INTEGRAL FUNCTIONS WITH NEGATIVE ZEROS

R. P. BOAS, JR.

1. Introduction. If $f(z)$ is an integral function of non-integral order with only real negative zeros, there is a close connection between the rates of growth of the function and of $n(r)$, the number of zeros of absolute value not exceeding r. The best known theorem is that of Valiron [**12**], which may be stated as follows.

THEOREM 1. *If $f(z)$ is an integral function with real negative zeros, of order less than 1, with $f(0) = 1$, the conditions*

$$\log f(r) \sim A\pi \csc \pi\rho\, r^{\rho}, \qquad r \to \infty,\ A > 0, \tag{1.1}$$

and

$$n(r) \sim Ar^{\rho} \tag{1.2}$$

are equivalent.

Either (1.1) or (1.2) implies that $f(z)$ is of order ρ, $0 < \rho < 1$, and from either condition it can be deduced [**1**; **5**] that

$$\log f(re^{i\theta}) \sim \pi A \csc \pi\rho\, e^{i\rho\theta} r^{\rho} \tag{1.3}$$

for $|\theta| < \pi$, uniformly in $|\theta| \leqslant \pi - \delta < \pi$.

When $\rho = \frac{1}{2}$, Theorem 1 implies, after a change of variable, a statement about a canonical product of order 1 with real zeros (not necessarily even).

THEOREM 2. *If $f(z)$ is a canonical product of order 1 with real zeros, the conditions*

$$\log |f(iy)| \sim \pi A |y|, \qquad |y| \to \infty, \tag{1.4}$$

and

$$n(r) \sim 2Ar \tag{1.5}$$

are equivalent.

There is another condition which was shown by Paley and Wiener [**8**, p. 70] to be equivalent to those of Theorem 2.

THEOREM 3. *Under the hypotheses of Theorem 2, if $f(0) = 1$, the condition*

$$\lim_{R\to\infty} \int_{-R}^{R} x^{-2} \log |f(x)|\, dx = -\pi^2 A \tag{1.6}$$

is equivalent to (1.4) *and* (1.5).

In terms of functions of order $\frac{1}{2}$, Theorem 3 becomes the following:

Received April 25, 1952. The author is a fellow of the John Simon Guggenheim Memorial Foundation.

THEOREM 4. *If $f(z)$ is of order $\frac{1}{2}$, all its zeros are real and negative, and $f(0) = 1$, the conditions*

$$(1.7) \qquad \lim_{R\to\infty}\int_0^R x^{-3/2}\log|f(-x)|\,dx = -\pi^2 A,$$

$$\log f(r) \sim A\pi r^{\frac{1}{2}},$$

$$n(r) \sim A r^{\frac{1}{2}}$$

are equivalent.

My object is to investigate what becomes of Theorem 4 for a general order ρ, $0 < \rho < 1$. The result is as follows.

THEOREM 5. *If $f(z)$ is of order less than 1, all its zeros are real and negative, and $f(0) = 1$, the conditions* (1.1) *and* (*for any σ, $0 < \sigma < 1$*)

$$(1.8) \qquad \int_0^r x^{-1-\sigma}\{\log|f(-x)| - \pi\cot\pi\sigma\, n(x)\}\,dx \sim \pi A(\rho-\sigma)^{-1}(\cot\pi\rho - \cot\pi\sigma) r^{\rho-\sigma}$$

are equivalent.

When $\sigma = \rho$, (1.8) is to be interpreted as (1.9), below. The conclusion implies in particular that $f(z)$ is of order ρ. For $\rho = \sigma = \frac{1}{2}$, Theorem 5 reduces to Theorem 4.

It is also true (and can be proved somewhat more simply) that the integral on the left-hand side of (1.8) is $O(r^{\rho-\sigma})$ if and only if $\log f(r) = O(r^\rho)$.

Special cases of (1.8) which are natural generalizations of (1.7) are

$$\int_0^\infty x^{-1-\rho}\{\log|f(-x)| - \pi\cot\pi\rho\, n(x)\}dx = -\pi^2 A\csc^2\pi\rho \qquad (\sigma = \rho),$$

$$(1.9) \qquad \int_0^r x^{-3/2}\log|f(-x)|\,dx \sim \pi A(\rho-\tfrac{1}{2})^{-1}\cot\pi\rho\, r^{\rho-\frac{1}{2}} \qquad (\sigma = \tfrac{1}{2} < \rho),$$

$$\int_r^\infty x^{-3/2}\log|f(-x)|\,dx \sim \pi A(\tfrac{1}{2}-\rho)^{-1}\cot\pi\rho\, r^{\rho-\frac{1}{2}} \qquad (\sigma = \tfrac{1}{2} > \rho).$$

For $\rho \neq \frac{1}{2}$, we see from (1.9) that

$$\int_0^\infty x^{-1-\rho}\log|f(-x)|\,dx$$

converges if and only if

$$\int_0^\infty x^{-1-\rho}n(x)\,dx$$

converges, which is equivalent to $\sum r_n^{-\rho} < \infty$, where $-r_n$ are the zeros of $f(z)$. In this case, of course, $A = 0$.

A consequence of Theorem 5 is that (1.1) implies

$$\int_0^r x^{-1-\rho}\log|f(-x)|\,dx \sim \pi A\cot\pi\rho\log r \qquad (\rho \neq \tfrac{1}{2}),$$

https://doi.org/10.4153/CJM-1953-019-9 Published online by Cambridge University Press

$$\int_0^r x^{-1-\sigma} \log |f(-x)|\, dx \sim \pi A(\rho - \sigma)^{-1} \cot \pi\rho\, r^{\rho-\sigma} \qquad (\rho > \sigma),$$

$$\int_r^\infty x^{-1-\sigma} \log |f(-x)|\, dx \sim \pi A(\sigma - \rho)^{-1} \cot \pi\rho\, r^{\rho-\sigma} \qquad (\rho < \sigma).$$

We may compare these relations with Titchmarsh's result [**10**] that

$$\log |f(-x)| \sim \pi A \cot \pi\rho\, x^\rho$$

in a set of unit linear density; a converse theorem was given by Titchmarsh [**10**] and by Bowen and Macintyre [**2**].

Theorem 3 was proved by Paley and Wiener by using Wiener's general Tauberian theorems; a proof that (1.6) implies (1.4), using methods from the theory of functions, was given by Levinson [**6**, p. 33], but no such proof of the converse appears to have been given previously. The proof of Theorem 5 incidentally contains a new proof of Theorem 3 by function-theory methods.

In Theorem 1 the inference (1.2) implies (1.1) is easy; the converse is more difficult. It was first proved by Valiron [**11**], and later by Titchmarsh [**10**] and by Paley and Wiener [**8**], by Tauberian methods; proofs depending more on the theory of functions have been given by Valiron [**12**], Pfluger [**9**], Levinson [**6**] for $\rho = \frac{1}{2}$, Delange [**4**; **4a**], Bowen [**1**], and Heins [**5**]; the last two are the simplest. For further developments along the lines of Theorem 1 see the papers cited and also Bowen and Macintyre [**2**; **3**] and Noble [**7**].

2. Theorem 5: first part. We begin by proving that (1.8) implies (1.1). Consider the integral

$$I = \int_C r(r - z)^{-1} z^{-1-\sigma} \log f(z)\, dz, \tag{2.1}$$

where C is the contour made up of the circle $|z| = R > r$, with a cut along the negative real axis from $z = -R$ to $z = 0$ and back again; the multiple-valued functions are to be positive for large positive values of z. Initially C has indentations to avoid the zeros of $f(z)$ and the origin, but the contributions of the indentations tend to zero with the diameters of the indentations, and we may disregard them. We also suppose that $-R$ is not one of the zeros of $f(z)$. The integrand is regular except for a pole at $z = r$, and consequently we have

$$I = -2\pi i r^{-\sigma} \log f(r). \tag{2.2}$$

To evaluate the integral along the cut we note that if we take $\arg f(z)$ to be zero for $x > 0$, we have $\arg f(-x) = \pi n(x)$, $x > 0$, on the upper side of the cut, and $\arg f(-x) = -\pi n(x)$ on the lower side. Hence the contribution of the cut is

$$2i \int_0^R r(r + x)^{-1} \phi(x)\, dx,$$

where

$$\phi(x) = x^{-1-\sigma}\{\sin \pi\sigma \log |f(-x)| - \pi \cos \pi\sigma\, n(x)\}.$$

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The integral around the circle approaches zero, at least as $R \to \infty$ through an appropriate sequence of values, because if $f(z)$ is of order λ, say, for any positive ϵ we have $\log |f(z)| < R^{\lambda+\epsilon}$ for all large R, $\log |f(z)| > - R^{\lambda+\epsilon}$ for a sequence of values of R tending to ∞; and $|\arg f(z)| \leqslant R^{\lambda+\epsilon}$ because $n(t) = O(t^{\lambda+\epsilon})$ and so

$$\arg f(z) = \Im \log f(z) = y \int_0^\infty \frac{n(t)\, dt}{(t+x)^2 + y^2} = O(R^{\lambda+\epsilon})$$

(cf. Valiron [**12**], Bowen and Macintyre [**2**]). Hence

$$\int_0^\infty r(r+x)^{-1}\phi(x)\, dx = -\pi r^{-\sigma} \log f(r), \tag{2.3}$$

where the integral is to be understood as

$$\lim \int_0^R$$

when $R \to \infty$ through a certain sequence of values.

If $\rho = \sigma$,

$$\int_0^\infty \phi(x)\, dx$$

converges and (since $r/(r+x)$ is monotonic) we may let $r \to \infty$ under the integral sign in (2.3) to obtain (1.1) from (1.8).

If $\rho < \sigma$, put

$$\Phi(x) = \int_0^x \phi(t)\, dt;$$

then (1.8) gives us

$$\Phi(x) \sim Br^{\rho-\sigma}, \qquad B = \pi A(\rho - \sigma)^{-1}(\cot \pi\rho - \cot \pi\sigma).$$

By (2.3) we have

$$-\pi r^{-\sigma} \log f(r) = \int_0^\infty r(r+x)^{-1} d\Phi(x) = \int_0^\infty r(r+x)^{-2}\Phi(x)\, dx,$$

and since $\Phi(x) \sim Bx^{\rho-\sigma}$,

$$\int_0^\infty r(r+x)^{-2}\Phi(x)\, dx \sim B \int_0^\infty rx^{\rho-\sigma}(r+x)^{-2}\, dx = Br^{\rho-\sigma}\pi(\sigma-\rho) \csc \pi(\sigma - \rho),$$

and (1.1) follows. If $\rho > \sigma$ we write

$$\Phi(x) = \int_x^\infty \phi(t)\, dt$$

and proceed similarly.

3. Theorem 5: second part. We now show that (1.1) implies (1.8). By (1.3), (1.1) implies

$$\log f(z) \sim A\pi z^\rho \csc \pi\rho, \qquad -\pi < \theta < \pi, \tag{3.1}$$

uniformly in $|\theta| \leqslant \pi - \delta < \pi$. Consider the integral

$$-i \int_C z^{-1-\sigma} \log f(z)\, dz$$

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over the contour used in §2. The integrand is regular inside the contour and so the integral is zero. The integral along the cut is

$$2\int_0^R x^{-1-\sigma}\{\sin \pi\sigma \log |f(-x)| - \pi \cos \pi\sigma\, n(x)\}\, dx.$$

The integral around the circle is

$$\int_{-\pi}^{\pi} z^{-\sigma} \log f(z)\, d\theta. \tag{3.2}$$

By (3.1), if we can let $R \to \infty$ under the integral sign in (3.2), we shall have

$$\lim_{R\to\infty} R^{\sigma-\rho}\int_{-\pi}^{\pi} z^{-\sigma}\log f(z)\, d\theta = 2A\pi(\rho-\sigma)^{-1}\csc \pi\rho \sin \pi(\rho-\sigma), \tag{3.3}$$

which will establish (1.8). Now the convergence in (3.1) is uniform in $(-\pi+\delta, \pi-\delta)$, and so

$$\lim_{R\to\infty} R^{\sigma-\rho}\int_{-\pi+\delta}^{\pi-\delta} z^{-\sigma}\log f(z)\, d\theta = 2\pi A(\rho-\sigma)^{-1}\csc \pi\rho \sin(\pi-\delta)(\rho-\sigma). \tag{3.4}$$

The remainder of the integral contributes

$$R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right)\{\log |f(Re^{i\theta})| \cos \sigma\theta + \arg f(Re^{i\theta}) \sin \sigma\theta\}\, d\theta. \tag{3.5}$$

The part involving $\arg f(Re^{i\theta})$ is $O(\delta)$ as $\delta \to 0$, uniformly in R, since $n(R) = O(R^\rho)$ implies $\arg f(Re^{i\theta}) = O(R^\rho)$ as before.

By Jensen's theorem and Theorem 1,

$$R^{-\rho}\int_{-\pi}^{\pi}\log |f(Re^{i\theta})|\, d\theta = 2\pi\int_0^R t^{-1}n(t)\, dt \to 2\pi A/\rho,$$

and by (3.1),

$$R^{-\rho}\int_{-\pi+\delta}^{\pi-\delta}\log |f(Re^{i\theta})|\, d\theta \to 2\pi A\rho^{-1}\sin(\pi-\delta)\rho \csc \pi\rho;$$

so

$$R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right)\log |f(Re^{i\theta})|\, d\theta \to 2\pi\rho^{-1}\{1 - \sin(\pi-\delta)\rho \csc \pi\rho\} = O(\delta). \tag{3.6}$$

Furthermore, the parts of (3.5) and (3.6) involving $\log^+ |f(Re^{i\theta})|$ are uniformly $O(\delta)$ since $\log^+ |f(Re^{i\theta})| = O(R^\rho)$ uniformly in θ. Then

$$\left|R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right)\log^- |f(Re^{i\theta})| \cos \sigma\theta\, d\theta\right| \leqslant \left|R^{-\rho}\left(\int_{-\pi}^{-\pi+\delta} + \int_{\pi-\delta}^{\pi}\right)\log^- |f(Re^{i\theta})|\, d\theta\right| = O(\delta).$$

Thus the part of the left-hand side of (3.3) omitted from (3.4) is uniformly $O(\delta)$, and hence (3.3) is true.

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Northwestern University

https://doi.org/10.4153/CJM-1953-019-9 Published online by Cambridge University Press