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SOME SPECTRAL PROPERTIES OF POLAR DECOMPOSITIONS

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1. Introduction and notation. The results in this paper respond to two rather natural questions about a polar decomposition $A = UP$, where U is a unitary matrix and P is positive semidefinite. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. The questions are:
(A) When will $|\lambda_1|, \dots, |\lambda_n|$ be the eigenvalues of P?
(B) When will $\lambda_1/|\lambda_1|, \dots, \lambda_n/|\lambda_n|$ be the eigenvalues of U?
The complete answer to (A) is "if and only if U and P commute." In an important special case the answer to (B) is "if and only if U^2 and P commute."

Since these matters are best couched in terms of two different inertias, we begin with a unifying definition of inertia which views all inertias from a single perspective.

For each square complex matrix A and each complex number z let $m(A, z)$ denote the multiplicity of z as a root of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

For each set X of complex numbers $i(A, X)$, the *inertia of A in X*, is defined by

$$i(A, X) = \sum \{m(A, z) : z \in X\}.$$

For each partition $\mathscr{P}$ of the complex numbers $\mathbf{C}$ the *inertia of A with respect to $\mathscr{P}$*, denoted $I(A, \mathscr{P})$, is the function

$$X \in \mathscr{P} \rightarrow i(A, X).$$

(It follows that ord $A = \sum \{i(A, X) : X \in \mathscr{P}\}$.)

The most commonly considered inertia uses the partition

$$\mathscr{P} = \{ \{\operatorname{Re} z > 0\}, \{\operatorname{Re} z < 0\}, \{\operatorname{Re} z = 0\} \},$$

and the traditional notation (cf. e.g. **[9]**) is

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$$\begin{aligned}\pi(A) &= i(A, \{\text{Re } z > 0\})\\ \nu(A) &= i(A, \{\text{Re } z < 0\})\\ \delta(A) &= i(A, \{\text{Re } z = 0\})\\ \text{In}(A) &= (\pi(A), \nu(A), \delta(A)).\end{aligned}$$

We shall be concerned with the two partitions

$$\mathscr{P}_1 = \{\Gamma_r: 0 \leqq r < \infty\}, \quad \text{where } \Gamma_r = \{\,|z| = r\}, \quad \text{and}$$

$$\mathscr{P}_2 = \{\,\{0\}\,\} \cup \{e^{i\theta}\mathbf{R}_+: 0 < \theta \leqq 2\pi\},$$

$$\text{where } \mathbf{R}_+ = \{0 < x < \infty\}.$$

We shorten the above notations by setting

$$C[A] = I(A, \mathscr{P}_1) \quad \text{and } C_r[A] = i(A, \Gamma_r) \quad \text{for } 0 \leqq r < \infty$$

$$R[A] = I(A, \mathscr{P}_2) \quad \text{and}$$

$$R_\theta[A] = \begin{cases} i(A, \{0\}) \text{ if } \theta = 0 \\ i(A, e^{i\theta}\mathbf{R}_+) \quad \text{if } 0 < \theta \leqq 2\pi. \end{cases}$$

(We have chosen "C" for "circle" and "R" for "ray".) The inertia $R[A]$ has appeared in [3] and [**10**] with different notation.

Questions (A) and (B) respectively can now be posed:

(1.1) When is $C[A] = C[P]$?

(1.2) When is $R[A] = R[U]$?

The answer to (1.1) is if and only if A is normal. This is Theorem 2.1. Since the spectral theorem readily shows that $C[A] = C[P]$ when A is normal, the interesting part is that A not normal implies $C[A] \neq C[P]$. The eigenvalues of P are, by definition, the singular values of A. Singular values have been extensively studied (cf. e.g. [**6**], Chapter 6 of [**1**], and their bibliographies), and our answer to (1.1) is a new result about singular values.

If $R[A] = R[U]$ holds then A must be nonsingular. (Furthermore, if A is singular P is, but U is not, uniquely determined (cf. p. 68 of [**5**])). Our answer to (1.2) appears in Theorems 1.6 and 2.7, and it is complete only under the extra hypothesis that the spectrum of U lies in a closed halfplane whose boundary contains the origin. Although a characterization of those A whose unitary part U has this halfplane property seems to be lacking, some information is known about this class. It is necessary that $\sigma(A)$, the spectrum of A, lie in the same halfplane. (Proof. Let $Q = \sqrt{P}$, the positive semidefinite square root of P. Then

$$\sigma(A) = \sigma(UP) = \sigma(QUQ).$$

Since $V(QUQ)$, the numerical range of QUQ, clearly lies in π, the halfplane containing $\sigma(U)$, we have $\alpha(A) = \sigma(QUQ) \subset V(QUQ) \subset \pi$). It is sufficient that A be nonsingular and $V(A)$ lie in such a halfplane. (Proof. Set $X = A$ and $Y = P$ in the lemma of **[11]** and **[12]** which says

$$\sigma(XY^{-1}) \subset V(X)/V(Y) \text{ if } 0 \notin V(Y).$$

This argument is used in **[12]** to give a new proof of a very slightly different result due to S. K. Berberian.) These results suggest conjecturing that if $\sigma(A) \subset \pi$, a halfplane with the origin in its boundary, then $\sigma(U) \subset \pi$. However, the polar decomposition $A = UP$ where

$$A = \sqrt{2}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\sigma(A) = \{\sqrt{2}\}, \sigma(U) = \left\{\frac{1 \pm i}{\sqrt{2}}\right\}$$

can be used to construct a counterexample. Consider the polar decomposition:

$$\alpha A \oplus \bar{\alpha} A = (\alpha U \oplus \bar{\alpha} U)(P \oplus P)$$

where $\alpha = e^{i\theta}$ and $\pi < 4\theta < 3\pi$.

Here

$$\sigma(\alpha A \oplus \bar{\alpha} A) = \{\sqrt{2}\alpha, \sqrt{2}\bar{\alpha}\} \subset \{\text{Re } z > 0\},$$

but

$$\sigma(\alpha U \oplus \bar{\alpha} U) = \exp\left\{\pm i\theta \pm \frac{i\pi}{4}\right\}$$

lies in no halfplane.

The complete answer to (1.2) when $\sigma(U)$ is restricted to a halfplane with the origin on its boundary is "if and only if P and U^2 commute". But it is not the answer in general, as the examples given in Section 3 show. In fact, the slightest relaxation of the restriction on $\sigma(U)$ permits the construction of an $A = UP$ satisfying $R[A] = R[U]$ for which P and U^2 do not commute (cf. Example 3.2).

When $n = 3$ it is possible to characterize the $A = UP$ which satisfy $R[A] = R[U]$ in terms of algebraic restrictions which the entries of P and U must satisfy (cf. Example 3.3), and thus completely answer (1.2) when $n = 3$. The algebraic conditions involved are complicated enough that no clear view of the full answer to (1.2) for $n = 3$ has emerged. On the contrary, the variety of examples we give suggests that (1.2) has a complicated answer for all $n > 2$.

The results of **[6]** and **[7]** are related to this paper. They answer the following question completely: Given complex numbers $\lambda_1, \ldots, \lambda_n$, which spectra Spec($U$) and Spec($P$) occur as U, P vary over all polar decompositions UP of all $n \times n$ matrices having spectrum $\lambda_1, \ldots, \lambda_n$?

2. The main theorems. Let $M_n(\mathbf{C})$ denote the set of $n \times n$ complex matrices. If $M \in M_n(\mathbf{C})$ then $\sigma(M)$ denotes its spectrum,

$$V(M) = \{x^*Mx: x \in \mathbf{C}^n \text{ is a unit vector}\}$$

is its numerical range,

$$v(M) = \max\{\, |z| : z \in V(M)\,\}$$

is its numerical radius, and

$$\mathbf{R}_+V(M) = \{rz: r \in \mathbf{R}_+, z \in V(M)\,\}$$

is its angular numerical range. If M is positive semidefinite its positive semidefinite square root is denoted $\sqrt{M}$. By $M_1 \oplus \ldots \oplus M_k$ we mean the block diagonal matrix

$$\operatorname{diag}(M_1, \ldots, M_k).$$

The identity matrix in $M_n(\mathbf{C})$ is I_n.

We shall need the following well known theorem.

THEOREM 2.0. *Let $M \in M_n(\mathbf{C})$ and suppose that $\lambda \in \sigma(M)$ lies in the boundary of $V(M)$. Then there exists a unitary $W \in M_n(\mathbf{C})$ such that*

$$W^*MW = \lambda I_m \oplus L$$

where $m = m(M, \lambda)$ and $L \in M_{n-m}(\mathbf{C})$.

Proof. This is just a special case of the general theorems in Section 20 of **[2]**. For a more elementary proof, observe that, since M may be replaced by $uM + vI$ with $u, v \in \mathbf{C}$ and $|u| = 1$, there is no loss in assuming that

$$\lambda = 0 \quad \text{and} \quad V(M) \subset \{\operatorname{Re} z \geqq 0\}.$$

But then Re M is positive semidefinite and Theorem 2 on page 80 of **[9]** gives the desired result.

In the introduction we discussed polar decompositions $A = UP$, but here the symbol A disappears, and our theorems just discuss the product UP of a unitary U with a positive semidefinite P. We begin by answering question (1.1) with a theorem which is also a result about the singular values of UP. Roughly speaking, it says that UP is normal if and only if its singular values are just the absolute values of its eigenvalues. This equivalence can be stated more succinctly in terms of the circle inertia $C[\,\cdot\,]$, as we do in the following theorem where it appears as the equivalence of (2.2) and (2.3).

THEOREM (2.1). *Let U, $P \in M_n(\mathbf{C})$ be unitary and positive semidefinite respectively. Then the following are equivalent.*

(2.2) $C[UP] = C[P]$.
(2.3) *UP is normal.*
(2.4) *U and P commute.*
(2.5) *There is a unitary similarity which diagonalizes both U and P.*

Proof. The equivalence of (2.3), (2.4), and (2.5) is well known (use the spectral theorem and cf. p. 935 of **[4]**), and (2.5) readily implies (2.2).

It now suffices to show that (2.2) implies (2.3). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of UP. This means that if λ is an eigenvalue of UP, it occurs $m(UP, \lambda)$ times in the list $\lambda_1, \ldots, \lambda_n$. Then by hypothesis the eigenvalues of P are $|\lambda_1|, \ldots, |\lambda_n|$, and in this listing each eigenvalue p of P occurs $m(P, p)$ times. Hence

$$\operatorname{trace}(\,(UP)^*(UP)\,) = \operatorname{trace}(P^2) = \sum_{i=1}^{n} |\lambda_i|^2.$$

Since the lefthand member of this equation is the euclidean norm of UP, Schur's theorem (cf. p. 229 of **[8]**) implies that UP is normal.

The idea of using Schur's theorem is Charles R. Johnson's. Our original proof showed that (2.2) $\Rightarrow$ (2.4) and was analogous to that of Theorem (2.7). It proceeded by showing that UP has eigenvalues of modulus equal to its numerical radius and by using Theorem (2.0) to split them off as a diagonal direct summand. This reduced the question to considering the other direct summand, and induction finished the matter.

It is now clear that if both $C[A] = C[P]$ and $R[A] = R[U]$ hold, then A must be invertible (because $R[A] = R[U]$) and normal. The converse is true also, as the spectral theorem shows.

Now we take up question (1.2). Here is a sufficient condition for $R[A] = R[U]$ where UP is a polar decomposition of A.

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THEOREM 2.6. *Let U, $P \in M_n(\mathbf{C})$ with U unitary and P positive definite. If U^2 and P commute then $R[UP] = R[U]$.*

Proof. Set $Q = \sqrt{P}$. Then QUQ is normal because:

$$(QUQ)^*QUQ = QU^*PU^2U^*Q = QU^*U^2PU^*Q$$
$$= QUQ(QUQ)^*.$$

(Similarly, QUQ normal implies that U^2 and P commute, but we won't need this fact until the proof of Theorem 2.7.) Since U and $QUQ = Q^*UQ$ are both normal $R[U] = R[QUQ]$ by Theorem 6.5 of [**3**]. Since UP and QUQ are similar

$$R[QUQ] = R[UP].$$

This theorem and the following one give a complete answer to question (1.2) in the special case that the spectrum of $U = A(\sqrt{A^*A})^{-1}$ lies in a certain kind of halfplane.

THEOREM 2.7. *Let U, $P \in M_n(\mathbf{C})$ with U unitary and P positive definite. Suppose $\sigma(U)$ lies in a halfplane whose boundary L contains the origin. Then $R[UP] = R[U]$ implies U^2 commutes with P.*

Proof. By the remark in the proof of Theorem 2.6 it suffices to show that QUQ, where $Q = \sqrt{P}$, is normal. Let n be the smallest positive integer for which QUQ need not be normal. Since the hypotheses and the conclusion of this theorem are invariant under unitary similarity we assume without loss, that

$$U = \alpha D \oplus E,$$

where

$$D = I_p \oplus (-I_q),$$
$$p = m(U, \alpha), \quad q = m(U, -\alpha),$$

and $\alpha \in \sigma(U)$ satisfies

$$\text{dist}(\alpha, L) = \min\{\text{dist}(\lambda, L): \lambda \in \sigma(U)\}.$$

Then $p > 0$, q may be zero, and $V(E)$, which, since E is unitary, is the convex hull of $\sigma(E)$, misses $\alpha\mathbf{R}$. Since UP and QUQ are similar,

$$R[QUQ] = R[UP] = R[U].$$

Thus

$$i(QUQ, \alpha\mathbf{R}_+) = p \quad \text{and} \quad i(QUQ, -\alpha\mathbf{R}_+) = q.$$

Since $V(QUQ)$ lies in $\mathbf{R}_+V(U)$, it lies between the rays $\alpha\mathbf{R}_+$ and $-\alpha\mathbf{R}_+$, and by Theorem 2.0 there exists an $n \times n$ unitary matrix V and a $(p + q) \times (p + q)$ real diagonal matrix F with $\mathrm{In}(F) = (p, q, 0)$ such that

$$(2.8) \qquad V^*QUQV = \alpha F \oplus G,$$

where $R[G] = R[E]$. (Of course, if $n = p + q$, E and G do not occur and the normality of QUQ is proven. So we may suppose that $p + q < n$). If we partition $S = QV$ into four blocks $(S_{ij})_{i,j = 1,2}$ such that S_{11} is $(p + q) \times (p + q)$, then (2.8) yields

$$\alpha S_{11}^*DS_{11} + S_{21}^*ES_{21} = \alpha F.$$

Thus

$$V(S_{21}^*ES_{21}) \subset \alpha\mathbf{R}.$$

But clearly

$$V(S_{21}^*ES_{21}) \subset \mathbf{R}_+V(E) \cup \{0\},$$

and, since $V(E) \cap \alpha\mathbf{R} = \emptyset$, we know that

$$V(S_{21}^*ES_{21}) = \{0\}.$$

Hence $S_{21} = 0$ (for note: if $x = S_{21}y \neq 0$ and $\|y\| = 1$ then,

$$0 = \|x\|^{-2}x^*Ex \in V(E),$$

a contradiction). Then, since S is invertible, S_{11} must be also, and so, to see that $S_{12} = 0$ it suffices to examine

$$0 = \alpha S_{12}^*DS_{11} + S_{22}^*ES_{21},$$

which also comes from the partitioned version of (2.8).

Let Q_iV_i, where Q_i is positive definite and V_i is unitary, be the polar decomposition of S_{ii}. Since S is invertible its polar decomposition is unique; hence

$$Q = Q_1 \oplus Q_2 \quad \text{and} \quad V = V_1 \oplus V_2.$$

From (2.8) we obtain

$$G = V_2^*Q_2EQ_2V_2.$$

It suffices to prove that G is normal because then the normality of QUQ will follow from (2.8) and the fact that F is diagonal. Clearly, G is normal if and only if Q_2EQ_2 is. Now $U_2 = E$ and $P_2 = Q_2^2$ satisfy the hypotheses of this theorem because

$$\sigma(U_2) = \sigma(E) \subset \sigma(U) \quad \text{and}$$

$$R[U_2P_2] = R[G] = R[E] = R[U_2]$$

(for note that U_2P_2 is similar to G). Since U_2, P_2 have order less than n, $Q_2U_2Q_2$ is normal. Hence there does not exist a least n for which the theorem fails.

3. Some examples and remarks. Since the spectrum of every 1×1 or 2×2 unitary matrix lies in a halfplane with the origin on its boundary, Theorems (2.6) and (2.7) answer question (1.2) completely when $n \leqq 2$, i.e., $R[UP] = R[U]$ if and only if U^2 and P commute. However, when $n > 2$ there exist classes of unitary U's and positive definite P's such that $R[UP] = R[U]$ but U^2 and P do not commute. It is actually unnecessary to use $M_n(\mathbf{C})$, there are examples U, P in $M_n(\mathbf{R})$ with U orthogonal and P symmetric.

Example 3.1. Perhaps the most perspicuous example where $R[UP] = R[U]$, but U^2 and P do not commute is: $n > 2$,

$$U = (u_{ij}) \text{ with } u_{ij} = \begin{cases} \delta_{i+1,j} & \text{for } i = 1, \dots, n-1 \\ \delta_{1,j} & \text{for } i = n \end{cases}$$

and P is a positive definite diagonal matrix but not a multiple of I_n. Then

$$\det(U - \lambda I) = (-1)^n(\lambda^n - 1) \quad \text{and}$$

$$\det(UP - \lambda I) = (-1)^n(\lambda^n - \det P).$$

When $n = 3$ it is not hard to see that $R[UP] = R[U]$ will hold for this U and many non-diagonal positive definite P's also.

Example 3.2. Let $\alpha < -1/2$ and $\beta \geqq 0$ satisfy $\alpha^2 + \beta^2 = 1$, and set

$$m = 1 - 4\alpha^2.$$

Then

$$f_\alpha(x) = (8\alpha^5)x^2 + m(m^2 - 12\alpha^4)x + 2\alpha m^2(1 - \alpha^2)$$

will have both roots positive if $\alpha > -1$ is near enough -1, because the roots of $f_{-1}(x)$ are $x = 0, 9/8$ and the product of the roots of $f_\alpha(x)$ is

$$2\alpha m^2(1 - \alpha^2)/(8\alpha^5) > 0 \quad \text{if } -1 < \alpha < -\frac{1}{2}.$$

Let $e = e(\alpha)$ denote the square root of the smaller positive root of $f_\alpha(x)$. set

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}, \quad P = \begin{bmatrix} -2\alpha & e & e \\ e & 1 & 0 \\ e & 0 & 1 \end{bmatrix}.$$

Clearly U is unitary with

$$\sigma(U) = \{1, \alpha \pm i\beta\}.$$

Since e approaches the root $x = 0$ of $f_{-1}(x)$ as α decreases to -1, $\det P = -2\alpha - 2e^2$ is positive for $\alpha > -1$ small enough. If $\det P > 0$ then the two other leading principal minors of P are positive. Hence P is positive definite if $\alpha > -1$ is small enough. The characteristic polynomial of UP is

$$\lambda^3 + (m - 2\alpha e^2)\lambda + 2\alpha + 2e^2 = 0.$$

Set

$$s = \frac{m(\alpha + e^2)}{\alpha(m - 2\alpha e^2)},$$

which will be positive if $\alpha > -1$ is small enough. The roots of the characteristic polynomial are $-2\alpha s$, $s(\alpha \pm i\beta)$, as can be verified by substituting and using

$$m^3(\alpha + e^2)^2 = \alpha^2(m - 2\alpha e^2)^3,$$

which is equivalent to $f_\alpha(e^2) = 0$. Hence $R[UP] = R[U]$, but U^2 and P do not commute if $-1 < \alpha < -1/2$. As α decreases toward -1, $\sigma(U)$ comes arbitrarily near to lying in a halfplane with the origin on its boundary without actually doing so. Thus, if $C \subset \Gamma_1$ is any arc of more than π radians and $n > 2$ there is an $n \times n$ unitary V with $\sigma(V) \subset C$ and an $n \times n$ positive definite Q such that $R[VQ] = R[V]$ but V^2 and Q do not commute. (Let U, P be as above. Set

$$Q = P \oplus I_{n-3} \quad \text{and} \quad V = e^{i\theta}(U \oplus I_{n-3})$$

where $\alpha > -1$ and θ are selected so that $\sigma(V) \subset C$.) It is worth noting that U is real orthogonal and P is real symmetric.

Example 3.3. Let G_n denote the direct product $\mathbf{R}_+ \times \Gamma \times \mathcal{U}_n$, where $\mathbf{R}_+$ and $\Gamma = \Gamma_1$ are the usual multiplicative groups, and $\mathcal{U}_n$ is the group of unitary similarities of $M_n(\mathbf{C})$ with the operation of composition. Let G_n act on the set

$$S_n = \{ (U, P) \in M_n(\mathbf{C}) \colon U \text{ is unitary and } P \text{ is positive definite}\}$$

by

$$g(U, P) = (e^{i\theta} W^* U W, r W^* P W)$$

where

$$g = (r, e^{i\theta}, W^* \otimes W) \in G_n.$$

Clearly $R[UP] = R[U]$ if and only if $R[VQ] = R[V]$ for every (V, Q) in $G_n(U, P)$, the orbit of $(U, P) \in S_n$. Since every orbit in S_3 contains a pair (V, Q) of the form

$$(3.4) \quad \begin{cases} V = \operatorname{diag}(1, \alpha, \beta) \text{ where } |\alpha| = |\beta| = 1. \\ Q = \begin{bmatrix} 1 & c & d \\ c & a & z \\ d & \bar{z} & b \end{bmatrix} \begin{matrix} \text{where } c, d, e \geqq 0, \\ z = e(\xi + i\eta), \ \xi, \eta \in \mathbf{R} \text{ and } \xi^2 + \eta^2 = 1, \end{matrix} \end{cases}$$

the problem of describing

$$\{ (U, P) \in S_3 \colon R[UP] = R[U] \}$$

reduces to that of describing those $(V, Q) \in S_3$ which satisfy $R[VQ] = R[V]$ and have the form (3.4). We shall indicate how a set of algebraic equations and inequalities in the symbols α, β, a, b, c, d, e, can be written down which characterize the set of such pairs (V, Q). Setting the coefficients of the monic polynomial whose roots are r, $s\alpha$, $t\beta$ equal to the corresponding coefficients of the monic characteristic polynomial of VQ yields

$$(3.5) \quad r + s\alpha + t\beta = 1 + a\alpha + b\beta$$

$$(3.6) \quad rs\alpha + rt\beta + st\alpha\beta = (a - c^2)\alpha + (b - d^2)\beta + (ab - e^2)\alpha\beta$$

$$(3.7) \quad rst = ab + 2cde\xi - ad^2 - bc^2 - e^2.$$

The problem is to characterize the $(V, Q) \in S_3$ of the form (3.4) such that these equations admit a solution $r, s, t > 0$. Henceforth for each $w \in \mathbf{C}$ we shall write $w_1 = \operatorname{Re} w$ and $w_2 = \operatorname{Im} w$. Now if

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$$

then the real and imaginary parts of (3.5) can be used to express s and t in terms of r:

$$(3.8)\quad \begin{cases} s = a + \beta_0(r - 1) \text{ where } \beta_0 = -\beta_2/(\alpha_1\beta_2 - \alpha_2\beta_1) \\ t = b + \alpha_0(r - 1) \text{ where } \alpha_0 = \alpha_2/(\alpha_1\beta_2 - \alpha_2\beta_1). \end{cases}$$

(If $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$ then $1, \alpha, \beta$ lie in a halfplane and Theorems 2.6 and 2.7 give a characterization of (V, Q).) When these equations are used to eliminate s and t from (3.6) a quadratic equation for r results:

$$(3.9)\quad Ar^2 + Br + C = 0$$

where

$$A = \beta_0\alpha + \alpha_0\beta + \alpha_0\beta_0\alpha\beta,$$

$$B = \alpha(a + \beta_0)(1 + \beta\alpha_0) + \beta(b - \alpha_0)(1 + \alpha\beta_0),$$

$$C = \alpha(a - \beta_0)\beta(b - \alpha_0) - (a - c^2)\alpha - (b - d^2)\beta - (ab - c^2)\alpha\beta.$$

We consider only the case where the rank of

$$(3.10)\quad M = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix}$$

is 2. (In the other cases extra restrictions on the entries of V and Q arise from the vanishing of minors of M.) In that case we can solve the linear system

$$(3.11)\quad A_ir^2 + B_ir + C_i = 0, \quad i = 1, 2$$

via Cramer's rule and conclude that (3.9) has a positive root r if and only if

$$(3.12)\quad \begin{cases} A_1B_2 - A_2B_1 \neq 0, \quad r = -\dfrac{(A_1C_2 - A_2C_1)}{(A_1B_2 - A_2B_1)} > 0, \text{ and} \\ (B_1C_2 - B_2C_1)(A_1B_2 - A_2B_1) = (A_1C_2 - A_2C_1)^2. \end{cases}$$

(This last equation is a consistency condition stating that r^2 computed by squaring the above formulas for r is the same as r^2 computed by applying Cramer's rule to (3.11).)

We conclude that:

If V, Q satisfy (3.4) and M, given by (3.10), is of rank 2, then ξ can be selected so that Q is positive definite and $R[VQ] = R[Q]$ if and only if (3.12) holds, s and t (given by (3.8)) are positive, $a - c^2 > 0$, and these r, s, t satisfy

$$ab - 2cde - ad^2 - bc^2 - e^2 \leqq rst$$
$$\leqq \mathrm{ab} + 2cde - ad^2 - bc^2 - e^2.$$

Remark (3.13). We actually can eliminate the symbols r, s, t from this characterization by using (3.8) and then (3.12), but that would make the notation more complicated. We have eliminated ξ, and, by solving the last equation in (3.12), another variable could, at least theoretically, be eliminated by expressing it in terms of the others. That would leave algebraic conditions involving six parameters.

Remark (3.14). The last two inequalities above involving rst are designed so $-1 \leqq \xi \leqq 1$ can be chosen so that $rst = \det Q$. Since we require r, s, t to be positive, we get $\det Q > 0$ and together with $a - c^2 > 0$ we see that Q is positive definite.

Remark (3.15). The algebraic conditions given by this characterization are complicated enough to raise doubts about how useful they are. For example, they do not seem to give much insight into (1.2).

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