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SEMIGLOBAL EXTENSION OF MAXIMALLY COMPLEX SUBMANIFOLDS

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Abstract

Let A be a domain of the boundary of a (weakly) pseudoconvex domain Ω of $\mathbb{C}^n$ and M a smooth, closed, maximally complex submanifold of A. We find a subdomain E of $\mathbb{C}^n$, depending only on Ω and A, and a complex variety $W \subset E$ such that $bW = M$ in E. Moreover, a generalization to analytic sets of depth at least 4 is given.

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1. Introduction

In the last fifty years, the *boundary problem*, that is the problem of characterizing real submanifolds which are boundaries of 'something' analytic, has been widely treated.

The first result of this kind is due to Wermer [20]: compact real curves in $\mathbb{C}^n$ are boundaries of complex varieties if and only if they satisfy a global integral condition, the *moments condition*. For higher dimension the problem was solved, by Harvey and Lawson [9], proving that an obviously necessary condition (*maximal complexity*) is also sufficient for compact manifolds in $\mathbb{C}^n$. Later on, characterizations for closed (not necessarily compact) submanifolds in q-concave open subsets of $\mathbb{CP}^n$ were provided by Dolbeault and Henkin and by Dihn in [5–7]. A new approach to the problem in $\mathbb{CP}^n$ has been recently set forth by Harvey and Lawson [11–14].

Our goal is to drop the compactness hypothesis. The results in [4] deal with the global situation of submanifolds contained in the boundary of a special class of strongly pseudoconvex unbounded domains in $\mathbb{C}^n$. In this paper we consider the boundary problem for complex analytic varieties in a 'semiglobal' setting.

The boundary problem has been considered in this context by Chirka [3], who proved the following 'relative' Harvey–Lawson result.

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THEOREM [3]. *Let K be a compact, polynomially convex subset of $\mathbb{C}^n$ and let $M \subset \mathbb{C}^n \setminus K$ be a maximally complex submanifold of real dimension $2p-1$ such that $M \cup K$ is compact. Then there is a p-dimensional analytic subset A in $\mathbb{C}^n \setminus (K \cup M)$ such that $A \cup K \cup M$ is compact, $M \subset \overline{A}$ and A is a regular submanifold with boundary M outside a set $E \subset \overline{A} \setminus K$ of $(2p-1)$-Hausdorff measure* 0.

In our situation, we let $\Omega \subset \mathbb{C}^n$ be a smooth, (weakly) pseudoconvex open domain in $\mathbb{C}^n$ with boundary $b\Omega$, and we let M be a smooth, maximally complex $(2m+1)$-dimensional real closed submanifold $(m \geq 1)$ of some open domain $A \subset b\Omega$ with $K = b\Omega \setminus A$. We want to find a domain E of $\mathbb{C}^n$, not depending on M, and a complex subvariety W of E such that $bW = M$ in E (possibly in the sense of currents).

In the paper we use a parameter version of Harvey and Lawson's theorem to construct a solution (E, W) to the problem above, where E can be characterized, roughly, in terms of the envelope of K with respect to the algebra of functions holomorphic in a neighborhood of $\overline{\Omega}$. We refer to Section 3 for a precise statement of the results, which in some ways echoes that of Lupacciolu on the extension of CR-functions (see [16, Theorem 2]).

If A is not relatively compact, our theorem can be restated in terms of 'principal divisors hull', leading to a global result for unbounded strictly pseudoconvex domains, different from the results in [4]. Indeed, this method of proof allows us to drop the Lupacciolu hypothesis in [4] and extend the maximally complex submanifold to a domain, which can anyhow not be the whole of Ω. If the Lupacciolu hypothesis holds, then the domain of extension is in fact all of Ω. So this result is actually a generalization of the one in [4].

The crucial question of the maximality of the domain E we construct is not answered; in some simple cases the domain is indeed maximal (see Example 4.1).

In the last section, by the same methods, the extension result is proved for analytic sets (see Theorem 5.1).

It is worth noticing that in [18] related results are obtained via a bump lemma and cohomological methods. That approach may be generalized to complex spaces.

2. Definitions and notations

In this paper we will always consider, unless otherwise stated, $\mathbb{C}^n$ with coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$.

A smooth real $(2m+1)$-dimensional submanifold M of $\mathbb{C}^n$ is said to be a *CR manifold* if its complex tangent $T_p^{\mathbb{C}}M$ has constant dimension at each point p. If $m > 0$ and $\dim_{\mathbb{C}} T_p^{\mathbb{C}}M = m$, that is, it is the maximal possible, M is said to be *maximally complex*. Observe that a smooth hypersurface of $\mathbb{C}^n$ is always maximally complex.

If $m = 0$ and $M = \gamma$ is a compact curve, we say that γ satisfies the *moments condition* if

$$\int_\gamma \omega = 0,$$

for any holomorphic $(1,0)$-form ω.

It is easy to observe that the (smooth) boundary of a complex variety of $\mathbb{C}^n$ of dimension $m+1$ is maximally complex if $m>0$ (respectively satisfies the moments condition if $m=0$).

3. Main result

Let $\Omega\subset\mathbb{C}^n$ be a (weakly) pseudoconvex open domain in $\mathbb{C}^n$. Let A be a relatively compact subdomain of $b\Omega$, and $K=b\Omega\setminus A$. We set $\widehat{K}$ to be the hull of K with respect to the algebra $\mathcal{O}(\overline{\Omega})$ of the functions holomorphic in a neighborhood of $\overline{\Omega}$:

$$\widehat{K}=\{x\in\Omega : |f(x)|\le\|f\|_K,\ \forall f\in\mathcal{O}(\overline{\Omega})\}.$$

(From now on, when the algebra is not specified, we always assume that the hull is with respect to $\mathcal{O}(\overline{\Omega})$.) Observe that, when Ω is bounded and strongly pseudoconvex (and thus admits a fundamental system of Stein neighborhoods, see [19]), $\widehat{K}$ coincides with the intersection of the hulls $\widehat{K}_\alpha$ of K with respect to $\mathcal{O}(\Omega_\alpha)$, where Ω_α is any fundamental system of Stein neighborhoods. Moreover, in this case we have the following result by Alexander and Stout [1].

Theorem [1]. *If K is a compact subset of $b\Omega$, then for any connected component U of $b\Omega\setminus K$ there is a (unique) component $\widetilde{U}$ of $\Omega\setminus\widehat{K}$ such that $b\widetilde{U}\cap b\Omega$ coincides with U.*

The following is our result in the case when Ω is strongly pseudoconvex; below (see Theorem 3.4), we give the statement under the weaker hypothesis of Ω being just pseudoconvex. The reason for the separation is that in the strongly pseudoconvex case several improvements are possible with respect to Theorem 3.4; moreover, in order to state the latter we will need a refinement of Alexander and Stout's result.

Theorem 3.1. *Let Ω be strongly pseudoconvex, $A\subset b\Omega$ a domain with boundary $bA=K$ and $\widetilde{A}$ the component of $\Omega\setminus\widehat{K}$ corresponding to A. Then for any maximally complex $(2m+1)$-dimensional closed real submanifold M of A, $m\ge 1$, there exists an $(m+1)$-dimensional complex variety W in $\widetilde{A}$, with isolated singularities, such that $bW\cap(A\setminus\widehat{K})=M\cap(A\setminus\widehat{K})$.*

We remark, however, that Theorem 3.1 can be more easily proved as a consequence of Chirka's relative Harvey–Lawson result [3], since Ω in this case admits a Stein basis. The fact that the singularities of W are isolated depends on the regularity of W in a neighborhood of M as shown in the following lemma, which is an easy consequence of Lewy's extension result for CR functions (see for example [4] for a proof).

Lemma 3.2. *There exist a tubular neighborhood I of A and an $(m+1)$-dimensional complex submanifold with boundary $W_I\subset\overline{\Omega}\cap I$ such that $A\cap bW_I=M$.*

For Theorem 3.4 we are going to drop the strongly pseudoconvexity assumption, and in this case the Alexander–Stout result does not hold as it is stated; hence we shall premise some considerations and prove an analogous lemma for the weakly pseudoconvex case.

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If K' is a compact subset of $\overline{\Omega}$, we say that K' is $\mathcal{O}(\overline{\Omega})$-convex if it coincides with its hull $\widehat{K'}$ with respect to the algebra $\mathcal{O}(\overline{\Omega})$: in particular, for any $K \subset b\Omega$ we have that $\widehat{K}$ is $\mathcal{O}(\overline{\Omega})$-convex.

Let $\widehat{K}$ be a $\mathcal{O}(\overline{\Omega})$-convex compact subset of $\overline{\Omega}$, with $K = \widehat{K} \cap b\Omega$ (from now on we will always assume this hypothesis on K). The following lemma, which is a direct consequence of the result by Alexander and Stout [1], shows that there is a correspondence between the connected components of $b\Omega \setminus K$ and those of $\Omega \setminus \widehat{K}$.

Lemma 3.3. *For any connected component of A of $b\Omega \setminus K$ there exists a connected component $\widetilde{A}$ of $\Omega \setminus \widehat{K}$ such that $b\widetilde{A} \cap b\Omega = A$.*

Proof. Choose any pair A, B of (distinct) connected components of $b\Omega \setminus K$. We are going to show that $\overline{\Omega} \setminus \widehat{K}$ disconnects A from B, that is, there is no continuous arc γ in $\overline{\Omega} \setminus \widehat{K}$ connecting a point of A to one of B (this implies the thesis). Fix a tubular neighborhood $\mathcal{U}$ of $b\Omega$ in Ω, such that the projection $\pi : \mathcal{U} \to b\Omega$ realizing the distance (that is with the property $d(p, \pi(p)) = d(p, b\Omega)$ for all $p \in \mathcal{U}$) is well defined. For a small $\varepsilon > 0$, we let K_ε be the intersection of $\pi^{-1}(K)$ with a closed ε-neighborhood of K in Ω. If Ω_n is an exhaustion of Ω by smooth, strictly pseudoconvex subdomains, then, for n large enough, the compact set $K^n_\varepsilon = b\Omega_n \cap K_\varepsilon$ is nonempty and $b\Omega_n \setminus K^n_\varepsilon$ contains two connected components A_n and B_n corresponding to A and B. Define a compact subset $H^n_\varepsilon \subset \Omega_n$ as $H^n_\varepsilon = \widehat{(K^n_\varepsilon)}_{\mathcal{O}(\overline{\Omega}_n)}$; by Alexander and Stout's theorem [1] we have that A_n is disconnected from B_n in $\overline{\Omega}_n \setminus H^n_\varepsilon$. Since $\mathcal{O}(\overline{\Omega})|_{\overline{\Omega}_n} \subset \mathcal{O}(\overline{\Omega}_n)$, we also have that $H^n_\varepsilon \subset L^n_\varepsilon = \widehat{K}^n_\varepsilon \cap \Omega_n$. If now we let $S^n_\varepsilon = K_\varepsilon \cap (\Omega \setminus \Omega_n)$, we clearly have that $\widehat{K}_\varepsilon$ contains $S^n_\varepsilon \cup L^n_\varepsilon$ and thus $S^n_\varepsilon \cup H^n_\varepsilon$. The latter set disconnects A from B, hence the same holds true for $\widehat{K}_\varepsilon$.

If now we let $\varepsilon \to 0$, the K_ε are a decreasing sequence of compact sets whose intersection is K, hence the hulls $\widehat{K}_\varepsilon$ are a sequence of compact sets, each one disconnecting A from B, decreasing to $\widehat{K}$. If there existed an arc $\gamma \subset \overline{\Omega} \setminus \widehat{K}$, connecting A to B, then $\gamma \cap \widehat{K}_\varepsilon$ would be a sequence of nonempty compact subsets of γ decreasing to $\emptyset$, which is a contradiction. □

Now, let us get back to the weakly pseudoconvex setting. Let $\widehat{K}_P \subset \mathbb{C}^n$ be the polynomial hull of K, and let $E = (\mathbb{C}^n \setminus \widehat{K}_P) \cup \widetilde{A}$; then E is an open domain containing A. With this definition, our main result can be stated as follows.

Theorem 3.4. *Let $\Omega \subset \mathbb{C}^n$ be a (weakly) pseudoconvex open domain in $\mathbb{C}^n$, and let $\widehat{K}$ be a $\mathcal{O}(\overline{\Omega})$-convex compact subset of $\overline{\Omega}$, with $K = \widehat{K} \cap b\Omega$. Let A be a relatively compact connected component of $b\Omega \setminus K$. For any maximally complex $(2m + 1)$-dimensional closed real submanifold M of A, $m \geq 2$, there exists an $(m + 1)$-dimensional complex variety $W \subset E$ such that $bW = M$ (where the boundary is taken in E and in the sense of currents).*

In the more general situation of Theorem 3.4, Lemma 3.2 may not hold, and for this reason there is no statement about singularities corresponding to the one in Theorem 3.1. In order to obtain the extension, however, we follow the same general

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method as in [4] and we slice M with suitable families of complex manifolds. The main differences with respect to [4] are due to the fact that we have to cut Ω with level-sets of holomorphic functions instead of hyperplanes. This creates some additional difficulties: first of all it is no longer possible to use the parameter which defines the level-sets as a coordinate; secondly, when $m = 1$, the intersections between tubular domains (see Lemmas 3.11, 3.14 and 3.15) may not be connected.

In order to apply the slicing technique we will need the following lemma.

Lemma 3.5. *Let $z^0 \in \Omega \setminus \widehat{K}$. Then there exist an open neighborhood $\Omega' \supset \overline{\Omega}$ and $f \in \mathcal{O}(\Omega')$ such that the following conditions hold.*

(1) $f(z^0) = 0$.
(2) $\{f = 0\}$ *is a regular complex hypersurface of* $\Omega' \setminus \widehat{K}$.
(3) $\{f = 0\}$ *intersects M transversally in a compact manifold.*

Remark 3.6. If f is such a function for z^0, for any point z' sufficiently near to z^0, $f(z) - f(z')$ satisfies conditions (1), (2) and (3) for z'.

Proof. By the definition of $\widehat{K}$, since $z^0 \in \Omega \setminus \widehat{K}$ there is a neighborhood Ω' such that $z^0 \notin \widehat{K}_{\mathcal{O}(\Omega')}$. Hence we can find a holomorphic function g in Ω' such that $g(z^0) = 1$ and $\|g\|_K < 1$; $h(z) = g(z) - 1$ is a holomorphic function whose zero set does not intersect $\widehat{K}$. Since regular level sets are dense, by choosing a suitable small vector v and redefining h as $h(z + v) - h(z^0 + v)$ we can safely assume that h satisfies both (1) and (2). Moreover, $\{h = 0\} \cap b\Omega \Subset b\Omega \setminus K$, showing compactness. Then we may suppose that M is not contained in $\{z_1 = z_1^0\}$ and, for ε small enough, we consider the function $f(z) = h(z) + \varepsilon(z_1 - z_1^0)$. It is not difficult to see (by applying Sard's lemma) that (3) holds for generic ε. □

Now, in the following subsection we deal with the proofs of Theorems 3.4 and 3.1 when $m \geq 2$; later on we treat the case $m = 1$. This is due to the fact that in the latter case proving that we can apply Harvey–Lawson to $\{f = 0\} \cap M$ is not automatic.

3.1. Dimension of M greater than or equal to five: $m \geq 2$. First of all, note that Chirka's theorem [3] already provides a complex variety W defined on $\mathbb{C}^n \setminus \widehat{K}_P$ such that $bW = M$, hence from now on we focus on the extension to $\widetilde{A}$.

For any $z^0 \in \widetilde{A}$, Lemma 3.5 provides a holomorphic function such that the level-set $f_0 = \{f = 0\}$ contains z^0 and intersects M transversally in a compact manifold M_0. The intersection is again maximally complex (it is the intersection of a complex manifold and a maximally complex manifold, see [9]), so we can apply the Harvey–Lawson theorem to obtain a holomorphic chain $W_0 \subset \mathbb{C}^n$ such that $bW_0 = M_0$. For τ in a small neighborhood U of 0 in $\mathbb{C}$, the hypersurface $f_\tau = \{f - \tau = 0\}$ intersects M transversally along a compact submanifold M_τ which, again by the Harvey–Lawson theorem, bounds a holomorphic chain W_τ.

We claim the following proposition holds.

Proposition 3.7. *The union $W_U = \bigcup_{\tau \in U} W_\tau$ is a complex subvariety of an open set $\widetilde{U} \subset E$ such that $\widetilde{A} \cap \widetilde{U} = \widetilde{A} \cap \bigcup_{\tau \in U} f_\tau$.*

We need some intermediate results. Let us consider a generic projection $\pi : \widetilde{U} \to \mathbb{C}^m$ and set $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$, with holomorphic coordinates (w', w), $w' \in \mathbb{C}^m$, $w = (w_1, \dots, w_{n-m}) \in \mathbb{C}^{n-m}$. Let $V_\tau = \mathbb{C}^m \setminus \pi(M_\tau)$.

For $\tau \in U$, $w' \in \mathbb{C}^m \setminus \pi(M_\tau)$ and $\alpha \in \mathbb{N}^{n-m}$, we define

$$I^\alpha(w', \tau) \doteqdot \int_{(\eta',\eta)\in M_\tau} \eta^\alpha \omega_{BM}(\eta' - w'),$$

where ω_{BM} is the Bochner–Martinelli kernel.

The following lemma is proved in [4] in a situation where M is contained in the boundary of a strictly pseudoconvex domain, and where thus Lemma 3.2 implies that each W_τ is a regular manifold in a neighborhood of M_τ. With our hypotheses the proof in [4] essentially carries over, but for the convenience of the reader we present it here with the appropriate modifications.

Lemma 3.8. *Let $F(w', \tau)$ be the multiple-valued function which represents W_τ on $\mathbb{C}^m \setminus \pi(M_\tau)$ and denote by $P^\alpha(F(w', \tau))$ the sum of the αth powers of the values of $F(w', \tau)$. Then*

$$P^\alpha(F(w', \tau)) = I^\alpha(w', \tau).$$

In particular, the cardinality $P^0(F(w', \tau))$ of $F(w', \tau)$ is finite.

Proof. Let V_0 be the unbounded component of V_τ; since the Harvey–Lawson solution W_τ is compact, one has $P^\alpha(F(w', \tau)) = 0$ for $w' \in V_0$. We follow [9] to show that also $I^\alpha(F(w', \tau)) = 0$ on V_0. Indeed, if w' is far enough from $\pi(M_\tau)$, then $\beta = \eta^\alpha \omega_{BM}(\eta' - w')$ is a regular $(m, m-1)$-form on some ball B_R of $\mathbb{C}^n$ containing M_τ. Since β is then $\overline{\partial}$-closed, in B_R there exists a $(m, m-2)$-form γ such that $\overline{\partial}\gamma = \beta$. Thus we can write, in the language of currents,

$$[M_\tau](\beta) = [M_\tau]_{m,m-1}(\overline{\partial}\gamma) = \overline{\partial}[M_\tau]_{m,m-1}(\gamma) = 0;$$

in fact, since M_τ is maximally complex, $[M_\tau] = [M_\tau]_{m,m-1} + [M_\tau]_{m-1,m}$ and $\overline{\partial}[M_\tau]_{m,m-1} = 0$ (see [9]). Moreover, since $[M_\tau](\beta)$ is analytic in the variable w', $[M_\tau](\beta) = 0$ for all $w' \in V_0$.

Now, by the regularity statement of Harvey and Lawson's result we have that, outside a closed set of M_τ with measure zero, $W_\tau \cup M_\tau$ is locally a regular complex manifold with boundary. Hence, for w' belonging to a open, dense subset U of the regular points of $\pi(M_\tau)$, there is a small ball $B_\varepsilon(w') \subset \mathbb{C}^m$ such that $W_\tau \cap \pi^{-1}(B_\varepsilon(w'))$ is a finite union of graphs of holomorphic functions.

To conclude the proof we need to show that the jumps of the functions $P^\alpha(F(w', \tau))$ and $I^\alpha(w', \tau)$ across the common boundary of two components of V_τ are the same. By density, it is sufficient to check this conditions along the set U. Then, let $z' \in \pi(M_\tau) \cap U$ be a regular point in the common boundary of two components V_1 and V_2. As observed in the previous paragraph, we can (locally in a neighborhood of z') write W_τ as a finite union of graphs of holomorphic functions, whose boundaries M_τ^i are either in M_τ or empty. In the first case, the M_τ^i are *CR* graphs over $\pi(M_\tau)$ in the neighborhood of z'. We may thus consider the jump j_i of $P^\alpha(F(w',\tau))$ due to a single function. We remark that the jump for a function f is $j_i = f(z')^\alpha$, and the total jump will be the sum of them.

To deal with the jump of $I^\alpha(w',\tau)$ across z', we split the integration set in the sets M^i_τ (thus obtaining the integrals I^α_i) and $M_\tau \setminus \bigcup_i M^i_\tau (I^\alpha_0)$. Because of the Plemelj formulas for hypersurfaces (see [9]) the jumps of I^α_i are precisely j_i. Moreover, since the form $\eta^\alpha \omega_{BM}(\eta' - z')$ is C^∞ in a neighborhood of $M_\tau \setminus \bigcup_i M^i_\tau$, the jump of I^α_0 is 0. So $P^\alpha(F(w',k)) = I^\alpha(w',k)$. □

Remark 3.9. Lemma 3.8 implies, in particular, that all the functions $P^\alpha(F(w',\tau))$ are continuous in τ. Indeed, they are represented as integrals of a fixed form over a submanifold M_τ which varies continuously with the parameter τ.

Lemma 3.10. *$P^\alpha(F(w',\tau))$ is holomorphic in the variable $\tau \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m}$.*

Proof. Let us fix a point $(w',\underline{\tau})$ such that $w' \notin M_{\underline{\tau}}$ (this condition remains true for $\tau \in B_\epsilon(\underline{\tau})$, for $\epsilon > 0$ small enough). Consider as domain of $P^\alpha(F)$ the set $\{w'\} \times B_\epsilon(\underline{\tau})$. In view of Morera's theorem, we need to prove that for any simple curve $\gamma \subset B_\epsilon(\underline{\tau})$,

$$\int_\gamma P^\alpha(F(w',\tau))\, d\tau = 0.$$

Let $\Gamma \subset B_\epsilon(\underline{\tau})$ be an open set such that $b\Gamma = \gamma$. By $\gamma * M_\tau$ $(\Gamma * M_\tau)$ we mean the union of M_τ along γ (along Γ). Note that these sets are submanifolds of $\mathbb{C} \times \mathbb{C}^n$. The projection $\pi : \Gamma * M_\tau \to \mathbb{C}^n$ on the second factor is injective and $\pi(\Gamma * M_\tau)$ is an open subset of M bounded by $\pi(b\Gamma * M_\tau) = \pi(\gamma * M_\tau)$. By Lemma 3.8 and Stokes' theorem

$$\begin{aligned}
\int_\gamma P^\alpha(F(w',\tau))\, d\tau &= \int_\gamma I^\alpha(w',\tau)\, d\tau \\
&= \int_\gamma \Big(\int_{(\eta',\eta) \in M_\tau} \eta^\alpha \omega_{BM}(\eta' - w') \Big)\, d\tau \\
&= \iint_{\gamma * M_\tau} \eta^\alpha \omega_{BM}(\eta' - w') \wedge d\tau \\
&= \iint_{\Gamma * M_\tau} d(\eta^\alpha \omega_{BM}(\eta' - w') \wedge d\tau) \\
&= \iint_{\Gamma * M_\tau} d\eta^\alpha \wedge \omega_{BM}(\eta' - w') \wedge d\tau \\
&= \iint_{\pi(\Gamma * M_\tau)} d\eta^\alpha \wedge \omega_{BM}(\eta' - w') \wedge \pi_*\, d\tau \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that in $d\eta^\alpha$ appear only holomorphic differentials, η^α being holomorphic. However, since all the holomorphic differentials supported by $\pi(\Gamma * M_\tau) \subset M$ already appear in $\omega_{BM}(\eta' - w') \wedge \pi_*\, d\tau$ (due to the fact that M is maximally complex and supports only $m+1$ holomorphic differentials) the integral is zero. $\square$

Proof of Proposition 3.7. Let us fix a regular point $(w_0', w_0) \in f_{\tau_0} \subset \widetilde{U}$. In a neighborhood of this point $W = W_U$ is a manifold, since the construction depends continuously on the initial data. We want to show that W is indeed analytic in $\widetilde{U}$.

Let us fix $j \in \{1, \ldots, n-m\}$ and consider α of the form $(0, \ldots, 0, \alpha_j, 0, \ldots, 0)$; let P_j^α be the corresponding $P^\alpha(F(w', \tau))$. Observe that for any j we can consider a finite number of P_j^α (it suffices to use $h = \max_j P_j^0(F(w', \tau))$ of them). By a linear combination of the P_j^α with rational coefficients, we obtain the elementary symmetric functions

$$S_j^0(w', k), \ldots, S_j^h(w', \tau)$$

in such a way that for any point $(w', w) \in W$ there exists $\tau \in U$ such that $(w', w) \in W_\tau$; thus, defining

$$Q_j(w', w, \tau) \doteqdot S_j^h(w', \tau) + S_j^{h-1}(w', \tau) w_j + \cdots + S_j^0(w', \tau) w_j^h,$$

we have, in other words,

$$W \subset V = \bigcup_{\tau \in U} \bigcap_{j=1}^{n-m} \{Q_j(w', w, \tau) = 0\}.$$

Define $\widetilde{V} \subset \mathbb{C}^n(w', w) \times \mathbb{C}(\tau)$ as

$$\widetilde{V} = \bigcap_{j=1}^{n-m} \{Q_j(w', w, \tau) = 0\}$$

and

$$\widetilde{W} = W_\tau * U \subset \widetilde{V}.$$

Observe that, since the functions S_j^α are holomorphic, $\widetilde{V}$ is a complex subvariety of $\mathbb{C}^n \times U$. Since $\widetilde{V}$ and $\widetilde{W}$ have the same dimension, in a neighborhood of (w_0', w_0, τ) $\widetilde{W}$ is an open subset of the regular part of $\widetilde{V}$, thus a complex submanifold. We denote by $\mathrm{Reg}(\widetilde{W})$ the set of points $z \in \widetilde{W}$ such that $\widetilde{W} \cap \mathcal{U}$ is a complex submanifold in a neighborhood $\mathcal{U}$ of z. It is easily seen that $\mathrm{Reg}(\widetilde{W})$ is an open and closed subset of $\mathrm{Reg}(\widetilde{V})$, so a connected component. Observing that the closure of a connected component of the regular part of a complex variety is a complex variety we obtain that $\widetilde{W}$ is a complex variety, $\widetilde{W}$ being the closure of $\mathrm{Reg}(\widetilde{W})$ in $\widetilde{V}$.

Finally, since the projection $\pi : \widetilde{W} \to W$ is a homeomorphism and so proper, it follows that W is a complex subvariety as well. $\square$

Now we prove that the varieties $\widetilde{W}_U$ that we have found, which are defined in the open subsets of type $\widetilde{U}$ (see Proposition 3.7), patch together in such a way to define a complex variety on the whole of E, thus completing the proof of Theorem 3.4.

LEMMA 3.11. *Let $\widetilde{U}_f$ and $\widetilde{U}_g$ be two open subsets as in Proposition 3.7 and let W_f and W_g be the corresponding varieties. Let $z^1 \in \widetilde{U}_f \cap \widetilde{U}_g$. Then W_f and W_g coincide in a neighborhood of z^1.*

PROOF. Note that we are only interested in showing that the solutions in $\widetilde{U}_f \cap \widetilde{A}$ and $\widetilde{U}_g \cap \widetilde{A}$ agree, since we already know that the solution exists in $\mathbb{C}^n \setminus \widehat{K}_P$. So, let $\lambda = f(z^1)$ and $\tau = g(z^1)$ and consider

$$L(\lambda', \tau') = \{f = \lambda'\} \cap \{g = \tau'\} \subset \widetilde{A}$$

for (λ', τ') in a neighborhood of (λ, τ). Note that for almost every (λ', τ'), $L(\lambda', \tau')$ is a complex submanifold of codimension two of $\widetilde{U}_f \cap \widetilde{U}_g$. Moreover, $W_f \cap L(\lambda', \tau')$ and $W_g \cap L(\lambda', \tau')$ are both solutions of the Harvey–Lawson problem for $M \cap L(\lambda', \tau')$, consequently they must coincide. Since the complex subvarieties $L(\lambda', \tau')$ which are regular form a dense subset, W_f and W_g coincide on the connected component of $\widetilde{U}_f \cap \widetilde{U}_g$ containing z^1. □

REMARK 3.12. The above proof does not work in the case $m = 1$ since $M \cap L(\lambda', \tau')$ is generically empty.

In order to conclude the proof of Theorem 3.1, observe, first of all, that in this case each W_τ, and hence the whole W, is contained in (the closure of) $\widetilde{A}$: this is a consequence of the existence of a strip (Lemma 3.2) and of the Stein manifold version of the Harvey–Lawson result. Moreover, from [10] it also follows that each W_τ has isolated singularities; now we have to show that the set S of the singular points of the whole W is a discrete subset of $\Omega \setminus \widehat{K}$. Let $z^1 \in \Omega \setminus \widehat{K}$, and choose a function h holomorphic in a neighborhood of $\overline{\Omega}$ such that $h(z^1) = 1$ and $K \subset \{|h| \leq \frac{1}{2}\}$, and consider $f = h - \frac{3}{4}$. Observe that $z^1 \in \{\mathrm{Re}\, f > 0\}$ and $K \subset \{\mathrm{Re}\, f < 0\}$. Choose a defining function φ for $b\Omega$, strongly plurisubharmonic in a neighborhood of Ω and let us consider the family

$$(\phi_\lambda = \lambda\varphi + (1 - \lambda)\mathrm{Re}\, f)_{\lambda \in [0,1]}$$

of strongly plurisubharmonic functions. For λ near 1, $\{\phi_\lambda = 0\}$ does not intersect the singular locus. Let $\overline{\lambda}$ be the largest value of λ for which $\{\phi_\lambda = 0\} \cap S \neq \emptyset$. Then the analytic set S touches the boundary of the Stein domain

$$\{\phi_{\overline{\lambda}} < 0\} \cap \Omega \subset \Omega.$$

Hence $\{\phi_{\overline{\lambda}} = 0\} \cap S$ is a set of isolated points in S. By repeating the same argument, we conclude that S is made up of isolated points, thus completing the proof of Theorem 3.1 for $m \geq 2$. In the following subsection we deal with the case $m = 1$.

3.2. Dimension of M equal to three: $m = 1$. From now on, we are assuming that Ω is strictly pseudoconvex, so that Lemma 3.2 applies. The first goal is to show that when we slice M transversally with complex hypersurfaces, we obtain one-dimensional real submanifolds which satisfy the moments condition.

Again, we fix our attention to a neighborhood of the form

$$\widetilde{U} = \bigcup_{\tau\in U} g_\tau.$$

Let us choose an arbitrary holomorphic (1, 0)-form ω in $\mathbb{C}^n$.

LEMMA 3.13. *The function*

$$\Phi_\omega(\tau) = \int_{M_\tau} \omega$$

is holomorphic in U.

PROOF. Using again Morera's theorem, we need to prove that for any simple curve $\gamma \subset U$, $\gamma = b\Gamma$,

$$\int_\gamma \Phi_\omega(\tau)\, d\tau = 0.$$

By Stokes' theorem,

$$\begin{aligned}
\int_\gamma \Phi_\omega(\tau)\, d\tau &= \int_\gamma\Big(\int_{M_k} \omega\Big)\, d\tau \\
&= \iint_{\gamma * M_\tau} \omega \wedge d\tau \\
&= \iint_{\Gamma * M_\tau} d(\omega \wedge d\tau) \\
&= \iint_{\Gamma * M_\tau} \partial\omega \wedge d\tau \\
&= \iint_{\pi(\Gamma * M_\tau)} \partial\omega \wedge \pi_*\, d\tau \\
&= 0.
\end{aligned}$$

The last equality is due to the fact that $\pi(\Gamma * M_\tau) \subset M$ is maximally complex and thus supports only (2, 1)- and (1, 2)-forms while $\partial\omega \wedge \pi_*\, d\tau$ is a (3, 0)-form. □

LEMMA 3.14. *Let* g *be a holomorphic function on a neighborhood of* $\overline{\Omega}$, *and suppose that* $\{|g| > 1\} \cap \widehat{K} = \emptyset$. *Then there exists a variety* W_g *on* $\Omega \cap \{|g| > 1\}$ *such that* $bW_g \cap b\Omega = M \cap \{|g| > 1\}$.

LEMMA 3.15. *Given two functions* g_1 *and* g_2 *as above, the varieties* W_{g_1} *and* W_{g_2} *agree on* $\{|g_1| > 1\} \cap \{|g_2| > 1\}$.

PROOF OF LEMMA 3.14. We are going to use several times open subsets of the type $\widetilde{U}$ as in Proposition 3.7, so we need to fix some notation. Given an open subset $U \subset \mathbb{C}$, define $\widetilde{U}$ by

$$\widetilde{U} = \bigcup_{\tau\in U} \{f = \tau\}.$$

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From now on we use open subsets of the form $U = B(\overline{\tau}, \delta)$, where $B(\overline{\tau}, \delta)$ is the disc centred at $\overline{\tau}$ of radius δ. We say that $\{f = \overline{\tau}\}$ is the *core* of $\widetilde{U}$ and δ is its *amplitude*.

For a fixed $d > 1$ consider the compact set $H_d = \overline{\Omega} \cap \{|g| \geq d\}$; we show that W_g is well defined on H_d. Let us fix also a compact set $C \subset \Omega$ such that W_I (see Lemma 3.2) is a closed submanifold in $H_d \setminus C$.

Consider all the open subsets $V_\alpha = \widetilde{U}_\alpha \cap \Omega$, constructed using only the function $f = g - 1$ up to addition of the function $\varepsilon(z_j - z_j^0)$ (see Lemma 3.5). If we do not allow ε to be greater than a fixed $\overline{\varepsilon} > 0$, then by a standard argument of semicontinuity and compactness we may suppose that the amplitude of each $\widetilde{U}$ is greater than a positive δ.

We claim that it is possible to find a countable covering of H_d consisting of a countable sequence V_i of those V_α in such a way to have the following conditions.

(1) $V_0 \subset H_d \setminus C$.
(2) If

$$B_l = \bigcup_{i=1}^{l} V_i$$

then $V_{l+1} \cap B_l \cap \Omega \neq \emptyset$.

The only claim we have to prove is the existence of V_0, since the second statement follows by a standard compactness argument.

Set $L = \max_{H_d} \operatorname{Re} g$. Since $\operatorname{Re} g$ is a nonconstant pluriharmonic function, the level set $\{\operatorname{Re} g = L\}$ is a compact subset of $b\Omega \cap H_d$. Then we can choose $\eta > 0$ such that $\{\operatorname{Re} g = L - \eta\} \cap \Omega$ is contained in $H_d \setminus C$, and this allows us to define V_0.

Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be two such open sets and $z^0 \in \widetilde{U}_1 \cap \widetilde{U}_2$. We can suppose that the cores of $\widetilde{U}_1$ and $\widetilde{U}_2$ contain z^0. They are of the form

$$f + \varepsilon_1(z_j - z_j^0) = \tau(\varepsilon_1) \quad \text{and} \quad f + \varepsilon_2(z_j - z_j^0) = \tau(\varepsilon_2).$$

For $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, we consider the open sets $\widetilde{U}_\varepsilon$ whose core, passing by z^0, is given by the equation $f + \varepsilon(z_j - z_j^0) = \tau(\varepsilon)$. We must show that the set

$$\Lambda = \{\varepsilon \in (\varepsilon_1, \varepsilon_2) : \exists W_\varepsilon \text{ such that } W_\varepsilon \cap (\widetilde{U}_1 \cap \widetilde{U}_\varepsilon) = W_1 \cap (\widetilde{U}_1 \cap \widetilde{U}_\varepsilon)\}$$

is open and closed, where W_ε is a variety in $\widetilde{U}_\varepsilon$.

The set Λ is open. Indeed, if $\varepsilon \in \Lambda$, then for ε' in a neighborhood of ε the core of $\widetilde{U}_{\varepsilon'}$ is contained in $\widetilde{U}_\varepsilon$ and so its intersection with M is maximally complex. Because of Lemma 3.13 the condition holds also for all the level sets in $\widetilde{U}_{\varepsilon'}$ and then we can apply again the Harvey–Lawson theorem [9] and the arguments of Proposition 3.7 in order to obtain $W_{\varepsilon'}$. Moreover, there is a connected component of $U_\varepsilon \cap U_{\varepsilon'}$ which contains z^0 and touches the boundary of Ω, where the W_ε and $W_{\varepsilon'}$ both coincide with W_I (see Lemma 3.2). By virtue of the analytic continuation principle, they must coincide in the whole connected component.

The set Λ is closed. Indeed, since each $\widetilde{U}$ has an amplitude of at least δ, we again have that, for $\overline{\varepsilon} \in \overline{\Lambda}$, the intersection of $\widetilde{U}_{\overline{\varepsilon}}$ and $\widetilde{U}_\varepsilon$ must include (for $\varepsilon \in \Lambda$, $|\varepsilon - \overline{\varepsilon}|$ sufficiently small) a connected component containing z^0 and touching the boundary. We then conclude as in the previous case. □

PROOF OF LEMMA 3.15. Let us consider the connected components of $W_{g_1} \cap \{|g_2| > 1\}$. For each connected component W_1 two cases are possible.

(1) W_1 touches the boundary of Ω: $W_1 \cap b\Omega \neq \emptyset$.

(2) The boundary of W_1 is inside Ω:

$$bW_1 \Subset \{|g_1| = 1\} \cup \{|g_2| = 1\} \subset \Omega.$$

In the former case, the result easily follows in view of the analytic continuation principle (remember that on a strip near the boundary W_{g_1} and W_{g_2} coincide).

The latter case is actually impossible. Indeed, suppose on the contrary that the component W_1 satisfies (2). Restrict g_1 and g_2 to W_1 and choose $t > 1$ such that

$$W_t \doteqdot \{|g_i| > t, i = 1, 2\} \Subset W_1.$$

The boundary bW_t of W_t consists of points where either $|g_1| = t$ or $|g_2| = t$. Choose a point z_0 of the boundary where $|g_1| = t$ and $|g_2| > t$. Then $|g_2|$ is a plurisubharmonic function on the analytic set

$$A = \{g_1 = g_1(z_0)\} \cap \{|g_2| \geq t\}.$$

Since $W_t \Subset W_1$, the boundary of the connected component of A through z_0 is contained in $\{|g_2| = t\}$. This is a contradiction, because of the maximum principle for plurisubharmonic functions. □

4. Some remarks

4.1. Maximality of the solution. As stated above, we do not have a complete answer to the problem of the maximality of $\widetilde{A}$. Nevertheless, here is a simple example where the domain constructed is actually maximal.

EXAMPLE 4.1. Let $\Omega \subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $0 \in \Omega$, and let h be a pluriharmonic function defined in a neighborhood U of $\overline{\Omega}$ such that $h(0) = 0$ and $h(z) = h(z_1, \ldots, z_{n-1}, 0)$ (that is h does not depend on z_n). Put

$$H = \{z \in U : h(z) = 0\}$$

and let

$$A = b\Omega \cap \{z \in U : h(z) > 0\}.$$

Then

$$\widetilde{A} = \Omega \cap \{z \in U : h(z) > 0\}.$$

In order to show that $\widetilde{A}$ is maximal for our problem, it suffices to find, for any $z \in H \cap \Omega$, a complex manifold $W_z \subset \widetilde{A}$ such that $M_z = \overline{W_z} \cap A$ is smooth and W_z cannot be extended through any neighborhood of z. We may suppose $z = 0$.

So, let $f \in \mathcal{O}(\overline{\Omega})$ be such that $\operatorname{Re} f = h$, $f(0) = 0$. We define

$$W_0 = \{z \in \widetilde{A} : z_n = e^{1/f(z)}\};$$

W_0 extends as a closed submanifold of $U \setminus \{f = 0\}$. Moreover, observe that each point of $\{f = 0\}$ is a cluster point of W_0. Suppose on the contrary that W_0 extends

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through a neighborhood V of 0 by a complex manifold W'_0. Then $\{f = 0\} \cap V \subset W'_0$, thus $\{f = 0\} \cap V = W'_0 \cap V$. This is a contradiction.

4.2. The unbounded case. Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain, and $A \subset b\Omega$ an unbounded open subset of $b\Omega$.

Consider the set

$$\mathcal{A} = \{A' \Subset b\Omega \mid A' \subset A, A' \textit{ domain}\}.$$

For an arbitrary $A' \in \mathcal{A}$ ($bA' = K'$), let $D_{A'}$ be the compact connected component of $\Omega \setminus \widehat{K'}$. Set

$$D = \bigcup_{A' \in \mathcal{A}} D_{A'}.$$

From Theorem 3.4 it follows that for every maximally complex closed $(2m + 1)$-dimensional real submanifold M of A, there is an $(m + 1)$-dimensional complex closed subvariety W of D, with isolated singularities, such that $bW \cap A = M$. Hence the domain D is a possible solution of our extension problem.

When $A = b\Omega$, we may restate the previous result in a more elegant way. In the same situation as above, consider

$$\mathbb{C}^n \subset \mathbb{CP}^n, \quad \mathbb{C}^n = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}_\infty$$

and define the *principal divisors hull* $\widehat{C}_{\mathcal{D}}$ of $C = \overline{\Omega} \cap \mathbb{CP}^{n-1}_\infty$ by

$$\widehat{C}_{\mathcal{D}} = \{z \in \Omega \mid \forall f \in \mathcal{O}(\overline{\Omega})\ \overline{L}_{f,z} \cap C \neq \emptyset\},$$

where $\overline{L}_{f,z}$ is the closure of the connected component (in $\overline{\Omega}$) of the level-set $\{f = f(z)\}$ passing through z. Then

$$D = \Omega \setminus \widehat{C}_{\mathcal{D}}.$$

Indeed, if $z \in D$, then there exist an open subset $A' \subset b\Omega$ and a function $f \in \mathcal{O}(\overline{\Omega})$ such that $\overline{L}_{f,z} \cap b\Omega$ is a compact submanifold of A'. In particular $z \notin \widehat{C}_{\mathcal{D}}$. Conversely, if $z \notin \widehat{C}_{\mathcal{D}}$ then there is a function $g \in \mathcal{O}(\Omega')$ ($\Omega' \supset \Omega$ domain) such that $N = \overline{L}_{g,z} \cap C = \emptyset$, that is it is a compact submanifold of $b\Omega$. By choosing a relatively compact open subset $A' \subset b\Omega$ large enough to contain N it follows that $z \in D_{A'} \subset D$.

5. Generalization to analytic sets

Let Ω, A and K be as before. We want now to consider the extension problem for analytic sets.

Let us recall that if $\mathcal{F}$ is a coherent sheaf on a domain U in $\mathbb{C}^n$, $x \in U$ and

$$0 \to \mathcal{O}_x^{m_k} \to \cdots \to \mathcal{O}_x^{m_0} \to \mathcal{F}_x \to 0$$

is a resolution of $\mathcal{F}_x$, then the *depth* of $\mathcal{F}$ at the point x is the integer $p(\mathcal{F}_x) = n - k$.

We will say that $M \subset A$ is a *k-deep trace* of an analytic subset if there is:

(i) an open set $U \subset \mathbb{C}^n$ ($U \cap b\Omega = A$);

(ii) an $(m + 1)$-dimensional irreducible analytic set W_M, whose ideal sheaf $\mathcal{I}_{W_M}$ has depth at least k at each point of U, such that $W_M \cap b\Omega = M$.

In this case, we say that the real dimension of M is $2m + 1$.

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THEOREM 5.1. *For any* $(2m+1)$*-dimensional* 4*-deep trace of analytic subset* $M \subset A$*, there exists an* $(m+1)$*-dimensional complex variety* W *in* $\Omega \setminus \widehat{K}$*, such that* $bW \cap (A \setminus \widehat{K}) = M \cap (A \setminus \widehat{K})$*.*

Observe that in this situation we already have a strip U on which the set M extends. So we only need to generalize Lemma 3.5 and the results in Section 3.1.

LEMMA 5.2. *Let* $z^0 \in \Omega \setminus \widehat{K}$*. Then there exist an open Stein neighborhood* $\Omega_\alpha \supset \Omega$ *and* $f \in \mathcal{O}(\Omega_\alpha)$ *such that the following conditions hold.*

(1) $f(z^0) = 0$.
(2) $\{f = 0\}$ *is a regular complex hypersurface of* $\Omega_\alpha \setminus \widehat{K}$*.*
(3) $\{f = 0\}$ *intersects* M *in a compact set and* W_M *in an analytic subset (of depth at least* 3*).*

PROOF. The proof of the first two conditions is exactly the same as before. Hence we focus on the third.

Again in the same way as before, we obtain compactness of the intersection with M. Then we may suppose that W_M is not contained in $\{z_1 = z_1^0\}$ and, for ε small enough, let $f : \Omega_\alpha \to \mathbb{C}$ be the function $f(z) = h(z) + \varepsilon(z_1 - z_1^0)$, where Ω_α and h are as defined in Lemma 3.5. Consider the stratification of W_M in complex manifolds. By Sard's lemma, the set of ε for which the intersection of $\{f(z) = 0\}$ with a fixed stratum is transversal is open and dense. Hence the set of ε for which the intersection of $\{f(z) = 0\}$ with each stratum is transversal is also open and dense; in particular, it is nonempty. The conclusion follows. □

The previous lemma enables us to extend each analytic subset

$$W_0 = W_M \cap \{f = 0\}$$

to an analytic set defined on the whole of

$$\Omega \cap \{f = 0\}.$$

Indeed, on a strictly pseudoconvex corona the depth of W_0 is at least 3 and thus W_0 extends in the hole (see for example [2, 17]). Obviously the extension lies in $\{f = 0\}$.

Observe that, up to a arbitrarily small modification of $b\Omega$, we can suppose that it intersects each stratum of the stratification of W_M transversally. In this situation M is a smooth submanifold with negligible singularities of Hausdorff codimension at least two (see [7]).

Again, we consider a generic projection $\pi : \widetilde{U} \to \mathbb{C}^m$ and we use holomorphic coordinates (w', w), $w = (w_1, \ldots, w_{n-m})$ on

$$\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}.$$

Keeping the notation used in Section 3.1, let $V_\tau = \mathbb{C}^m \setminus \pi(M_\tau)$.

For $\tau \in U$, $w' \in \mathbb{C}^m \setminus \pi(M_\tau)$ and $\alpha \in \mathbb{N}^{n-m}$, we define

$$I^\alpha(w', \tau) \doteqdot \int_{(\eta', \eta) \in \mathrm{Reg}(M_\tau)} \eta^\alpha \omega_{BM}(\eta' - w'),$$

ω_{BM} being the Bochner–Martinelli kernel.

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Observe that the previous integral is well defined and converges. In fact, $W_\tau = W_M \cap \{f = \tau\}$ is an analytic set and thus, by Lelong's theorem, its volume is bounded near the singular locus. Hence, by Fubini's theorem, the regular part of $M_\tau = W_\tau \cap b\Omega$ has finite volume up to a small modification of $b\Omega$.

Lemma 5.3. *Let $F(w', \tau)$ be the multiple-valued function which represents $\widetilde{M}_\tau$ on $\mathbb{C}^m \setminus \pi(M_\tau)$. If we denote by $P^\alpha(F(w', \tau))$ the sum of the αth powers of the values of $F(w', \tau)$, then*

$$P^\alpha(F(w', \tau)) = I^\alpha(w', \tau).$$

In particular, $F(w', \tau)$ is finite.

Proof. Let V_0 be the unbounded component of V_τ (where, of course, the function $P^\alpha(F(w', \tau))$ vanishes). Following [9], it is easy to show that $I^\alpha(F(w', \tau))$ also vanishes on V_0. Indeed, if w' is far enough from $\pi(\mathrm{Reg}(M_\tau))$, then $\beta = \eta^\alpha \omega_{BM}(\eta' - w')$ is a regular $(m, m-1)$-form on some ball B_R of $\mathrm{Reg}(M_\tau)$. Hence, since in B_R there exists γ such that $\overline{\partial}\gamma = \beta$, we may write in the sense of currents

$$[\mathrm{Reg}(M_\tau)](\beta) = [\mathrm{Reg}(M_\tau)]_{m,m-1}(\overline{\partial}\gamma) = \overline{\partial}[\mathrm{Reg}(M_\tau)]_{m,m-1}(\gamma).$$

We claim that $\overline{\partial}[\mathrm{Reg}(M_\tau)]_{m,m-1}(\gamma) = 0$ and, in order to prove this, we first show that $[\mathrm{Reg}(M_\tau)]$ is a closed current. Indeed, observe that $d[\mathrm{Reg}(M_\tau)]$ is a flat current, since it is the differential of an $\mathrm{L}^1_{\mathrm{loc}}$ current (see [8]). Moreover,

$$S = \mathrm{supp}(d[\mathrm{Reg}(M_\tau)]) \subset \mathrm{Sing}(M_\tau),$$

so, denoting by $\dim_{\mathcal{H}}$ the Hausdorff dimension and by $\mathcal{H}_s$ the s-Hausdorff measure,

$$\dim_{\mathcal{H}}(S) \le \dim_{\mathcal{H}}(\mathrm{sing}(M_\tau)) \le \dim_{\mathcal{H}}(\mathrm{Reg}(M_\tau)) - 2$$

and consequently

$$\mathcal{H}_{\dim_{\mathcal{H}}(\mathrm{Reg}(M_\tau))-1}(S) = 0.$$

By Federer's support theorem (see [8]), this implies that

$$d[\mathrm{Reg}(M_\tau)] = 0.$$

Now, since $\mathrm{Reg}(M_\tau)$ is maximally complex,

$$[\mathrm{Reg}(M_\tau)] = [\mathrm{Reg}(M_\tau)]_{m,m-1} + [\mathrm{Reg}(M_\tau)]_{m-1,m}.$$

Since $\overline{\partial}[\mathrm{Reg}(M_\tau)]_{m,m-1}$ is the only component of bidegree $(m, m-2)$ of $d[\mathrm{Reg}(M_\tau)]$ and $d[\mathrm{Reg}(M_\tau)] = 0$, we have

$$\overline{\partial}[\mathrm{Reg}(M_\tau)]_{m,m-1} = 0.$$

Moreover, since $[\mathrm{Reg}\ M_\tau](\beta)$ is analytic in the variable w', $[\mathrm{Reg}\ M_\tau](\beta) = 0$ for all $w' \in V_0$.

The rest of the proof proceeds as in Lemma 3.8. □

Lemma 5.4. *$P^\alpha(F(w', \tau))$ is holomorphic in the variable $\tau \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m-1}$.*

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PROOF. The only difference with the proof for the case of manifolds is the fact that I is an integration over the regular part of $\Gamma * M_\tau$ and not all over $\Gamma * M_\tau$. It is easy to see that Stokes' theorem is valid also in this situation, so the chain of integrals in Lemma 3.10 holds in this case, too. □

The rest of the proof of Theorem 5.1 proceeds as in the proof of Theorem 3.4 (see Section 3.1).

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