcal{P}} E/P$ defined by $f(a) = (a/P)_{P\in\mathcal{P}}$, is a homomorphism of the effect algebras.

Assume $f(a) \le f(b)$, so that $a/P \le b/P$ for any $P \in \mathcal{P}$, and set $g = a - b$; then $g \le u$. If $g \not\le 0$ then $a/A(g) \le b/A(g)$ and there is an element $e \in A(g)$ such that $a - e \le b$, so $g = a - b \le e$, which implies $e \in U(g) \cap A(g)$, a contradiction.

Therefore, $f(u) = f(v)$ if and only if $u = v$, which proves that f is an injective homomorphism such that $f(a) \le f(b)$ if and only if $a \le b$, which shows that E is a subdirect product of $\{E/P : P \in \mathcal{P}\}$.

Assume now that $a \wedge b$ is defined in E. Then by Proposition 6.10, for any $P \in \mathcal{P}$ we have $(a \wedge b)/P = (a/P) \wedge (b/P)$. Consequently $f(a \wedge b) = f(a) \wedge f(b)$ and in a similar way, $f(a \vee b) = f(a) \vee f(b)$. ☐

We recall that if E is a lattice effect algebra with the RDP then E is an MV-algebra, and we have a known subdirect representation of any MV-algebra as a subdirect product of linear MV-algebras [1]. In such a case, for $\mathcal{P}$ it is sufficient to take $\mathcal{P} = \{V(a) : a > 0\}$.

PROPOSITION 7.3. *Let P be a proper ideal of a perfect effect algebra E. Then the quotient effect algebra E/P is perfect.*

PROOF. By Proposition 6.1, E/P satisfies the RDP. We claim that E/P has the Rad-property. In view of Proposition 4.1,

$$\mathrm{Infinit}(E)/P \subseteq \mathrm{Infinit}(E/P) \subseteq \mathrm{Rad}(E/P).$$

Assume $x/P \in \mathrm{Rad}(E/P)$. Then either $x \in \mathrm{Rad}(E)$ or $x' \in \mathrm{Rad}(E)$. If $x \in \mathrm{Rad}(E) = \mathrm{Infinit}(E)$ then $x/P \in \mathrm{Infinit}(E/P) \subseteq \mathrm{Rad}(E/P)$. If $x' \in \mathrm{Rad}(E)$, then $x'/P \in \mathrm{Infinit}(E/P)$, so $x'/P \in \mathrm{Rad}(E/P)$, a contradiction, which proves that $x/P \in \mathrm{Infinit}(E/P)$ and $\mathrm{Rad}(E/P) = \mathrm{Infinit}(E/P)$.

If now x/P is an arbitrary element of E/P then by the above argument, either $x \in \mathrm{Rad}(E)$ or $x' \in \mathrm{Rad}(E)$, which proves that either $x/P \in \mathrm{Infinit}(E)/P \subseteq \mathrm{Infinit}(E/P)$ or $x' \in \mathrm{Rad}(E)$ giving $x'/P \in \mathrm{Infinit}(E)/P \subseteq \mathrm{Infinit}(E/P)$, so E/P is perfect. ☐

As a corollary, we have the following result.

THEOREM 7.4. *Every perfect effect algebra E is a subdirect product of antilattice perfect effect algebras.*

PROOF. By Theorem 7.2, E is a subdirect product of antilattice effect algebras E/P, where P is a proper prime ideal. In view of Proposition 7.3, E/P is perfect. ☐

In the paper [3], the authors introduced the class $\mathcal{BP}_0$ of MV-algebras. Every MV-algebra is a lattice effect algebra with the RDP, and vice versa. Inspired by this, we say that an effect algebra with the RDP belongs to the class $\mathcal{BP}_0$ of effect algebras if and only if, for every maximal ideal M of E, we have $E = M \cup M'$.

PROPOSITION 7.5. *Let an effect algebra E belong to the class $\mathcal{BP}_0$. Let M be a maximal ideal and P a proper ideal of E. Then $P \subseteq M$ if and only if $M/P \neq E/P$.*

PROOF. Assume $P \subseteq M$. If there exists $x \in M$ such that $x/P = 1/P$, then there are two elements e and f in P such that $x - e = 1 - f$. Hence $1 - x = f - e \in P \subseteq M$, which is a contradiction.

Conversely, let $M/P \neq E/P$ and take $x \in P$. If $x' \in M$ then $x'/P = 0$, so $x/P = 1/P$, giving a contradiction. Therefore, $x \in M$. □

THEOREM 7.6. *If E is an effect algebra from the class $\mathcal{BP}_0$ with the property that for any $x \in E$ and any $P \in \mathcal{P}(E)$ we have $x/P \leq (1-x)/P$ or $(1-x)/P \leq x/P$, then E is a subdirect product of antilattice perfect effect algebras.*

PROOF. According to Theorem 7.2, E is a subdirect product of antilattice effect algebras $\{E/P : P \in \mathcal{P}(E),\ P \neq E\}$. In what follows, we show that every E/P is a perfect effect algebra. Let N be a maximal ideal of E containing P and let x be an arbitrary element in E. If $x \in N$ then $(1-x)/P \leq x$ or $x/P \leq (1-x)/P$. Since N is a maximal ideal of E, by Proposition 6.4 and Proposition 7.5, the first case would imply $(1-x)/P \in N/P$ which is a contradiction. Hence $x/P \leq (1-x)/P$, which implies that $x/P + x/P$ is defined in E/P. Therefore there are two elements $x_1, x_2 \in N$ such that $x_0 = x_1 + x_2 \in N$ and $x_1, x_2 \in [x]_P$. Repeating this process with the element x_0, we have $x_0/P + x_0/P \in E/P$, so $4(x/P)$ is defined in E/P, and so on. Therefore $2^n(x/P) \in E/P$ for any integer $n \geq 1$, which implies $x/P \in \mathrm{Infinit}(E/P) \subseteq \mathrm{Rad}(E/P)$. Summarizing, we have proved that for every $x \in E$, either $x/P \in \mathrm{Infinit}(E/P)$ or $(1-x)/P \in \mathrm{Infinit}(E/P)$.

In particular, if $x/P \in \mathrm{Rad}(E/P)$ then $x \in \kappa(M)$, where $M \in \mathcal{M}(E/P)$ and $\kappa(M) \supseteq P$ is defined in Proposition 6.4, which proves by the above that $x/P \in \mathrm{Infinit}(E/P)$ and therefore E/P has the Rad-property and E/P is perfect. □

We observe that, for example, every effect algebra E with the RDP satisfies the following condition: for any $x \in E$ and any prime ideal P of E, $x/P \leq (1-x)/P$ or $(1-x)/P \leq x/P$. On the other hand, if E/P is perfect then, by Proposition 5.3, this condition holds. It would be interesting to know whether the assumption of this condition in Theorem 7.6 is necessary in order to obtain the conclusion of the theorem.

Acknowledgment

The author is very indebted to the referees for their valuable suggestions which improved the readability of the paper.

The paper has been supported by Science and Technology Assistance Agency under the contract No. APVT-51-032002 and by the Center of Excellence SAS – Physics of Information – I/2/2005, Bratislava, Slovakia.

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