# AN EMBEDDING FOR $\pi_2$ OF A SUBCOMPLEX OF A FINITE CONTRACTIBLE TWO-COMPLEX

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**Introduction and statement of results.** A longstanding open question in low dimensional topology was raised by J. H. C. Whitehead in 1941 [**9**]: "Is any subcomplex of an aspherical, two-dimensional complex itself aspherical?" The asphericity of classical knot complements [**7**] provides evidence that the answer to Whitehead's question might be "yes". Indeed, each classical knot complement has the homotopy type of a two-complex which can be embedded in a finite contractible two-complex. This property is shared by a large class of four-manifolds; these are the ribbon disc complements, whose asphericity has been conjectured, and even claimed, but never proven. (See [**4**] for a discussion.) It is reasonable and convenient to formulate the following.

RESTRICTED WHITEHEAD CONJECTURE (RWC). Subcomplexes of finite contractible two-complexes are aspherical.

The RWC has a purely algebraic reformulation, which we now describe.

By a *normal factorization* of a group $G$, we mean an expression $G = R_1 \ldots R_k$ where $k$ is a positive integer and $R_1, \ldots, R_k$ are normal subgroups of $G$. A normal factorization $F = R_1 \ldots R_k$ of a finitely generated free group $F$ is here said to be *efficient* if there exist pairwise disjoint finite subsets $\mathbf{r}_1, \ldots, \mathbf{r}_k$ of $F$ such that $|\mathbf{r}_1| + \ldots + |\mathbf{r}_k| = \operatorname{rank} F$ and, for $j = 1, \ldots, k$, $R_j$ is normally generated in $F$ by $\mathbf{r}_j$.

If $A$ and $B$ are subgroups of a group $G$, then $[A, B]$ denotes the subgroup of $G$ generated by all commutators $[a, b] = aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$. If $A$ and $B$ are normal in $G$, then so is $[A, B]$, and $[A, B]$ is contained in $A \cap B$.

ALGEBRAIC RWC (ARWC). If $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group, then $R \cap S \subseteq [R, S]$.

Using [**3**, Theorem 1], it is a simple matter to show that the conclusion of the ARWC holds in the case where the normal factorization involves just two factors. (See Lemma 4 below.) In the general case, our main result is the following.

THEOREM 1. *If $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group $F$, then $R \cap S \subseteq [R, S]F_n$ for all $n \geq 1$.*

Here, for a group $G$, $G_n$ denotes the $n$th term of the lower central series of $G$, defined inductively by $G_1 = G$ and $G_{n+1} = [G, G_n]$.

To see that the RWC is implied by the ARWC, suppose that $X$ is a subcomplex of a finite contractible two-complex $Y$. It suffices to prove that $\pi_2 X = 0$. One easily reduces to the case where $Y$ has a single two-cell and $Y$ is obtained from $X$ by attaching two-cells. The case where $X$ has a single zero-cell can be handled using the Lyndon Identity Theorem [**5**]; this was done in greater generality by Cockroft in [**2**]. Suppose that $X$ is a union of subcomplexes $X_r$ and $X_s$ which intersect in the one-skeleton $Y^1$, and where each of $X_r$ and $X_s$ has at least one two-cell. By induction, each of $X_r$ and $X_s$ is aspherical. Now, $Y = X \cup X_t$, where $X_t$ consists of the one-skeleton $Y^1$ together with the two-cells of $Y - X$. Let $F = \pi_1 Y^1$, a finitely generated free group; let $R$, $S$, $T$ denote the kernel of the homomorphism on fundamental groups induced by the inclusions of $Y^1$ in $X_r$, $X_s$, $X_t$

respectively. Then $F = RST$ is an efficient normal factorization of $F$, since $Y$ is contractible. As normal generators for $R$, $S$, $T$, take the based homotopy classes of the attaching maps for the two-cells of $X_r$, $X_s$, $X_t$ respectively. A result of Gutierrez and Ratcliffe [**3**, Theorem 1] then provides that $\pi_2 X \cong (R \cap S)/[R, S]$, and so the ARWC implies that $\pi_2 X = 0$. Even without the ARWC, the promised embedding for $\pi_2$ is an immediate consequence of Theorem 1.

COROLLARY. *If $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group $F$ and $Q = F/[R, S]$, then $(R \cap S)/[R, S]$ embeds naturally in $Q_\omega = \bigcap \{Q_n : n \geq 1\}$.*

Conversely, the above remarks show how to use a counterexample to the ARWC to construct a counterexample to the RWC.

In Theorem 2, we determine the structure of $\operatorname{Gr} Q$, where $Q$ is the group in the Corollary to Theorem 1. In particular, the groups $Q_n/Q_{n+1}$ are finitely generated free abelian for all positive integers $n$.

Aside from the result of [**3**] which was used above, the main general tool employed in the proof of Theorem 1 is the graded integral Lie algebra $\operatorname{Gr} G$ that is constructed from the lower central series of a group $G$. Of special utility is the theorem of Magnus [**6**] which states that if $F$ is a free group, then $\operatorname{Gr} F$ is a free Lie algebra. Further, the homogeneous components of $\operatorname{Gr} F$ are finitely generated free abelian, with ranks given by an explicit formula due to Witt [**10**].

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**On free Lie algebras.** Very little is new here. The primary reference for the material in this section is [**1**, Chapter II]. All algebras are to be taken over the integers.

A magma is a pair $(M, .)$, where $M$ is a set and . is a binary operation on $M$. If $\mathbf{x}$ is a set, then $M(\mathbf{x})$ denotes the free magma on $\mathbf{x}$. Thus $M(\mathbf{x})$ is the disjoint union of sets $\mathbf{x}_n$ $(n \geq 1)$, where $\mathbf{x}_1 = \mathbf{x}$, and $\mathbf{x}_n$ is defined inductively as the disjoint union of the sets $\mathbf{x}_m \times \mathbf{x}_{n-m}$ $(m = 1, \ldots, n-1)$. The operation in $M(\mathbf{x})$ is given by $x \,.\, y = (x, y)$ for $x \in \mathbf{x}_m$ and $y \in \mathbf{x}_{n-m}$. If $(N, .)$ is a magma, then any function of $\mathbf{x}_1 = \mathbf{x}$ into $N$ extends uniquely to a magma homomorphism $(M(\mathbf{x}), .) \to (N, .)$.

The free integral Lie algebra on the set $\mathbf{x}$ is denoted by $L(\mathbf{x})$. There is a canonical embedding of $\mathbf{x}$ into $L(\mathbf{x})$. This yields a magma homomorphism $\xi : (M(\mathbf{x}), .) \to (L(\mathbf{x}), [\ ,\ ])$, where $[\ ,\ ]$ denotes the Lie bracket in $L(\mathbf{x})$. We identify $\mathbf{x}$ with $\xi(\mathbf{x})$. Any function of $\mathbf{x}$ into a Lie algebra $L$ extends uniquely to a Lie algebra homomorphism $L(\mathbf{x}) \to L$. In particular, $L(\mathbf{x})$ is generated as a Lie algebra by $\mathbf{x}$. The Lie algebra $L(\mathbf{x})$ is graded by the positive integers: $L(\mathbf{x}) = \bigoplus \{L^n(\mathbf{x}) : n \geq 1\}$, where $L^n(\mathbf{x})$ is the subgroup of $L(\mathbf{x})$ spanned by the images of the elements of $\mathbf{x}_n$. In particular, $[L^m(\mathbf{x}), L^n(\mathbf{x})] \subseteq L^{m+n}(\mathbf{x})$, and $\mathbf{x}$ is a $\mathbb{Z}$-basis for the free abelian group $L^1(\mathbf{x})$. If $\mathbf{x}$ is finite, then $L^n(\mathbf{x})$ is free abelian, with finite rank given in terms of $n$ and the order of $\mathbf{x}$. (The explicit formula is given in [**10**]; see also [**1**, II.3.3, Theorem 2].)

If $\mathbf{y} \subset \mathbf{x}$, then $M(\mathbf{x} - \mathbf{y})$ is naturally viewed as a subset of $M(\mathbf{x})$. The elements of $M(\mathbf{x}) - M(\mathbf{x} - \mathbf{y})$ are here said to *involve* $\mathbf{y}$. Similarly, a homogeneous element of $L(\mathbf{x})$ is said to *involve* $\mathbf{y}$ if it is of the form $\xi(\mu)$, where $\mu \in M(\mathbf{x}) - M(\mathbf{x} - \mathbf{y})$.

LEMMA 1. *The ideal $I$ generated by* $\mathbf{y}$ *in* $L(\mathbf{x})$ *is the* $\mathbb{Z}$*-span of those elements of* $L(\mathbf{x})$ *that involve* $\mathbf{y}$. *If* $\mu_1, \ldots, \mu_m \in M(\mathbf{x}-\mathbf{y})$, $e_1, \ldots, e_m \in \mathbb{Z}$, *and* $e_1\xi(\mu_1) + \ldots + e_m\xi(\mu_m) \in I$, *then* $e_1\xi(\mu_1) + \ldots + e_m\xi(\mu_m) = 0$.

*Proof.* Let $\mathbf{Y}$ denote the set of elements of $L(\mathbf{x})$ that involve $\mathbf{y}$. We show by induction on $n$ that if $\eta \in \mathbf{Y} \cap L^n(\mathbf{x})$, then $\eta \in I$. Clearly, $\mathbf{Y} \cap L^1(\mathbf{x}) = \mathbf{y}$. For $n > 1$, if $\eta \in \mathbf{Y} \cap L^n(\mathbf{x})$, then there exists $\mu \in \mathbf{x}_n - M(\mathbf{x}-\mathbf{y})$ such that $\eta = \xi(\mu)$. There exist unique $m \in \{1, \ldots, n-1\}$, $\nu_m \in \mathbf{x}_m$, and $\nu_{n-m} \in \mathbf{x}_{n-m}$ such that $\mu = \nu_m \,.\, \nu_{n-m}$. Since $\mu \in M(\mathbf{x}) - M(\mathbf{x}-\mathbf{y})$, either $\nu_m$ or $\nu_{n-m}$ is in $M(\mathbf{x}) - M(\mathbf{x}-\mathbf{y})$. By induction, either $\xi(\nu_m)$ or $\xi(\nu_{n-m})$ is in $I$. Since $I$ is an ideal of $L(\mathbf{x})$, $\eta = [\xi(\nu_m), \xi(\nu_{n-m})] \in I$. This completes the induction, and proves that $I$ contains the $\mathbb{Z}$-span of $\mathbf{Y}$.

By [**1**, II.2.9, Proposition 10], $L(\mathbf{x})$ decomposes as the internal direct sum of $I$ and $L(\mathbf{x}-\mathbf{y})$. The second statement of the lemma follows, since $L(\mathbf{x}-\mathbf{y})$ is the $\mathbb{Z}$-span of $\xi(M(\mathbf{x}-\mathbf{y}))$. That $I$ equals the $\mathbb{Z}$-span of $\mathbf{Y}$ follows from the fact that $L(\mathbf{x})$ is spanned by $\xi(M(\mathbf{x}))$. ■

The direct product $K \times L$ of Lie algebras $K$ and $L$ has as underlying abelian group the direct product of $K$ and $L$, with Lie bracket given by $[(x, y), (x', y')] = ([x, x'], [y, y'])$ for $x, x' \in K$ and $y, y' \in L$.

Let $\mathbf{a}$ and $\mathbf{b}$ be disjoint sets. Let $I$ and $J$ be the ideals of $L(\mathbf{a} \cup \mathbf{b})$ generated by $\mathbf{a}$ and $\mathbf{b}$ respectively. The function $\mathbf{a} \cup \mathbf{b} \to L(\mathbf{a})$ which restricts to the identity on $\mathbf{a}$ and which carries each element of $\mathbf{b}$ to zero induces a split Lie algebra epimorphism of $L(\mathbf{a} \cup \mathbf{b})$ onto $L(\mathbf{a})$, with kernel $J$. (It is split using [**1**, II.2.9, Proposition 10] as above.) There is an analogous split Lie algebra epimorphism of $L(\mathbf{a} \cup \mathbf{b})$ onto $L(\mathbf{b})$, with kernel $I$. Taken together, these induce a Lie algebra epimorphism of $L(\mathbf{a} \cup \mathbf{b})$ onto $L(\mathbf{a}) \times L(\mathbf{b})$, with kernel $I \cap J$.

**On group commutators.** Let $A$ and $B$ be normal subgroups of a group $G$. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be disjoint sets, and let $\mathbf{a} \to A$, $\mathbf{b} \to B$ and $\mathbf{c} \to G$ be functions. The function $\mathbf{a} \cup \mathbf{b} \cup \mathbf{c} \to G$ induces a magma homomorphism $\gamma : (M(\mathbf{a} \cup \mathbf{b} \cup \mathbf{c}), .) \to (G, [\ ,\ ])$.

LEMMA 2. *Let* $\mu \in M(\mathbf{a} \cup \mathbf{b} \cup \mathbf{c})$.

(i) *If* $\mu$ *involves* $\mathbf{a}$, *then* $\gamma(\mu) \in A$.

(ii) *If* $\mu$ *involves both* $\mathbf{a}$ *and* $\mathbf{b}$, *then* $\gamma(\mu) \in [A, B]$.

*Proof.* In both cases, one assumes that $\mu \in (\mathbf{a} \cup \mathbf{b} \cup \mathbf{c})_n$ and proceeds by induction on $n$. Details are left to the reader. ■

The graded integral Lie algebra associated to $G$ is $\operatorname{Gr} G = \bigoplus \{\operatorname{Gr}^n G : n \geq 1\}$, where $\operatorname{Gr}^n G = G_n / G_{n+1}$, and with Lie bracket determined by $[uG_{n+1}, vG_{m+1}] = [u, v]G_{n+m+1}$, for all $u \in G_n$ and $v \in G_m$. The first homogeneous component $\operatorname{Gr}^1 G = G/G_2$ generates $\operatorname{Gr} G$ as a Lie algebra. A group homomorphism $f : G \to H$ induces a homomorphism $\operatorname{Gr} f : \operatorname{Gr} G \to \operatorname{Gr} H$ of graded Lie algebras; the process Gr is a functor from groups to graded integral Lie algebras. It is easy to prove that the functor Gr preserves direct products: there is an isomorphism $\operatorname{Gr}(G \times H) \to \operatorname{Gr} G \times \operatorname{Gr} H$ which carries $(g, h)(G \times H_2) \in \operatorname{Gr}^1(G \times H)$ to $(gG_2, hH_2) \in \operatorname{Gr}^1 G \times \operatorname{Gr}^1 H$, for all $g \in G$ and $h \in H$.

Recall [**6**] (see also [**1**, II.5.4, Theorem 3]) that if $F = \operatorname{free}(\mathbf{x})$ is the free group with basis $\mathbf{x}$, then the function $\mathbf{x} \to \operatorname{Gr} F$ which carries $x \in \mathbf{x}$ to $xF_2 \in \operatorname{Gr}^1 F$ induces an isomorphism $L(\mathbf{x}) \to \operatorname{Gr} F$ of graded integral Lie algebras.

**The case of just two factors.** Throughout this section, we assume that $F = AB$ is an efficient normal factorization of a finitely generated free group $F$. Select pairwise disjoint finite normal generating sets $\mathbf{a}$ and $\mathbf{b}$ for $A$ and $B$ in $F$ such that $|\mathbf{a}| + |\mathbf{b}| = \operatorname{rank} F$.

LEMMA 3. *The homomorphism* $h: \text{free}(\mathbf{a}) \to F/B$ *given by* $h(a) = aB$ *induces an isomorphism* $\operatorname{Gr} h: \operatorname{Gr}(\text{free}(\mathbf{a})) \to \operatorname{Gr} F/B$ *of graded Lie algebras.*

*Proof.* Select a basis $\mathbf{x}$ for $F$, and let $X$ be the two-complex modeled on the presentation $(\mathbf{x} \mid \mathbf{b})$ for $F/B$. Thus, $\pi_1 X \cong F/B$, and $\chi(X) = 1 - |\mathbf{x}| + |\mathbf{b}| = 1 - |\mathbf{a}|$. Furthermore, $X$ is a subcomplex of the two-complex $Y$ modeled on the presentation $(\mathbf{x} \mid \mathbf{a}, \mathbf{b})$ for the trivial group $F/AB$. Since $Y$ is simply connected and $\chi(Y) = 1 - |\mathbf{x}| + |\mathbf{a}| + |\mathbf{b}| = 1$, $Y$ is contractible. This implies that $H_2 X = 0$, and hence that $H_2 F/B = 0$. (See [**8**].) Further, since $1 - |\mathbf{a}| = \chi(X) = 1 - \operatorname{rank} H_1 X + \operatorname{rank} H_2 X$, $\operatorname{rank} H_1 F/B = \operatorname{rank} H_1 X = |\mathbf{a}|$. Since $F = AB$, $F/B$ is normally generated by $\{aB : a \in \mathbf{a}\}$. As such, the homomorphism $h$ induces an epimorphism $H_1 h: H_1 \text{free}(\mathbf{a}) \to H_1 F/B$ of the abelianized groups. The fact that $H_1 \text{free}(\mathbf{a})$ and $H_1 F/B$ have the same finite rank then implies that $H_1 h$ is an isomorphism. By [**8**, Lemma 3.1], $\operatorname{Gr} h: \operatorname{Gr}(\text{free}(\mathbf{a})) \to \operatorname{Gr} F/B$ is an isomorphism. (Interesting but irrelevant is the further consequence [**8**, Theorem 7.4] that $h$ itself is injective.) ■

LEMMA 4. $A \cap B = [A, B]$.

*Proof.* We retain the notation of the proof of Lemma 3. Decompose $Y$ as a union of $X$ and a complementary two-complex modeled on the presentation $(\mathbf{x} \mid \mathbf{a})$ for $F/A$; then [**3**, Theorem 1] provides an epimorphism $\pi_2 Y \to (A \cap B)/[A, B]$. The result follows from the fact that $Y$ is contractible. ■

LEMMA 5. *There is an isomorphism of Lie algebras*

$$\Psi: L(\mathbf{a}) \times L(\mathbf{b}) \to \operatorname{Gr}(F/[A, B])$$

*such that* $\Psi((a, 0)) = aF_2 \in F/F_2 = F/[A, B]F_2 = \operatorname{Gr}^1(F/[A, B])$ *for all* $a \in \mathbf{a}$, *and* $\Psi((0, b)) = bF_2 \in F/F_2 = F/[A, B]F_2 = \operatorname{Gr}^1(F/[A, B])$ *for all* $b \in \mathbf{b}$.

*Proof.* The map $\Psi$ defines a Lie algebra homomorphism of the direct product since the images of $L(\mathbf{a})$ and $L(\mathbf{b})$ under $\Psi$ commute in $\operatorname{Gr}(F/[A, B])$.

Using the fact that $F = AB$ and $A \cap B = [A, B]$, one checks that the homomorphism $f: F/[A, B] \to F/A \times F/B$ given by $f(w[A, B]) = (wA, wB)$ is an isomorphism. There is thus an isomorphism of graded Lie algebras

$$\varphi: \operatorname{Gr}(F/[A, B]) \to \operatorname{Gr} F/A \times \operatorname{Gr} F/B$$

such that $\varphi(wF_2) = (wAF_2, wBF_2)$ for all $wF_2 \in \operatorname{Gr}^1(F/[A, B])$. The result follows from Lemma 3 and Magnus' isomorphism $L(\mathbf{a}) \cong \operatorname{Gr}(\text{free}(\mathbf{a}))$. ■

**Proof of Theorem 1.** Suppose that $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group $F$. Upon multiplication of the complementary factors, there is an efficient normal factorization $F = RST$ of $F$. There are pairwise disjoint finite subsets $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ such that $|\mathbf{r}| + |\mathbf{s}| + |\mathbf{t}| = \operatorname{rank} F$, where $R$, $S$, $T$ are normally generated in $F$ by $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ respectively. Let $\mathbf{u} = \mathbf{r} \cup \mathbf{s} \cup \mathbf{t}$.

LEMMA 6. *The function* $\varphi: \mathbf{u} \to \operatorname{Gr} F$ *given by* $\varphi(u) = uF_2$ *extends to an isomorphism* $\Phi: L(\mathbf{u}) \to \operatorname{Gr} F$ *of graded Lie algebras.*

*Proof.* Since $F$ is normally generated by $\mathbf{u}$, $\mathrm{Gr}^1 F = F/F_2$ is generated as an abelian group by $\varphi(\mathbf{u})$. It follows that $\Phi$ is surjective, since $\mathrm{Gr}^1 F$ generates $\mathrm{Gr}\,F$ as a Lie algebra. Since $\varphi(\mathbf{u}) \subset \mathrm{Gr}^1 F$, $\Phi$ is a homomorphism of graded Lie algebras: $\Phi(L^n(\mathbf{u})) = \mathrm{Gr}^n F$ for all $n \geqslant 1$. Since $|\mathbf{u}| = \mathrm{rank}\, F$, $L^n(\mathbf{u})$ and $\mathrm{Gr}^n F$ are free abelian groups of the same finite rank for all $n \geq 1$. This implies that $\Phi$ is injective. ■

LEMMA 7. $R \cap S = [R, ST] \cap [RT, S]$.

*Proof.* By Lemma 4, $R \cap S \subseteq (R \cap ST) \cap (RT \cap S) = [R, ST] \cap [RT, S]$. ■.

Let $q: F \to F/[R, S] = Q$ be the natural projection. The natural epimorphism $Q \to F/[R, ST]$ induces an epimorphism of graded Lie algebras $\mathrm{Gr}\,Q \to \mathrm{Gr}\,F/[R, ST]$. The structure of $\mathrm{Gr}\,F/[R, ST]$ is given by Lemma 5. Taken together, there is a composite epimorphism of Lie algebras

$$\rho: L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}) \cong \mathrm{Gr}\,F \to \mathrm{Gr}\,Q \to \mathrm{Gr}\,F/[R, ST] \cong L(\mathbf{r}) \times L(\mathbf{s} \cup \mathbf{t})$$

which carries each $r \in \mathbf{r}$ to $(r, 0)$ and each $x \in \mathbf{s} \cup \mathbf{t}$ to $(0, x)$. Using the natural epimorphism $Q \to F/[RT, S]$, there is an analogous composite epimorphism of Lie algebras

$$\sigma: L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}) \cong \mathrm{Gr}\,F \to \mathrm{Gr}\,Q \to \mathrm{Gr}\,F/[RT, S] \cong L(\mathbf{r} \cup \mathbf{t}) \times L(\mathbf{s}).$$

Let $I_\mathbf{r}$ and $I_\mathbf{s}$ denote the ideals of $L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t})$ generated by $\mathbf{r}$ and $\mathbf{s}$ respectively. By the discussion of direct products of Lie algebras following the proof of Lemma 1, it follows that $\ker \rho \cap \ker \sigma \subseteq I_\mathbf{r} \cap I_\mathbf{s}$.

Theorem 1 is now proved as follows. Let $w \in R \cap S$. By induction on $n$, we show that $w \in [R, S]F_n$ for all $n \geq 1$. The case $n = 1$ is trivial, and the case $n = 2$ follows from Lemma 7. Suppose that $n \geq 3$. By induction we may write $w = uv$, where $u \in [R, S]$ and $v \in F_{n-1}$. Since $[R, S] \subseteq R \cap S$, $v \in R \cap S$. Consider $vF_n \in \mathrm{Gr}^{n-1} F$. There exists a unique $\eta \in L^{n-1}(\mathbf{u})$ such that $vF_n = \Phi(\eta)$, where $\Phi$ is the isomorphism of Lemma 6. Since $\rho$ factors through $\mathrm{Gr}\,F/[R, ST]$, Lemma 7 implies that $\eta \in \ker \rho$. Similarly, $\eta \in \ker \sigma$. Thus it follows that $\eta \in I_\mathbf{r} \cap I_\mathbf{s}$.

Consider the magma homomorphisms $\xi: (M(\mathbf{u}), .) \to (L(\mathbf{u}), [\ ,\ ])$ and $\gamma: (M(\mathbf{u}), .) \to (F, [\ ,\ ])$ induced by the inclusions of $\mathbf{u}$ into $L(\mathbf{u})$ and $F$ respectively. By Lemma 1, there exist $\mu_1, \ldots, \mu_k \in \mathbf{u}_{n-1}$ and $\mathrm{e}_1, \ldots, \mathrm{e}_k \in \mathbb{Z}$ such that each $\mu_j$ involves $\mathbf{r}$ and such that $\eta = \mathrm{e}_1\xi(\mu_1) + \ldots + \mathrm{e}_k(\mu_k)$. Also by Lemma 1, the sum of those $\mathrm{e}_j\xi(\mu_j)$ for which $\mu_j$ involves $\mathbf{s}$ lies in $I_\mathbf{s}$. Since $\eta \in I_\mathbf{s}$, the sum of those $\mathrm{e}_j\xi(\mu_j)$ for which $\mu_j$ does not involve $\mathbf{s}$ lies in $I_\mathbf{s}$, and hence is zero, again by Lemma 1. We may thus assume that each $\mu_j$ involves both $\mathbf{r}$ and $\mathbf{s}$. Note that $\gamma(\mu_j)F_n = \Phi(\xi(\mu_j)) \in \mathrm{Gr}^{n-1} F$ for each $j$. By Lemma 2, each $\gamma(\mu_j)$ lies in $[R, S]$. Since $vF_n = \mathrm{e}_1\Phi(\xi(\mu_1)) + \ldots + \mathrm{e}_k\Phi(\xi(\mu_k)) = \gamma(\mu_1)^{\mathrm{e}_1} \ldots \gamma(\mu_k)^{\mathrm{e}_k}F_n$, we conclude that $v \in [R, S]F_n$. Thus $w = uv \in [R, S]F_n$. This completes the proof of Theorem 1.

**The structure of Gr** $Q$. As in the preceding discussions, $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group $F$, and $Q = F/[R, S]$. It has been noted that the composite epimorphism

$$L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}) \cong \mathrm{Gr}\,F \to \mathrm{Gr}\,Q$$

has kernel contained in $I_\mathbf{r} \cap I_\mathbf{s}$. The reverse inclusion follows from Lemma 1 and 2: $\mathrm{Gr}\,Q \cong L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t})/I_\mathbf{r} \cap I_\mathbf{s}$.

THEOREM 2. *There is a pull-back diagram*

$$\begin{array}{ccc} \operatorname{Gr} Q & \longrightarrow & L(\mathbf{s} \cup \mathbf{t}) \\ \downarrow & & \downarrow \\ L(\mathbf{r} \cup \mathbf{t}) & \longrightarrow & L(\mathbf{t}) \end{array}$$

*in the category of integral Lie algegras.*

Before we give the proof, note that, as a consequence, $\operatorname{Gr} Q$ embeds in the direct product $L(\mathbf{r} \cup \mathbf{t}) \times L(\mathbf{s} \cup \mathbf{t})$, which is torsion-free. For each $n \geq 1$, $Q_n/Q_{n+1} = \operatorname{Gr}^n Q$ is a homomorphic image of the finitely generated $\operatorname{Gr}^n F$.

COROLLARY. *For each positive integer* $n$, $Q_n/Q_{n+1}$ *is a finitely generated free abelian group.*

For the proof of Theorem 2, set $\mathbf{u} = \mathbf{r} \cup \mathbf{s} \cup \mathbf{t}$. Using [**1**, II.2.9, Proposition 10], there is a commutative square

$$\begin{array}{ccc} L(\mathbf{u}) & \xrightarrow{\alpha_2} & L(\mathbf{s} \cup \mathbf{t}) \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ L(\mathbf{r} \cup \mathbf{t}) & \xrightarrow{\beta_1} & L(\mathbf{t}) \end{array}$$

of split surjections of graded Lie algebras. For $k = 1, 2$, let $\alpha_k$ and $\beta_k$ be split by $j_k$ and $i_k$ respectively; these splittings are the obvious inclusions of subalgebras, so $j_1 i_1 = j_2 i_2$ and $\alpha_1 j_2 = i_1 \beta_2$. Let $\Pi$ denote the pull-back of $\beta_1$ and $\beta_2$; we have $\alpha_2 j_1 = i_2 \beta_1$ and

$$\Pi = \{(x, y) \in L(\mathbf{r} \cup \mathbf{t}) \times L(\mathbf{s} \cup \mathbf{t}) : \beta_1(x) = \beta_2(y)\}.$$

Since $\beta_1 \alpha_1 = \beta_2 \alpha_2$, a Lie algebra homomorphism $\alpha = \{\alpha_1, \alpha_2\} : L(\mathbf{u}) \rightarrow \Pi$ is induced. Recall that $\ker \alpha_1 = I_{\mathbf{r}}$ and $\ker \alpha_2 = I_{\mathbf{s}}$. The map $\alpha$ therefore induces a homomorphism of graded Lie algebras $a : L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t})/I_{\mathbf{r}} \cap I_{\mathbf{s}} \rightarrow \Pi$ given by

$$a(u + I_{\mathbf{r}} \cap I_{\mathbf{s}}) = (\alpha_1(u), \alpha_2(u)).$$

On the other hand, we define a function $b : \Pi \rightarrow L(\mathbf{r} \cup \mathbf{s} \cup \mathbf{t})/I_{\mathbf{r}} \cap I_{\mathbf{s}}$ by

$$b(x, y) = j_1(x) + j_2(y - i_2 \beta_2(y)) + I_{\mathbf{r}} \cap I_{\mathbf{s}}.$$

One checks that $a$ and $b$ are inverse functions, completing the proof. ∎

**Concluding remarks.** By Lemma 4 and the Corollary to Theorem 1, the ARWC would be implied by the following:

STRONGER CONJECTURE. If $R$ and $S$ are distinct factors from an efficient normal factorization of a finitely generated free group $F$, and if the factorization has at least three nontrivial factors, then the group $Q = F/[R, S]$ is residually nilpotent (i.e. $Q_\omega = 1$).

The conclusion of the Stronger Conjecture does not hold for efficient normal factorizations involving just two factors.

EXAMPLE. In the free group $F$ with basis $\{x, y\}$, let $R$ be the normal subgroup determined by $[x, y]y^{-1}$, and let $S$ be the normal subgroup determined by $[y, x]x^{-1}$. It is easy to show that $F = RS$: modulo $RS$, $1 \equiv xyx^{-1}y^{-1}y^{-1} \equiv xx^{-2}y^{-1}$, whence $x \equiv y^{-1}$, and

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so, for example, $1 \equiv [x, y]y^{-1} \equiv y^{-1}$. As in the proof of Lemma 5, $F/[R, S] \cong F/S \times F/R$. Now, $F/S$ is isomorphic to the semi-direct product $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}[1/2]$ via multiplication by two. In particular, $(F/S)_2 = (F/S)_\omega = \mathbb{Z}[1/2]$. Thus, neither $F/S$ nor $F/[R, S]$ is residually nilpotent.

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