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DETERMINING CROSSING NUMBERS OF GRAPHS OF ORDER SIX USING CYCLIC PERMUTATIONS

MICHAL STAŠ

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Abstract

We extend known results concerning crossing numbers by giving the crossing number of the join product $G + D_n$, where the connected graph G consists of one 4-cycle and of two leaves incident with the same vertex of the 4-cycle, and D_n consists of n isolated vertices. The proofs are done with the help of software that generates all cyclic permutations for a given number k and creates a graph for calculating the distances between all $(k-1)!$ vertices of the graph.

2010 *Mathematics subject classification*: primary 05C10; secondary 05C38.

Keywords and phrases: graph, drawing, crossing number, join product, rotation.

1. Introduction

The problem of reducing the number of crossings in a graph has applications in many areas, the most prominent being very large scale integration (VLSI) technology. The lower bound on the chip area is determined by the crossing number and the number of vertices of the graph. Reducing the crossing number is also important when considering the aesthetics of a graph because the graph should be easy to read and understand. Investigation of the crossing number of graphs is a classical and very difficult problem. In general, computing the crossing number of a given graph is an NP-complete problem.

The purpose of this article is to extend the known results in [2, 7–11] and [12] by adding another graph. The methods are based on combinatorial properties of cyclic permutations. Somewhat similar ideas were used in [4] and [11]. In [2, 3] and [12], properties of cyclic permutations were also verified with the help of software.

Let G be the graph consisting of one 4-cycle and two leaves incident with the same vertex of the 4-cycle. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \le i \le n$, denote the subgraph induced by the six edges incident with the vertex t_i. Thus, $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$.

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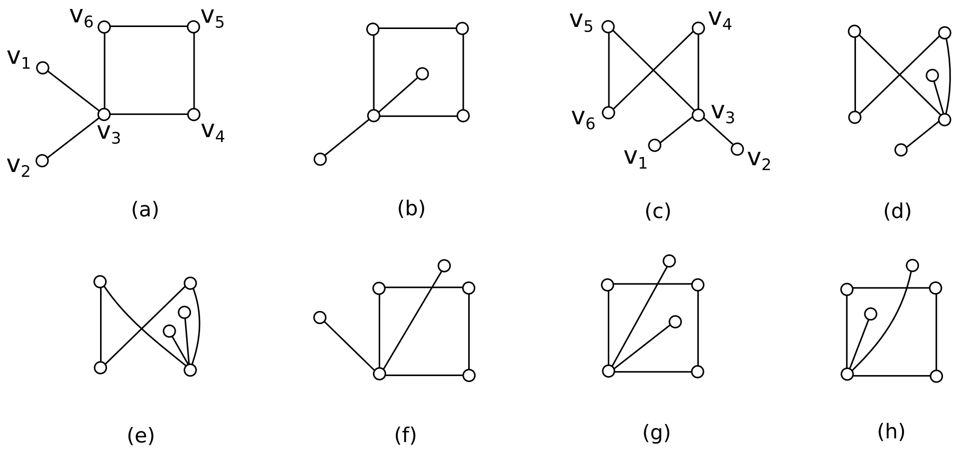

FIGURE 1. Two planar drawings of G and six drawings of G with $\mathrm{cr}_D(G) = 1$.

We will use definitions and notation of the crossing numbers of graphs from [6]. Some of our calculations are based on Kleitman's result on crossing numbers of complete bipartite graphs [5]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{for } m \leq 6.$$

2. Cyclic permutations and configurations

We will use the same definitions and notation for cyclic permutations and the corresponding configurations for a good drawing D of the graph $G + D_n$ as in [12]. The proofs are done with the help of software that generates all cyclic permutations (see [1]; the C++ version of the program is also located on the web site http://web.tuke.sk/fei-km/coga/). The list with the short names of $6!/6 = 120$ cyclic permutations on six elements are collected in [12, Table 1].

We will only consider drawings of the graph G for which there is a possibility of the existence of a subgraph $T^i \in R_D$, because of arguments in the proof of Theorem 3.4 below. Assume a good drawing D of the graph $G + D_n$ in which the edges of G do not cross each other. In this case, without loss of generality, from the drawings in Figure 1 we can choose the vertex notation of the graph as shown in Figure 1(a). Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Let us start with the subdrawing F^i induced by the edges incident with the vertices v_1 and v_5. These two edges separate the vertices v_6 and v_2, v_4 into two regions with unique incidences of these vertices. Since there are three possibilities for the subdrawing of F^i induced by the edges incident with the vertices v_1, v_5 and v_3, we obtain three different possible configurations of F^i denoted by A_1, A_2 and A_3. For our purposes, it does not matter which of the regions is unbounded, so we can assume that the drawings are as shown in Figure 2.

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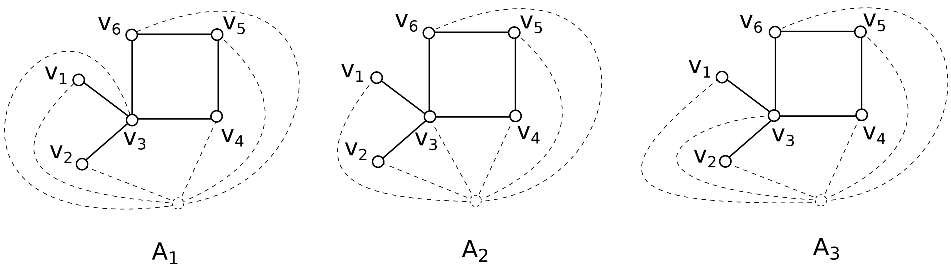

Figure 2. Drawings of three possible configurations from $\mathcal{M}$ of the subgraph F^i.

We represent a cyclic permutation by the permutation with 1 in the first position. Thus, the configurations A_1, A_2 and A_3 are represented by the cyclic permutations $P_{111} = (136542)$, $P_{120} = (165432)$ and $P_{96} = (165423)$, respectively. We denote by $\mathcal{M}_D$ the set of all configurations for the drawing D belonging to $\mathcal{M} = \{A_1, A_2, A_3\}$. The unique drawing of F^i contains six regions with the vertex t_i on its boundary. For example, if F^i has the configuration A_1, we can denote these six regions by $\omega_{1,2,3}$, $\omega_{2,3,4}$, $\omega_{4,5}$, $\omega_{5,6}$, $\omega_{3,6}$ and $\omega_{1,3}$ depending on which of the vertices are located on the boundary of the corresponding region.

If two different subgraphs F^i and F^j with configurations from $\mathcal{M}_D$ cross in a drawing D of $G + D_n$, then only the edges of T^i cross the edges of T^j. Thus, we will deal with the minimum numbers of crossings between two different subgraphs F^i and F^j depending on their configurations. Let X, Y be the configurations from $\mathcal{M}_D$. We denote by $\mathrm{cr}_D(X, Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X, Y) = \min\{\mathrm{cr}_D(X, Y)\}$ over all good drawings of the graph $G + D_n$ with $X, Y \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(X, Y)$ for all pairs $X, Y \in \mathcal{M}$.

By $\overline{P_i}$, we will understand the inverse cyclic permutation to the permutation P_i for $i = 1, \dots, 120$. Woodall [13] defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$ and the set of edges E, where two vertices are joined by an edge if the vertices correspond to permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (that is, an ordered set with six elements). Hence, if $d_{\mathrm{COG}}(\text{‘}\mathrm{rot}_D(t_i)\text{’}, \text{‘}\mathrm{rot}_D(t_j)\text{’})$ denotes the distance between two vertices corresponding to the cyclic permutations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$ in the graph COG, then

$$d_{\mathrm{COG}}(\text{‘}\mathrm{rot}_D(t_i)\text{’}, \text{‘}\overline{\mathrm{rot}_D(t_j)}\text{’}) = Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \leq \mathrm{cr}_D(T^i, T^j)$$

for any two different subgraphs T^i and T^j.

We are ready to find the necessary numbers of crossings between T^i and T^j for the configurations of F^i and F^j from $\mathcal{M}$. The configurations A_1 and A_2 are represented by the cyclic permutations $P_{111} = (136542)$ and $P_{120} = (165432)$, respectively. Since $\overline{P_{120}} = (123456) = P_1$, we have $\mathrm{cr}(A_1, A_2) \geq 4$ using $d_{\mathrm{COG}}(\text{‘}P_{111}\text{’}, \text{‘}P_1\text{’}) = 4$. The same reasoning gives $\mathrm{cr}(A_1, A_3) \geq 5$ and $\mathrm{cr}(A_2, A_3) \geq 5$. It is also clear that $\mathrm{cr}(A_i, A_i) \geq 6$ for $i = 1, 2, 3$.

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TABLE 1. The necessary number of crossings between T^i and T^j for the configurations X_k, X_l.

–	X_1	X_2	X_3
X_1	6	4	5
X_2	4	6	5
X_3	5	5	6

Assume a good drawing D of the graph $G + D_n$ with one crossing among edges of G (in which there is a $T^i \in R_D$). In this case, without loss of generality, we can choose the vertex notation of the graph as shown in Figure 1(c). By the arguments mentioned above, there are three possibilities for the subdrawing of F^i induced by the edges incident with the vertices v_1, v_5 and v_3, that is, we obtain three different possible configurations of F^i, say B_1, B_2 and B_3, with their rotations $P_{111} = (136542)$, $P_{120} = (165432)$ and $P_{96} = (165423)$, respectively. We denote by $\mathcal{N}_D$ the set of all configurations in the drawing D belonging to the set $\mathcal{N} = \{B_1, B_2, B_3\}$. The verification of the lower bounds for the number of crossings of two configurations from $\mathcal{N}$ proceeds in the same way as before. The resulting lower bounds for the number of crossings of configurations from $\mathcal{M}$ and $\mathcal{N}$ are summarised in Table 1. (Here, X_k and X_l are configurations of the subgraphs F^i and F^j, where $k, l \in \{1, 2, 3\}$, $X = A$ in case of $\mathcal{M}$ and $X = B$ in case of $\mathcal{N}$.)

3. The crossing number of $G + D_n$

For the proof of Theorem 3.4, we need the following statements related to some restricted drawings of the graph $G + D_n$. Note that if the edges of G do not cross each other in D, then $\mathrm{cr}_D(T^i \cup T^j) \geq 4$ for any two different subgraphs $T^i, T^j \in R_D$ by Table 1.

LEMMA 3.1 [12]. *Let D be a good and antipodal-free drawing of $G + D_n$ with $n > 2$. Suppose that $2|R_D| + |S_D| > 2n - 2\lfloor n/2 \rfloor$ and let $T^i, T^j \in R_D$ be two different subgraphs with $\mathrm{cr}_D(T^i \cup T^j) \geq 4$. If both conditions*

$$\mathrm{cr}_D(G \cup T^i \cup T^j, T^l) \geq 10 \quad \text{for any } T^l \in R_D \setminus \{T^i, T^j\}, \tag{3.1}$$

$$\mathrm{cr}_D(G \cup T^i \cup T^j, T^l) \geq 7 \quad \text{for any } T^l \in S_D \tag{3.2}$$

hold, then there are at least $6\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 2\lfloor n/2 \rfloor$ crossings in D.

If D is a good and antipodal-free drawing of $G + D_n$, and $T^i \in R_D$ is such that F^i has configuration $A_j \in \mathcal{M}_D$, then $\mathrm{cr}_D(G \cup T^i, T^l) \geq 3$ for any T^l, $l \neq i$ (see Figure 2). Moreover, $\mathrm{cr}_D(T^i, T^k) = 2$ with $T^k \in S_D$ can only occur for the configurations A_1 and A_2 of F^i.

LEMMA 3.2. *Let D be a good and antipodal-free drawing of $G + D_n$ with $n > 2$. Let $T^i \in R_D$ be a subgraph such that F^i has configuration $A_j \in \mathcal{M}_D$ for $j \in \{1, 2\}$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then:*

(a) $\mathrm{cr}_D(T^i \cup T^k, T^l) \geq 3$ *for any subgraph T^l with $l \neq i, k$;*
(b) $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \in S_D$ with $\mathrm{cr}_D(T^i, T^l) \geq 3$.*

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PROOF. Let us assume the configuration A_1 of F^i and remark that it is represented by the cyclic permutation $\mathcal{P}_{111} = (136542)$.

(a) The unique drawing of F^i contains six regions with the vertex t_i on their boundaries (see Figure 2). If there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) = 2$, then the vertex t_k must be placed in the quadrangular region $\omega_{2,3,4}$ with three vertices of G on its boundary. Thus, the configuration of the subgraph F^k can only be represented by one possible cyclic permutation $\mathcal{P}_{116} = (154632)$ and the edge t_iv_6 crosses the edge v_3v_4. By [1], we can verify that there is no cyclic permutation $\mathcal{P}_m$ different from $\mathcal{P}_{111}$ and $\mathcal{P}_{116}$ with $d_{\mathrm{COG}}(`\mathcal{P}_{111}', `\mathcal{P}_m') + d_{\mathrm{COG}}(`\mathcal{P}_{116}', `\mathcal{P}_m') < 3$. Thus, Woodall's result implies that there is no subgraph T^l with $\mathrm{cr}_D(T^i \cup T^k, T^l) < 3$ for any $l \neq i, k$.

(b) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 2$, so the subdrawing of F^k is represented by $\mathcal{P}_{116} = (154632)$. If there is a $T^l \in S_D$ with $\mathrm{cr}_D(T^k, T^l) = 1$, then the vertex t_l cannot be inside the 4-cycle of the graph G, but must be in the pentagonal region of $D(F^l)$ with four vertices of G on its boundary. Hence, the cyclic permutation representing the configuration of the subgraph F^l is either $\mathcal{P}_{105} = (136452)$ or $\mathcal{P}_{55} = (123654)$ (see Figure 4). Since $\overline{\mathcal{P}_{105}} = (125463) = \mathcal{P}_{75}$ and $\overline{\mathcal{P}_{55}} = (145632) = \mathcal{P}_{115}$, the distances $d_{\mathrm{COG}}(`\mathcal{P}_{111}', `\mathcal{P}_{75}') = 5$ and $d_{\mathrm{COG}}(`111', `\mathcal{P}_{115}') = 5$ imply that $\mathrm{cr}_D(T^i, T^l) \geq 5$. Thus, $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 1 + 5 + 1 = 7$.

Next, assume that $\mathrm{cr}_D(T^k, T^l) \geq 2$ for any $T^l \in S_D$. Since the case $\mathrm{cr}_D(T^i, T^l) \geq 4$ implies that $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 1 + 4 + 2 = 7$, let us consider a subgraph T^l with $\mathrm{cr}_D(T^i, T^l) = 3$. The vertex t_l must be in the region $\omega_{2,3,4}$ of the unique drawing of F^i. Consequently, $\mathrm{cr}_D(T^k, T^l) \geq 3$, that is, $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 1 + 3 + 3 = 7$.

From the symmetry of the configurations A_1 and A_2, we can use the same arguments for the configuration A_2 of F^i. This completes the proof. □

COROLLARY 3.3. *Let D be a good and antipodal-free drawing of $G + D_n$ with $n > 2$ and let $\mathcal{M}_D$ be a nonempty set with $\{A_1, A_2\} \subseteq \mathcal{M}_D$. If T^i, $T^j \in R_D$ are different subgraphs such that F^i, F^j have configurations from $\{A_1, A_2\}$, then*

$$\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 7 \quad \textit{for any } T^k \in S_D.$$

PROOF. Take configurations A_1 of F^i and A_2 of F^j. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the configuration of the subgraph F^k must be represented by the cyclic permutation $\mathcal{P}_{116} = (154632)$. Note that the configuration A_2 is represented by $\mathcal{P}_{120}$. Using $\overline{\mathcal{P}_{116}} = (123645) = \mathcal{P}_{31}$ and $d_{\mathrm{COG}}(`\mathcal{P}_{120}', `\mathcal{P}_{31}') = 4$, we obtain $\mathrm{cr}_D(T^j, T^k) \geq 4$. Hence, $\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 1 + 2 + 4 = 7$. We can apply the same idea if there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 2$. Next, assume that $\mathrm{cr}_D(T^i, T^k) \geq 3$ and $\mathrm{cr}_D(T^j, T^k) \geq 3$ for any $T^k \in S_D$. This forces $\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 1 + 3 + 3 = 7$ for any $T^k \in S_D$. This completes the proof. □

If we consider the set of configurations $\mathcal{N}_D$ with a subgraph $T^i \in R_D$, then a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$ can only exist for the configuration $B_2 \in \mathcal{N}_D$ of the subgraph F^i (see Figure 3). In this case, the subdrawing of $F^k = G \cup T^k$ is represented by $\mathcal{P}_{103} = (134652)$. Since $\overline{\mathcal{P}_{103}} = (125643) = \mathcal{P}_{85}$ and $d_{\mathrm{COG}}(`\mathcal{P}_{111}', `\mathcal{P}_{85}') = 4$, we obtain $\mathrm{cr}_D(T^j, T^k) \geq 4$ for any subgraph F^j having the configuration B_1. Thus, we can extend Corollary 3.3 for $\{B_1, B_2\} \subseteq \mathcal{N}_D$.

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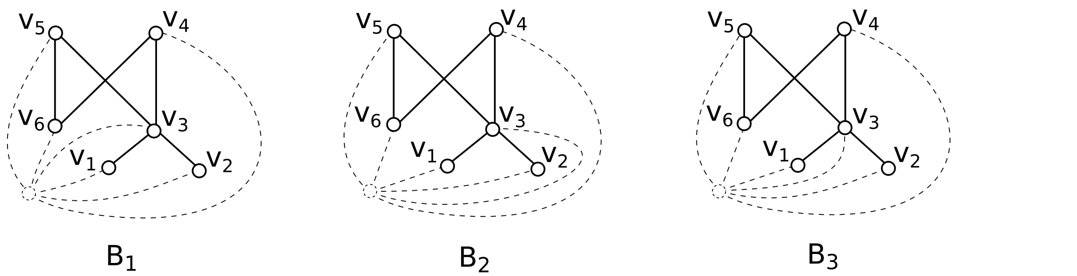

FIGURE 3. Drawings of three possible configurations from $\mathcal{N}$ of the subgraph F^i.

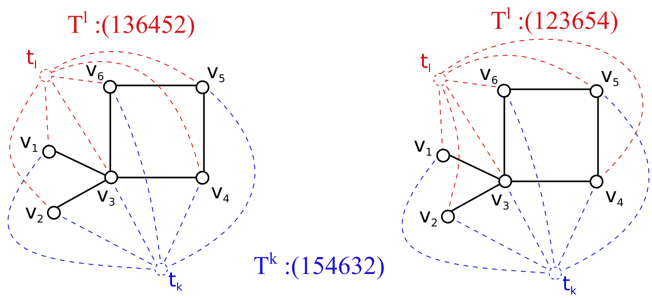

FIGURE 4. Two drawings of $G \cup T^k \cup T^l$ with $\mathrm{cr}_D(G \cup T^k, T^l) = 2$ for $T^k, T^l \in S_D$.

THEOREM 3.4. *If* $n \geq 1$, *then* $\mathrm{cr}(G + D_n) = 6\lfloor n/2\rfloor\lfloor(n-1)/2\rfloor + 2\lfloor n/2\rfloor$.

PROOF. In Figure 5(b), there is a drawing of $G + D_n$ with $6\,\lfloor n/2\rfloor\lfloor(n-1)/2\rfloor + 2\lfloor n/2\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 6\lfloor n/2\rfloor\lfloor(n-1)/2\rfloor + 2\lfloor n/2\rfloor$. We prove the reverse inequality by induction on n. The graph $G + D_1$ is planar and hence $\mathrm{cr}(G + D_1) = 0$. It is clear from Figure 5(a) that $\mathrm{cr}(G + D_2) \leq 2$. The graph $G + D_2$ contains a subdivision of $K_{3,4}$ and therefore $\mathrm{cr}(G + D_2) \geq 2$. So, $\mathrm{cr}(G + D_2) = 2$.

Suppose now that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + D_n) < 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$$

and that

$$\mathrm{cr}(G + D_m) \geq 6\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor + 2\left\lfloor\frac{m}{2}\right\rfloor \quad \text{for any } m < n.$$

We claim that the drawing D must be antipodal-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Then $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 2$. Since $\mathrm{cr}(K_{6,3}) = 6$, it follows that $\mathrm{cr}_D(T^k, T^{n-1} \cup T^n) \geq 6$ for $k = 1, 2, \ldots, n-2$. So,

$$\begin{aligned}
&\mathrm{cr}_D(G + D_n)\\
&\quad= \mathrm{cr}_D\,(G + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n) + \mathrm{cr}_D(G, T^{n-1} \cup T^n)\\
&\quad\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 2\left\lfloor\frac{n-2}{2}\right\rfloor + 6(n-2) + 2 = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.
\end{aligned}$$

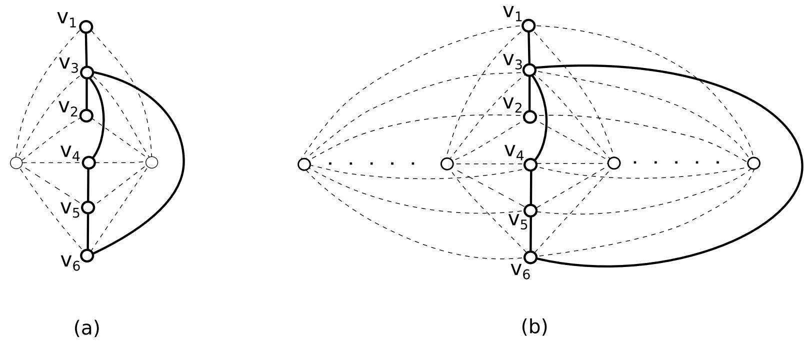

FIGURE 5. Good drawings of $G + D_2$ and $G + D_n$.

This contradiction confirms that D is antipodal-free. Our assumption on D together with $\mathrm{cr}(K_{6,n}) = 6\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{6,n}) < 2\left\lfloor \frac{n}{2} \right\rfloor.$$

If we denote $r = |R_D|$ and $s = |S_D|$, then

$$\mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, $r \geq 1$, $2r + s > 2n - 2\lfloor n/2\rfloor$ and $r > n - r - s$. For $T^i \in R_D$, we consider the possible configurations of F^i in the drawing D in three cases.

Case 1: $\mathrm{cr}_D(G) = 0$. Since $r \geq 1$, that is, there is a subgraph $T^i \in R_D$, we can choose the vertex notation of the graph as shown in Figure 1(a). We now have three possibilities for the set of configurations belonging to $\mathcal{M}_D$.

(a) $A_3 \in \mathcal{M}_D$. Without loss of generality, we can assume that $T^n \in R_D$ with the configuration A_3 of the subgraph F^n. The subdrawing of F^n induced by D can be obtained from the drawings in Figure 2. It is easy to verify that there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) \leq 2$. Moreover, $\mathrm{cr}_D(T^n, T^i) \geq 5$ for any $T^i \in R_D$ with $i \neq n$ by Table 1. By fixing the graph $G \cup T^n$,

$$\begin{aligned}
\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n) \\
&\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 4s + 3(n-r-s) + 0 \\
&= 6\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + (2r+s) + 3n - 5 \\
&\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + 2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1 + 3n - 5 \geq 6\left\lfloor \frac{n}{2} \right\rfloor\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

(b) $\{A_1, A_2\} \subseteq \mathcal{M}_D$. Without loss of generality, fix T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations from $\{A_1, A_2\}$. Then condition (3.1) is true by summing the values

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in all columns in the first two rows of Table 1, and condition (3.2) holds by Corollary 3.3. Consequently, we can apply Lemma 3.1.

(c) $\mathcal{M}_D = \{A_j\}$ for only one $j \in \{1, 2\}$. Without loss of generality, we can assume that the configuration of F^n is A_1. Write $S_D(T^n) = \{T^i \in S_D : \mathrm{cr}_D(F^n, T^i) = 3\}$ and $s_1 = |S_D(T^n)|$. Note that $S_D(T^n)$ is a subset of S_D and $s_1 \le s$, that is, $s - s_1 \ge 0$. Hence, there are two possibilities to consider.

(1) Suppose that $r > s_1$, that is, $r - 1 \ge s_1$. By fixing the graph $G \cup T^n$,

$$\begin{aligned}
\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)\\
&\ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 6(r-1) + 3s_1 + 4(s - s_1) + 3(n - r - s) + 0\\
&\ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 5(r-1) + 4s + 3(n - r - s)\\
&= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + (2r + s) + 3n - 5\\
&\ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2n - 2\left\lfloor\frac{n}{2}\right\rfloor + 1 + 3n - 5 \ge 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.
\end{aligned}$$

(2) Suppose that $r \le s_1$, that is, $r - 1 \le s_1 - 1$. Let T^k be a subgraph from the nonempty set $S_D(T^n)$. As $\mathcal{M}_D = \{A_1\}$, we have $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \ge 6 + 2 = 8$ for any $T^i \in R_D$ with $i \ne n$. From the proof of Lemma 3.2, the subgraph F^k can only have the configuration represented by the cyclic permutation $P_{116} = (154632)$. Thus, $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \ge 1 + 2 + 6 = 9$ for any $T^i \in S_D(T^n)$ with $i \ne k$. Again by Lemma 3.2, $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \ge 7$ for any $T^i \in S_D$ with $\mathrm{cr}_D(F^n, T^i) \ge 4$. Moreover, $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \ge 2 + 3 = 5$ for any $T^i \notin R_D \cup S_D$. Since $n - r - s \le r - 1 \le s_1 - 1$, by fixing the graph $G \cup T^n \cup T^k$,

$$\begin{aligned}
\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G \cup T^n \cup T^k) + \mathrm{cr}_D(G \cup T^n \cup T^k)\\
&\ge 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 8(r-1) + 9(s_1 - 1) + 7(s - s_1) + 5(n - r - s) + 3\\
&\ge 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 8(r-1) + 7(s_1 - 1) + 7(s - s_1) + 7(n - r - s) + 3\\
&= 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + r + 7n - 12 \ge 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.
\end{aligned}$$

The same arguments can be used for the case $\mathcal{M}_D = \{A_2\}$ due to their symmetry.

Case 2: $\mathrm{cr}_D(G) = 1$. Since the set R_D is nonempty, we only need to consider the two drawings of G shown in Figure 1(c) and (f).

(a) $\mathrm{cr}_D(G) = 1$ represented by Figure 1(c). In this case, the configurations belong to $\mathcal{N}_D$ and we can use similar subcases to those in Case 1 to obtain a contradiction.

(b) $\mathrm{cr}_D(G) = 1$ as in Figure 1(f). It is easy to check all possible drawings D for which the set R_D is nonempty and such that, if $T^i \in R_D$, then $\mathrm{cr}_D(G \cup T^i, T^j) \ge 4$ for any

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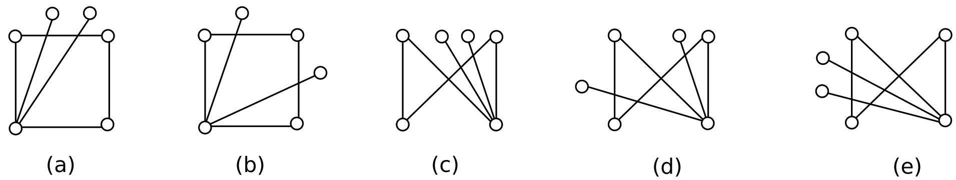

FIGURE 6. Five possible drawings of G with $\mathrm{cr}_D(G) \geq 2$ and $R_D \neq \emptyset$.

subgraph T^j with $j \neq i$. By fixing the graph $G \cup T^i$,

$$\begin{aligned}\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^i) + \mathrm{cr}_D(G \cup T^i)\\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.\end{aligned}$$

Case 3: $\mathrm{cr}_D(G) \geq 2$. We can use the same idea as in Case 2(b) for all five possible drawings of the graph G for which the set R_D is nonempty (see Figure 6).

Thus, we have shown that there is no good drawing D of the graph $G + D_n$ with fewer than $6\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 2\lfloor n/2 \rfloor$ crossings. This completes the proof of the theorem. □

In Figure 5(b), we can add the edge v_2v_4 to the graph G without additional crossings. This recovers an already known result for the crossing number of the graph $G_1 + D_n$ considered in [12].

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MICHAL STAŠ,
Department of Mathematics and Theoretical Informatics,
Faculty of Electrical Engineering and Informatics,
Technical University of Košice,
Letná 9, 042 00 Košice, Slovak Republic
e-mail: michal.stas@tuke.sk