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RESEARCH ARTICLE

Two stackelberg games in life insurance: Mean-variance criterion

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Abstract

We study two continuous-time Stackelberg games between a life insurance buyer and seller over a random time horizon. The buyer invests in a risky asset and purchases life insurance, and she maximizes a mean-variance criterion applied to her wealth at death. The seller chooses the insurance premium rate to maximize its expected wealth at the buyer's random time of death. We consider two life insurance games: one with term life insurance and the other with whole life insurance—the latter with pre-commitment of the constant investment strategy. In the term life insurance game, the buyer chooses her life insurance death benefit and investment strategy continuously from a time-consistent perspective. We find the buyer's equilibrium control strategy *explicitly*, along with her value function, for the term life insurance game by solving the extended Hamilton–Jacobi–Bellman equations. By contrast, in the whole life insurance game, the buyer pre-commits to a constant life insurance death benefit and a constant amount to invest in the risky asset. To solve the whole life insurance problem, we first obtain the buyer's objective function and then we maximize that objective function over constant controls. Under both models, the seller maximizes its expected wealth at the buyer's time of death, and we use the resulting optimal life insurance premia to find the Stackelberg equilibria of the two life insurance games. We also analyze the effects of the parameters on the Stackelberg equilibria, and we present some numerical examples to illustrate our results.

1. Introduction

Life insurance is a financial product that provides bequests to dependents, beloved ones, or charitable organizations after the policyholder's death. Since the 1960s, researchers have constructed various quantitative models to analyze life insurance acquisition and consumption/investment decision problems under uncertainty. Yaari (1965) is a starting point for modern research in this field. Richard (1975) combined Yaari's model with the optimization model in Merton (1969, 1971) to analyze an individual's portfolio choice and demand for life insurance, in which the individual's lifetime was assumed to be random but bounded. Researchers have extended these early papers of Yaari and Richard. In his dissertation, Ye (2006) augmented Richard's work by considering a more general random lifetime. Kwak *et al*. (2011) studied an optimal investment, consumption, and life insurance problem for a family with two generations. Bruhn and Steffensen (2011) found the optimal life insurance purchasing strategy to maximize the utility of consumption for a household within a continuous-time, finite-state Markov chain framework. De-Paz *et al*. (2014) studied the optimal life insurance decision problem with heterogeneous discounting. Wei *et al*. (2020) considered a life insurance and investment/consumption optimization problem for couples with correlated lifetimes. Liang and Young (2020) investigated the bequest goal problem with life insurance acquisition and model uncertainty. Maggistro *et al*. (2024) explored a two-agent portfolio optimization problem with life insurance under dynamic non-cooperative and cooperative game scenarios. See also Huang *et al*. (2008), Kraft and Steffensen (2008), Nielsen and Steffensen (2008), Bayraktar and Young (2013), Bayraktar *et al*. (2014, 2016), Zhang *et al*. (2021), Wang *et al*. (2021), Park *et al*. (2023), and Li *et al*. (2023) for other extensions.

Research on Stackelberg games in insurance has received much attention in the recent actuarial science literature. These leader–follower games enable us to have a deeper understanding of the interactions between buyers and sellers of insurance. Current research mainly focuses on two models. One is the one-period static model, and the corresponding Stackelberg equilibrium is called a Bowley solution. Chan and Gerber (1985) first proposed such a Stackelberg game in reinsurance by maximizing each player's expected utility of final wealth. Cheung *et al*. (2019) incorporated distortion risk measures into the reinsurance problem. Chi *et al*. (2020) considered the limited ceded risks for the reinsurer. Boonen *et al*. (2021) explored the effects of asymmetric information between the insurer and reinsurer on the Bowley solution. Li and Young (2021) computed the Bowley solution for a mean-variance Stackelberg game.

The other model is the continuous-time model, which leads to a so-called *Stackelberg differential game*. Chen and Shen (2018) applied Stackelberg game theory to the study of dynamic optimal reinsurance problems with proportional reinsurance and a diffusion surplus process. Chen and Shen (2019) extended their previous problem to a time-consistent mean-variance framework. Wang and Siu (2020) investigated a robust reinsurance contract under a principle-agent model with VaR constraints. Gu *et al*. (2020) studied an optimal excess-of-loss reinsurance contract with ambiguity, in which the insurer's surplus is described by a classical Cramér–Lundberg model. Cao *et al*. (2022) considered the Stackelberg differential game in a general spectrally negative Lévy framework incorporating model ambiguity concerning both the intensity and severity of insurable loss. Li and Young (2022) studied a mean-variance Stackelberg differential game with a mean-variance premium principle over a random time horizon. Guan *et al*. (2024) investigated a Stackelberg game between an insurer and a reinsurer under the α-maxmin mean-variance criterion and stochastic volatility. For related work, please see Hu *et al*. (2018), Bai *et al*. (2022), Cao *et al*. (2023), Cao and Young (2023), Zhang *et al*. (2024a), Zhang *et al*. (2024b), and Han *et al*. (2024).

To the best of our knowledge, ours is the first paper to analyze life insurance demand, along with the insurance premium, via a Stackelberg differential game. Moreover, we propose *two* continuous-time Stackelberg life insurance games: one with term life insurance and the other with whole life insurance. The buyer invests in a risky asset and purchases life insurance, and she maximizes a mean-variance criterion applied to her wealth at death. The seller chooses the insurance premium rate to maximize its expected wealth at the buyer's random time of death.

For the term life insurance game, we solve the buyer's problem from a time-consistent perspective (see Björk and Murgoci (2010) and Björk *et al*. (2014)), that is, the buyer continuously decides how much life insurance to purchase and how much money to invest into the risky asset. By solving the extended Hamilton–Jacobi–Bellman (HJB) equations, we find the explicit expressions of the equilibrium life insurance and investment strategy and of the corresponding value function. Given the time-consistent equilibrium strategy of the buyer, the seller maximizes its expected terminal wealth at the buyer's time of death. Then, using the optimal premium rate, we derive *explicit* expressions of the Stackelberg equilibrium controls and value functions. We show that a constant amount of term life insurance and constant investment in the risky asset form the equilibrium strategy.

In the whole life insurance game, we assume that the buyer pre-commits to buying a constant amount of whole life insurance and investing a constant amount in the risky asset.[1] To find the optimal constant strategy for the buyer of whole life insurance, we first find an expression for her objective function and

[1]Note that our use of "pre-commitment" differs from that of Zhou and Li (2000). Their pre-commitment strategy allows the control to depend on the state variables wealth and time. By contrast, we assume that the pre-commitment strategy is *constant* from the outset, and we call it a *constant pre-commitment strategy*.

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then we maximize the objective function over all amounts of whole life death benefit and investment in the risky asset. As in the term life insurance game, we then solve the seller's problem to find the optimal insurance premium rate. Finally, we obtain explicit expressions of the Stackelberg equilibria of this life insurance game.

For both games, we analyze the effects of the parameters on the equilibria. Moreover, we show that the properties of the equilibrium controls for the term life insurance game are more intuitively pleasing than those for the whole life insurance game. Also, we find that whole life insurance is more attractable to the buyer, as expected from a pre-commitment model versus a dynamic time-consistent model.

The rest of this paper is organized as follows. In Section 2, we solve a continuoue-time Stackelberg term life insurance game with investment. In Section 2.1, we describe the market for term life insurance and investment and define the buyer's problem, the seller's problem, and the Stackelberg equilibrium of the game. In Section 2.2, we solve the buyer's problem from a time-consistent perspective. Given the equilibrium choice of the buyer, we solve the seller's problem in Section 2.3. In Section 2.4, we present and analyze the Stackelberg equilibrium of the term life insurance game. In Section 3, we solve a continuous-time Stackelberg whole life insurance game with investment. We apply the same objective functions as those in Section 2. In Section 3.1, we solve the buyer's (constant) pre-commitment problem. Parallel to Sections 2.3 and 2.4, respectively, we solve the seller's problem in Section 3.2, and we present the Stackelberg equilibrium of the whole life insurance game in Section 3.3. Finally, in Section 4, we provide numerical examples to compare the equilibrium controls and the corresponding value functions between the term life insurance and whole life insurance games.

2. Term life insurance game

In this section, we consider a continuous-time Stackelberg game between a seller of (instantaneous) term life insurance and a buyer who purchases that life insurance. We also assume that the buyer invests her wealth in a risky asset. In Section 2.1, we describe the market for term life insurance and investment, we define the buyer's problem (namely, choosing her death benefit and dollar amount to invest in the risky asset), and we define the seller's problem (namely, choosing the optimal price for the term life insurance). In that section, we also define the Stackelberg equilibrium of the game.

In Section 2.2, we obtain the time-consistent equilibrium term life insurance and investment strategy for the buyer. Given the equilibrium choice of the buyer, we solve the seller's problem in Section 2.3. Finally, in Section 2.4, we present and analyze the Stackelberg equilibrium of the term life insurance game.

2.1 Background

We refer to the individual and the life insurance company as the "buyer" and "seller," respectively. Let τ denote the random time of death of the buyer, which is a stopping time with respect to a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P}\right)$. We assume τ follows an exponential distribution with constant hazard rate $\lambda > 0$, that is,

$$\mathbb{P}(\tau > t) = \mathrm{e}^{-\lambda t}, \quad t \geq 0.$$

Moreover, we assume that the individual can purchase term life insurance via a premium payable continuously at the rate of $h \geq 0$ per dollar of insurance.[2] Let $D(x, t) \geq 0$ represent the death benefit that the buyer purchases at time t^- (when her wealth at that time equals x). This means that if the individual spends $h\, D(x, t)$ at time t^- for this term life insurance, her beneficiaries will receive $D(x,t)$ if she dies at time t. If she survives, there is no payout and the term ends. We also assume that the individual has constant net income at the rate of c, that is, c is her income net of consumption.

[2]Because our model is (essentially) time-homogeneous, we assume that h is a non-negative constant from the start.

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Furthermore, we assume that the individual can invest her wealth in a risky asset whose price process $S = \{S_t\}_{t\geq 0}$ follows geometric Brownian motion:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t,$$

in which $\mu > 0$, $\sigma > 0$, and $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion defined on the probability space $\big(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P}\big)$. Let $\pi(x, t) \in \mathbb{R}$ denote the amount of wealth invested in the risky asset at time t when her wealth at that time equals x. The buyer's controlled wealth process $X = \{X_t\}_{t\in[0,\tau)}$, thus, follows:

$$\begin{cases} \mathrm{d}X_t = \big(c - hD(X_t, t) + \mu\pi(X_t, t)\big)\mathrm{d}t + \sigma\pi(X_t, t)\mathrm{d}B_t, & 0 \leq t < \tau, \\ X_\tau = X_{\tau^-} + D(X_{\tau^-}, \tau), \end{cases} \tag{2.1}$$

and the seller's controlled wealth process $Y = \{Y_t\}_{t\in[0,\tau)}$ in relation to the risk and reward arising from the single buyer follows:

$$\begin{cases} \mathrm{d}Y_t = hD(X_t, t)\,\mathrm{d}t, & 0 \leq t < \tau, \\ Y_\tau = Y_{\tau^-} - D(X_{\tau^-}, \tau). \end{cases} \tag{2.2}$$

Next, we define *admissible* life insurance and investment strategies for the buyer.

Definition 2.1 (Admissible life insurance and investment strategies). *A life insurance and investment strategy* $(\mathcal{D}, \Pi) = \big(\{D(X_t, t)\}, \{\pi(X_t, t)\}\big)_{t\in[0,\tau)}$ *is called admissible if it satisfies the following properties:*

1. *Equation (2.1) has a unique strong solution for any* $X_0 = x_0 \in \mathbb{R}$.
2. *D is a non-negative Borel measurable function, and π is a real-valued Borel measurable function.*
3. $\int_0^t \pi^2(X_s, s)\mathrm{d}s < \infty$ *with probability 1, for all* $t \in [0, \tau)$.

Let $\mathcal{A}$ denote the collection of admissible strategies, and note that admissible strategies are feedback strategies.

The individual has mean-variance preferences and optimizes over her lifetime $[0, \tau)$. In consequence, given a life insurance and investment strategy $(\mathcal{D}, \Pi)$, the buyer's objective function equals

$$J(x, t; \mathcal{D}, \Pi, h) = \mathbb{E}_{x,t}(X_\tau) - \frac{\gamma}{2}\,\mathbb{V}_{x,t}(X_\tau), \tag{2.3}$$

in which $\mathbb{E}_{x,t}$ and $\mathbb{V}_{x,t}$ denote expectation and variance, respectively, conditional on $X_t = x \in \mathbb{R}$ and $\tau > t$. In (2.3), the parameter $\gamma > 0$ represents her risk aversion toward variance. The individual seeks to maximize J in (2.3); however, the mean-variance objective can lead to a time-inconsistent problem because variance does not satisfy the property of iterated expectations. To handle this difficulty, we assume that the buyer essentially plays against future versions of herself and seeks to maximize her criterion, given her future choices and given the premium rate. In the following, we define the time-consistent Stackelberg follower action $\hat{\mathcal{D}}$ and $\hat{\Pi}$ of the buyer's game.

Definition 2.2 (Buyer's time-consistent Stackelberg follower action). *Suppose we are given a non-negative premium rate for life insurance h. For a given admissible strategy $(\hat{\mathcal{D}}(h), \hat{\Pi}(h))$, fix an arbitrary initial time $t \in [0, \tau)$, a positive number ε, a non-negative number D, and a real number π. Then, define the strategy $(\mathcal{D}^\varepsilon(h), \Pi^\varepsilon(h))$ by:*

$$D^\varepsilon(X_s, s; h) = \begin{cases} D, & t \leq s < \varepsilon \wedge \tau, \\ \hat{D}(X_s, s; h), & \varepsilon \wedge \tau \leq s < \tau, \end{cases}$$

and

$$\pi^\varepsilon(X_s, s; h) = \begin{cases} \pi, & t \leq s < \varepsilon \wedge \tau, \\ \hat{\pi}(X_s, s; h), & \varepsilon \wedge \tau \leq s < \tau. \end{cases}$$

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The strategy $(\hat{\mathcal{D}}(h), \hat{\Pi}(h))$ *is said to be a time-consistent Stackelberg follower action if, for all* $(x, t) \in \mathbb{R} \times [0, \tau)$*,* $D \geq 0$ *and* $\pi \in \mathbb{R}$*,*

$$\liminf_{\varepsilon \to 0^+} \frac{J(x, t; \hat{\mathcal{D}}(h), \hat{\Pi}(h), h) - J(x, t; \mathcal{D}^\varepsilon(h), \Pi^\varepsilon(h), h)}{\varepsilon} \geq 0,$$

and the equilibrium value function V *equals*

$$V(x, t; h) = J(x, t; \hat{\mathcal{D}}(h), \hat{\Pi}(h), h). \tag{2.4}$$

Next, we define the insurance company's (the leader's) optimal action.

Definition 2.3 (Seller's optimal Stackelberg leader action). *The price* $\hat{h}$ *is the optimal Stackelberg leader action if, for* $(y, t) \in \mathbb{R} \times [0, \tau)$*,*

$$\hat{h} = \arg\sup_{h \geq 0} \mathbb{E}_{y,t}(Y_\tau), \tag{2.5}$$

in which Y *follows the process in* (2.2) *with* $(\mathcal{D}, \Pi) = (\hat{\mathcal{D}}(h), \hat{\Pi}(h))$*, and* $\mathbb{E}_{y,t}$ *denotes expectation conditional on* $Y_t = y \in \mathbb{R}$ *and* $\tau > t$*.*[3]

Finally, we are ready to define the Stackelberg equilibrium of this term life insurance game.

Definition 2.4 *(Stackelberg equilibrium of the term life insurance game). The Stackelberg equilibrium of the term life insurance game is the following collection of strategies:* $\hat{h} \geq 0$ *from* (2.5) *in Definition* 2.3 *and* $(\hat{\mathcal{D}}, \hat{\Pi}) = (\hat{\mathcal{D}}(\hat{h}), \hat{\Pi}(\hat{h}))$ *from Definition* 2.2.

2.2 Buyer's problem

Before we find the equilibrium follower action for the buyer, we present a verification lemma for solving the buyer's problem. For the statement of the verification lemma, we define a differential operator corresponding to our problem $\mathcal{L}^{(D,\pi)}$, for all $\phi \in \mathcal{C}^{2,1}(\mathbb{R} \times [0, \tau))$, $D \geq 0$, and $\pi \in \mathbb{R}$, by:

$$\mathcal{L}^{(D,\pi)} \phi(x, t) = \phi_t(x, t) + (c - hD + \mu\pi)\phi_x(x, t) + \frac{1}{2}\sigma^2\pi^2\phi_{xx}(x, t). \tag{2.6}$$

We omit the proof of the following verification lemma for $V(\cdot, \cdot; h)$ because it is similar to the proof of Theorem 3.1 in Landriault *et al*. (2018).

Lemma 2.1 *(Verification lemma). Fix a value of the premium rate* $h \geq 0$*. Suppose there exist two real-valued functions* $\tilde{V}(\cdot, \cdot\,; h) \in \mathcal{C}^{2,1}(\mathbb{R} \times [0, \tau))$ *and* $G(\cdot, \cdot\,; h) \in \mathcal{C}^{2,1}(\mathbb{R} \times [0, \tau))$ *that satisfy the following conditions:*

1. *For all* $(x, t) \in \mathbb{R} \times [0, \tau)$*,*

$$\sup_{D \geq 0,\, \pi \in \mathbb{R}} \left\{ \mathcal{L}^{(D,\pi)}\, \tilde{V}(x, t; h) - \frac{\gamma}{2}\, \mathcal{L}^{(D,\pi)}\, G^2(x, t; h) + \gamma G(x, t; h)\, \mathcal{L}^{(D,\pi)}\, G(x, t; h) \right.$$
$$\left. - \lambda \left(\tilde{V}(x, t; h) - (x + D) + \frac{\gamma}{2} \left(G(x, t; h) - (x + D) \right)^2 \right) \right\} = 0. \tag{2.7}$$

Let $\hat{D}(x, t; h)$ *and* $\hat{\pi}(x, t; h)$ *denote the maximizers of the above HJB equation. Suppose the induced strategy* $(\hat{\mathcal{D}}(h), \hat{\Pi}(h))$ *is admissible.*

2. *For all* $(x, t) \in \mathbb{R} \times [0, \tau)$*,*

$$\mathcal{L}^{(\hat{D},\hat{\pi})} G(x, t; h) = \lambda \big(G(x, t; h) - (x + \hat{D}(x, t; h)) \big). \tag{2.8}$$

[3] As noted in an earlier footnote, $\hat{h}$ will be a constant, so we assume that from the outset.

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3. *For all* $(x, t) \in \mathbb{R} \times [0, \tau)$, *the following transversality condition holds:*

$$\lim_{s \to \infty} \mathrm{e}^{-\lambda(s-t)}\, \mathbb{E}_{x,t}\big(\phi\big(X_s, s; h\big)\big) = 0,$$

for $\phi = \tilde{V}$, G, *and* G^2.

Then, $\tilde{V}(\cdot, \cdot\,; h)$ *equals the individual's equilibrium value function* $V(\cdot, \cdot\,; h)$ *defined in (2.4), and* $(\hat{\mathcal{D}}(h), \hat{\Pi}(h))$ *is an equilibrium strategy. Moreover,* $G(x, t; h) = \mathbb{E}_{x,t}(X_\tau)$ *under this equilibrium strategy.*

In the following theorem, we use Lemma 2.1 to obtain the buyer's equilibrium strategy and corresponding value function.

Theorem 2.2. *For a given value of the premium rate* $h \geq 0$, *the equilibrium strategy* $(\hat{\mathcal{D}}(h), \hat{\Pi}(h)) = \big(\{\hat{D}(X_t, t; h)\}, \{\hat{\pi}(X_t, t; h)\}\big)_{t \in [0,\tau)}$ *equals the constant strategy:*

$$\hat{D}(X_t, t; h) \equiv \hat{D}(h) = \frac{1}{\gamma h}\left(\gamma c - \left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)\right)_+, \tag{2.9}$$

$$\hat{\pi}(X_t, t; h) \equiv \hat{\pi} = \frac{\mu}{\gamma \sigma^2}, \tag{2.10}$$

and the corresponding value function equals

$$V(x, t; h) = x + B(h),$$

in which

$$B(h) = \frac{1}{\gamma \lambda}\left\{\gamma c - \gamma(h - \lambda)\hat{D}(h) - \frac{(h-\lambda)^2}{2\lambda} + \frac{\mu^2}{2\sigma^2}\right\}. \tag{2.11}$$

Moreover,

$$G(x, t; h) = \mathbb{E}_{x,t}(X_\tau) = x + b(h),$$

in which

$$b(h) = \hat{D}(h) + \frac{h - \lambda}{\gamma \lambda}.$$

Proof. Because the problem is time-homogeneous, the value function is time-independent. Moreover, from the mean-variance criterion and from the dynamics of X in (2.1), we deduce that the value function equals $x + B$ for some value of B that depends on h. Through this reasoning, we obtain

$$V(t, x; h) = x + B(h), \quad \text{and} \quad G(t, x; h) = x + b(h),$$

in which b is another value that depends on h. By substituting the above expressions and their derivatives into (2.7), the HJB equation becomes

$$\sup_{D \geq 0, \pi}\left\{(\mu\pi + c - hD) - \frac{\gamma}{2}\sigma^2\pi^2 - \lambda\Big(B - D + \frac{\gamma}{2}\,(b - D)^2\Big)\right\} = 0. \tag{2.12}$$

Note that the expression in curly brackets is concave with respect to π and D; graphically, it is a downward opening paraboloid. Thus, the critical points yield the maximizers. In particular, the first-order condition for π gives us the optimal investment amount in (2.10). Similarly, from (2.12), the first-order condition for $\hat{D}$ gives us

$$\lambda - h + \gamma\lambda(b - \hat{D}) = 0. \tag{2.13}$$

Moreover, by substituting $G(t, x; h) = x + b(h)$ into (2.8), we derive another equation for $\hat{D}$:

$$\mu\hat{\pi} + c - h\hat{D} = \lambda(b - \hat{D}). \tag{2.14}$$

By eliminating $b(h)$ from (2.13) and (2.14) and by solving for $\hat{D}$ (truncated to lie in $\mathbb{R}_+$), we obtain $\hat{D}$ as in (2.9). Furthermore, by substituting $\hat{D}$ and $\hat{\pi}$ into the HJB equation, we derive the expression for $B(h)$ in (2.11). The expression for $b(h)$ follows from (2.13).

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To check that the proposed value function is, indeed, the value function as defined in (2.4) and that the corresponding strategy is the equilibrium strategy, we verify the conditions given in Lemma 2.1. Conditions (i) and (ii) hold by construction. For the transversality condition, note that, under the candidate constant investment and life insurance strategy, the buyer's wealth process $X = \{X_t\}_{t\in[0,\tau)}$ in (2.1) reduces to a diffusion process with constant coefficients. Thus, Condition (iii) holds. Hence, we complete the proof. □

Theorem 2.2 shows that the time-consistent equilibrium life insurance and investment strategy is a constant strategy, as expected from the form of the problem.

Remark 2.1. It makes sense that $\hat{D}(h) = 0$ when h is large, specifically, greater than or equal to $\gamma c + \lambda + \frac{\mu^2}{\sigma^2}$. When $h < \gamma c + \lambda + \frac{\mu^2}{\sigma^2}$, $\hat{D}(h)$ equals the sum of three interesting terms:

1. $\frac{c}{h}$, which equals the present value of a perpetuity of c per year under a "force of discount" equal to h. Alternatively, if we were to ignore the second and third terms, then we would have $\hat{D} = c/h$ or $c = h\hat{D}$; in other words, the income c would exactly cover the death benefit of $\hat{D}$.
2. $-\frac{h-\lambda}{\gamma h}$, which equals the negative of the proportional risk loading in h relative to λ, adjusted for the buyer's risk aversion. Thus, when the proportional risk loading in the premium is larger or when the individual is less risk-averse, then the buyer's death benefit will be reduced by a larger amount. Conversely, when the proportional risk loading is smaller or when the individual is more risk-averse, then the death benefit will be reduced by a smaller amount. In the unlikely case that insurance is a "good deal," that is, $h \le \lambda$, then $\hat{D}$ is greater than c/h.
3. $\frac{\mu^2}{\gamma h \sigma^2}$, which equals the square of the Sharpe ratio, adjusted for the buyer's risk aversion and for the premium rate. Thus, when the return of the risky asset is large relative to its volatility, then individual will buy more life insurance.

2.3 Seller's problem

In this section, we solve the seller's problem, namely, to choose a value of $h \ge 0$ to maximize its expected wealth at time τ. To that end, we compute from (2.2) and (2.9)

$$\begin{aligned} U(y,t) &:= \sup_{h\ge 0} \mathbb{E}_{y,t}(Y_\tau) = \sup_{h\ge 0} \mathbb{E}_{y,t}(Y_{\tau-} - \hat{D}(h)) \\ &= \sup_{h\ge 0} \int_0^\infty \lambda e^{-\lambda t}(Y_t - \hat{D}(h))dt = \sup_{h\ge 0} \int_0^\infty \lambda e^{-\lambda t}(y + h\hat{D}(h)\cdot t - \hat{D}(h))dt \\ &= y + \sup_{h\ge 0}\left(\frac{h}{\lambda} - 1\right)\hat{D}(h). \end{aligned}$$

U is independent of time t, so we write $U(y)$ in place of $U(y, t)$.

Theorem 2.3. *The optimal premium rate equals*

$$\hat{h} = \lambda\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}}. \tag{2.15}$$

Proof. If $\hat{D}(h) = 0$, then $U(y) = y$, and the seller can achieve average wealth strictly greater than y by choosing $h \in (\lambda, \gamma c + \lambda + \mu^2/\sigma^2)$ to guarantee $\hat{D}(h) > 0$. Thus, without loss of generality, we assume that h is such that $\hat{D}(h) > 0$. (In Theorem 2.4, we confirm that $\hat{D}(\hat{h}) > 0$.)

Because $\hat{D}(h)$ in (2.9) is strictly positive, we have

$$\mathbb{E}_y(Y_\tau) = y + \frac{h-\lambda}{\gamma\lambda h}\left(\gamma c - (h-\lambda) + \frac{\mu^2}{\sigma^2}\right).$$

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Let f denote the second term in the above expression modulo a factor of $1/(\gamma\lambda)$, that is,

$$f(h) = \frac{h-\lambda}{h}\left(\gamma c - (h-\lambda) + \frac{\mu^2}{\sigma^2}\right),$$

so $\arg\sup_{h\geq 0}\mathbb{E}_y(Y_\tau) = \arg\sup_{h\geq 0} f(h)$. Simple calculus yields

$$f'(h) = \frac{\lambda^2 - h^2}{h^2} + \frac{\lambda}{h^2}\left(\gamma c + \frac{\mu^2}{\sigma^2}\right) \propto (\lambda^2 - h^2) + \lambda\left(\gamma c + \frac{\mu^2}{\sigma^2}\right) =: g(h),$$

and

$$g'(h) = -2h,$$

which implies that the critical point equals the maximizer. By solving $f'(h) = 0$, we obtain $\hat{h}$ as given in (2.15). ☐

2.4 Stackelberg equilibrium

From Theorems 2.2 and 2.3, we obtain the following theorem, which presents the Stackelberg equilibrium for the term life insurance game (with investment).

Theorem 2.4. *The Stackelberg equilibrium strategy and value functions of the term life insurance game with investment are given by the following:*

1. *The equilibrium premium rate (per dollar of death benefit) for the term life insurance equals*

$$\hat{h} = \lambda\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}}, \tag{2.16}$$

and the seller's expected wealth at time τ equals

$$U(y) = y + \frac{1}{\gamma}\left(\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}} - 1\right)^2. \tag{2.17}$$

2. *In response to the seller offering the premium rate $\hat{h}$ in (2.16), the buyer's time-consistent equilibrium death benefit equals the constant*

$$\hat{D}(\hat{h}) = \frac{1}{\gamma}\left(\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}} - 1\right). \tag{2.18}$$

3. *Independent of the seller's actions, the buyer's time-consistent equilibrium amount to invest in the risky asset equals the constant*

$$\hat{\pi} = \frac{\mu}{\gamma\sigma^2}, \tag{2.19}$$

and her value function equals

$$V(x) = x + \frac{1}{\gamma\lambda}\left\{\gamma c - \frac{3\lambda}{2}\left(\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}} - 1\right)^2 + \frac{\mu^2}{2\sigma^2}\right\}. \tag{2.20}$$

Proof. The expressions in (2.16) and (2.19) come directly from (2.15) and (2.10), respectively. To obtain (2.18), substitute the expression for $\hat{h}$ in (2.16) into $\hat{D}(h)$ in (2.9) and simplify the result. We obtain (2.17) and (2.20) similarly. ☐

In the following corollary, we analyze properties of the Stackelberg equilibrium strategy.

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Corollary 2.5.

1. *The equilibrium premium rate for term life insurance $\hat{h}$ increases with respect to λ, γ, c, and μ, and it decreases with respect to σ.*
2. *The equilibrium death benefit for term life insurance $\hat{D}(\hat{h})$ increases with respect to c and μ, and it decreases with respect to λ, γ, and σ.*
3. *The equilibrium amount to invest in the risky asset $\hat{\pi}$ increases with respect to μ, decreases with respect to γ and σ, and is independent of λ and c.*

Proof. These properties are easy to see from the forms of $\hat{h}$ in (2.16), $\hat{D}(\hat{h})$ in (2.18), and $\hat{\pi}(\hat{h})$ in (2.16), except for the relationship between the death benefit and the risk aversion parameter. To complete the proof of this corollary, we show that $\hat{D}(\hat{h})$ decreases with respect to γ:

$$\begin{aligned}\frac{\partial \hat{D}(\hat{h})}{\partial \gamma} &= -\frac{1}{\gamma^2}\left(\sqrt{1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}}-1\right)+\frac{1}{\gamma}\cdot\frac{\frac{c}{\lambda}}{2\sqrt{1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}}} \\ &\propto -2\left(1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}-\sqrt{1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}}\right)+\frac{\gamma c}{\lambda} \\ &= 2\sqrt{1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}}-2-\frac{\gamma c+\frac{2\mu^2}{\sigma^2}}{\lambda}.\end{aligned}$$

Thus, $\frac{\partial \hat{D}(\hat{h})}{\partial \gamma} < 0$ if and only if

$$\begin{aligned} &2\sqrt{1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}} < 2+\frac{\gamma c+\frac{2\mu^2}{\sigma^2}}{\lambda} \\ \Longleftrightarrow\ &4\left(1+\frac{\gamma c+\frac{\mu^2}{\sigma^2}}{\lambda}\right) < 4+4\,\frac{\gamma c+\frac{2\mu^2}{\sigma^2}}{\lambda}+\left(\frac{\gamma c+\frac{2\mu^2}{\sigma^2}}{\lambda}\right)^2 \\ \Longleftrightarrow\ &0 < \frac{4\mu^2}{\lambda\sigma^2}+\left(\frac{\gamma c+\frac{2\mu^2}{\sigma^2}}{\lambda}\right)^2,\end{aligned}$$

which is true. □

Note that $\hat{h} > \lambda$ and $\hat{D}(\hat{h}) > 0$; therefore, the seller's expected wealth at time τ is strictly greater than y, as expected. It makes sense that $\hat{h}$ increases with increasing λ because as the individual's mortality hazard rate increases, she is more likely to die, which means life insurance should be more expensive. It is intuitively pleasing that $\hat{h}$ increases with increasing c because as the individual's income increases, she is able to spend more for life insurance. The death benefit $\hat{D}(\hat{h})$ also increases with increasing c; thus, we see that the death benefit acts as a type of income replacement, which matches actuarial wisdom. Moreover, as the financial market becomes more favorable, that is, as μ increases or σ decreases, the individual invests more in the risky asset and uses some of those expected earnings to purchase more life insurance.

We are somewhat surprised that the equilibrium death benefit decreases with respect to γ because we expected the death benefit to increase with increasing risk aversion toward variance. However, there is variability associated with the effect of the death benefit upon X_τ through the time of death τ, which is the length of time the individual will have to pay premium. Thus, because increasing γ leads to an increase in premium, this increased premium decreases the value of X_τ and might add to the variance.

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That said, the amount invested in the risky asset behaves as one expects with respect to γ; that is, $\hat{\pi}$ decreases with increasing γ.

Note that $\hat{D}(\hat{h})$ and $\hat{\Pi}$ are independent of both X_t and t; that is, even though the buyer continually decides how much life insurance to purchase and how much money to invest in the risky asset (as she plays the mean-variance "game" against future versions of herself), she purchases a constant amount of life insurance and invests a constant amount. In other words, she effectively buys whole life insurance via a continuous premium $\hat{h}$, payable for life, and continually rebalances her stock portfolio to keep its balance constant. Thus, in the following section, we consider a Stackelberg game similar to the one in this section, except that we assume the individual purchases whole life insurance and invests a constant amount in the risky asset, choosing both at time 0 to maximize her mean-variance criterion at her time of death. In other words, we assume the buyer "pre-commits" to her constant strategy at time 0.

3. Whole life insurance game

In this section, we compute the Stackelberg equilibrium of a whole life insurance game with investment. We apply the same objective functions as those in Section 2, that is, the buyer maximizes her terminal wealth under the mean-variance criterion, and the seller maximizes the expectation of its wealth at time τ. The buyer's and seller's controlled wealth processes satisfy the stochastic differential equations given in (2.1) and (2.2), respectively. In Section 3.1, we solve the buyer's problem assuming constant pre-commitment on her part. Parallel to Sections 2.3 and 2.4, respectively, we solve the seller's problem in Section 3.2, and we present the Stackelberg equilibrium of the whole life insurance game in Section 3.3.

3.1 Buyer's problem

By contrast with Section 2, in which we solve the buyer's problem from a time-consistent perspective, in this section, we maximize the mean-variance objective J in (2.3) assuming the individual commits to buying a constant amount of whole life insurance and to investing a constant amount in the risky asset. Under a slight abuse of notation, we write $J(x; D, \pi, h)$ in place of $J(x, t; \mathcal{D}, \Pi, h)$ because $D_t = D$ and $\pi_t = \pi$ for all $t \in [0, \tau)$, for some constants $D \geq 0$ and $\pi \in \mathbb{R}$.[4]

We begin with a verification lemma that we use to obtain an explicit expression for the objective function J. Afterward computing J, we maximize it directly over all-non-negative D and π to obtain the optimal life insurance and investment strategy and the corresponding value function.

Lemma 3.1. *(Verification lemma). Fix a value of the premium rate $h \geq 0$ and values of the death benefit and investment amount $(D, \pi) \in \mathbb{R}_+ \times \mathbb{R}$. Suppose there exist two real-valued functions $\tilde{J}(\,\cdot\,; D, \pi, h) \in \mathcal{C}^2(\mathbb{R})$ and $g(\,\cdot\,; D, \pi, h) \in \mathcal{C}^2(\mathbb{R})$ that satisfy the following conditions:*

1. *For all $x \in \mathbb{R}$,*

$$\mathcal{L}^{(D,\pi)}\,\tilde{J}(x; D, \pi, h) - \frac{\gamma}{2}\,\mathcal{L}^{(D,\pi)}\,g^2(x; D, \pi, h) + \gamma\, g(x; D, \pi, h)\,\mathcal{L}^{(D,\pi)}\,g(x; D, \pi, h)$$
$$= \lambda\left(\tilde{J}(x; D, \pi, h) - (x + D) + \frac{\gamma}{2}\left(g(x; D, \pi, h) - (x + D)\right)^2\right), \tag{3.1}$$

   in which $\mathcal{L}^{(D,\pi)}$ is given in (2.6).
2. *For all $x \in \mathbb{R}$,*

$$\mathcal{L}^{(D,\pi)} g(x; D, \pi, h) = \lambda\big(g(x; D, \pi, h) - (x + D)\big). \tag{3.2}$$

[4] Under a constant strategy, the model considered in this Section is time-homogenous, so we remove the t variable in the expressions for the objective function J.

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3. *For all $x \in \mathbb{R}$, the following transversality condition holds:*

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\, \mathbb{E}_x\big(\phi\big(X_t; D, \pi, h\big)\big) = 0, \tag{3.3}$$

for $\phi = \tilde{J}$, g, and g^2.

Then, $\tilde{J}(\cdot\,; D, \pi, h)$ equals the individual's objective function $J(\cdot\,; D, \pi, h)$ defined in (2.3) for a constant strategy. Moreover, $g(x; D, \pi, h) = \mathbb{E}_x(X_\tau)$.

Proof. Suppose $\tilde{J}$ and g satisfy equations (3.1) and (3.2).

Step 1. We first show that $g(x; D, \pi, h) = \mathbb{E}_x(X_\tau) = \mathbb{E}_x\big(\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, \big(X_t + D\big)\, \mathrm{d}t\big)$. Let $k > 0$ be a fixed number. By applying Itô's formula to $\mathrm{e}^{-\lambda k}\, g\big(X_k; D, \pi, h\big)$, we obtain

$$\mathrm{e}^{-\lambda k}\, g(X_k; D, \pi, h) = g(x; D, \pi, h) + \int_0^k \mathrm{e}^{-\lambda t}\, \big(\mathcal{L}^{(D,\pi)} g(X_t; D, \pi, h) - \lambda g(X_t; D, \pi, h)\big)\, \mathrm{d}t$$
$$+ \int_0^k e^{-\lambda t} \sigma \pi\, g_x(X_t; D, \pi, h) dB_t.$$

By taking the expectation of the above expression, conditional on $X_0 = x$, we obtain

$$\begin{aligned}\mathbb{E}_x\Big[\mathrm{e}^{-\lambda k}\, g(X_k; D, \pi, h)\Big] &= g(x; D, \pi, h) + \mathbb{E}_x\Big[\int_0^k \mathrm{e}^{-\lambda t}\, \big(\mathcal{L}^{(D,\pi)} g(X_t; D, \pi, h) - \lambda g(X_t; D, \pi, h)\big)\, \mathrm{d}t\Big] \\ &= g(x; D, \pi, h) - \mathbb{E}_x\Big[\int_0^k \lambda\, \mathrm{e}^{-\lambda t}\, (X_t + D)\, \mathrm{d}t\Big],\end{aligned}$$

in which the second equality follows from (3.2), and in which the arguments of the integrals are interpreted via their left limits.

By letting $k \to \infty$ and using (3.3), we obtain

$$g(x; D, \pi, h) = \mathbb{E}_x\Big[\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, \big(X_t + D\big)\, \mathrm{d}t\Big] = \mathbb{E}_x(X_\tau).$$

Here, we applied Lebesgue's dominated convergence theorem:

$$\mathbb{E}_x\Big[\int_0^k \lambda\, \mathrm{e}^{-\lambda t}\, |X_t + D|\, \mathrm{d}t\Big] = \mathbb{E}_x\Big[\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, |X_t + D| 1_{\{t<k\}}\, \mathrm{d}t\Big] < \mathbb{E}_x\Big[\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, |X_t + D|\, \mathrm{d}t\Big] < \infty.$$

Step 2. In this step, we show that, if $\tilde{J}$ solves (2.7), then $J(x; D, \pi, h) = \tilde{J}(x; D, \pi, h)$. From equation (3.2), we rewrite (2.7) as follows:

$$\mathcal{L}^{(D,\pi)}\Big(\tilde{J}(x; D, \pi, h) - \frac{\gamma}{2}\, g^2(x; D, \pi, h)\Big) + \gamma\, g(x; D, \pi, h)\lambda\big(g(x; D, \pi, h) - (x + D)\big)$$

$$= \lambda\, \Big(\tilde{J}(x; D, \pi, h) - (x + D) + \frac{\gamma}{2}\, \big(g(x; D, \pi, h) - (x + D)\big)^2\Big),$$

or equivalently,

$$\begin{aligned}&\mathcal{L}^{(D,\pi)}\Big(\tilde{J}(x; D, \pi, h) - \frac{\gamma}{2}\, g^2(x; D, \pi, h)\Big) \\ &= \lambda\, \Big(\tilde{J}(x; D, \pi, h) - (x + D) - \frac{\gamma}{2}\, \big(g^2(x; D, \pi, h) - (x + D)^2\big)\Big).\end{aligned} \tag{3.4}$$

By following the proof of Step 1, we obtain both

$$\tilde{J}(x; D, \pi, h) = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-\lambda t}\, \big(\lambda \tilde{J}\big(X_t; D, \pi, h\big) - \mathcal{L}^{(D,\pi)} \tilde{J}\big(X_t; D, \pi, h\big)\big)\, \mathrm{d}t\Big], \tag{3.5}$$

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and

$$g^2(x; D, \pi, h) = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t}\left(\lambda g^2\left(X_t; D, \pi, h\right) - \mathcal{L}^{(D,\pi)} g^2\left(X_t; D, \pi, h\right)\right) dt\right]. \tag{3.6}$$

By linearly combining (3.5) and (3.6) to form $\tilde{J} - \gamma g^2/2$, and by using (3.4), we obtain the following expression:

$$\begin{aligned}
&\tilde{J}(x; D, \pi, h) - \frac{\gamma}{2} g^2(x; D, \pi, h) \\
&= \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t}\left(\lambda\left(\tilde{J}\left(X_t; D, \pi, h\right) - \frac{\gamma}{2} g^2(X_t; D, \pi, h)\right)\right.\right. \\
&\qquad\left.\left. - \mathcal{L}^{(D,\pi)}\left(\tilde{J}\left(X_t; D, \pi, h\right) - \frac{\gamma}{2} g^2(X_t; D, \pi, h)\right)\right) dt\right] \\
&= \mathbb{E}_x\left[\int_0^\infty \lambda e^{-\lambda t}\left(\left(X_t + D\right) - \frac{\gamma}{2}\left(X_t + D\right)^2\right) dt\right] \\
&= \mathbb{E}_x\left(X_\tau\right) - \frac{\gamma}{2}\mathbb{E}_x\left((X_\tau)^2\right),
\end{aligned}$$

which implies

$$\tilde{J}(x; D, \pi, h) = \mathbb{E}_x(X_\tau) - \frac{\gamma}{2}\mathbb{V}_x(X_\tau) = J(x; D, \pi, h),$$

and completes the proof of this lemma. □

By using Lemma 3.1, we obtain the objective function in the following theorem.

Theorem 3.2 *Under a constant pre-commitment strategy, the objective function J in (2.3) equals*

$$J(x; D, \pi, h) = x + A(D, \pi; h),$$

for all $(x, D, \pi) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, in which

$$A(D, \pi; h) = a(D, \pi; h) - \frac{\gamma}{2} v(D, \pi; h), \tag{3.7}$$

with

$$a(D, \pi; h) = D + \frac{c - hD + \mu\pi}{\lambda}, \tag{3.8}$$

and

$$v(D, \pi; h) = \frac{\sigma^2\pi^2}{\lambda} + \left(\frac{c - hD + \mu\pi}{\lambda}\right)^2. \tag{3.9}$$

Moreover,

$$g(x; D, \pi, h) = \mathbb{E}_x(X_\tau) = x + a(D, \pi; h),$$

for all $(x, D, \pi) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, in which a is given in (3.8).

Proof. Because the dynamics for X in (2.1) are independent of initial wealth, and because we are considering the mean-variance criterion, we hypothesize that $J(x; D, \pi, h) = x + A(D, \pi; h)$ and $g(x; D, \pi, h) = x + a(D, \pi; h)$ for some functions A and a on $\mathbb{R}_+ \times \mathbb{R}$. By substituting $g(x; D, \pi, h) = x + a(D, \pi; h)$ into (3.2) (and by omitting the dependence of a on D and π for simplicity), we obtain

$$c - hD + \mu\pi = \lambda(a - D), \tag{3.10}$$

which leads to the expression for $a(D, \pi; h)$ in (3.8).

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Next, consider the equation in (3.1). Under our ansatz $J(x; D, \pi, h) = x + A(D, \pi; h)$ and $g(x; D, \pi, h) = x + a(D, \pi; h)$, we obtain

$$\mathcal{L}^{(D,\pi)}\, J(x; D, \pi, h) = \mathcal{L}^{(D,\pi)}\, g(x; D, \pi, h) = c - hD + \mu\pi, \tag{3.11}$$

and

$$\mathcal{L}^{(D,\pi)}\, g^2(x; D, \pi, h) = 2(c - hD + \mu\pi)(x + a) + \sigma^2\pi^2. \tag{3.12}$$

By substituting (3.11) and (3.12) into (3.1) and by using (3.10), we get

$$A = a - \frac{\gamma}{2}\left(\frac{\sigma^2\pi^2}{\lambda} + (a - D)^2\right),$$

which implies the expressions for $A(D, \pi; h)$ in (3.7) and $\nu(D, \pi; h)$ in (3.9).

By construction $J(x; D, \pi; h) = x + A(D, \pi; h)$ and $g(x; D, \pi; h) = x + a(D, \pi; h)$ satisfy (2.7) and (3.2). Thus, it remains to prove the transversality condition in (3.3) for $\phi = J$, g, and g^2. First, note that

$$X_t = x + (\mu\pi + c - hD)t + \sigma\pi B_t.$$

Thus,

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\left(\mathbb{E}_x(X_t) + A(D, \pi; h)\right) = 0 = \lim_{t\to\infty} \mathrm{e}^{-\lambda t}\left(\mathbb{E}_x(X_t) + a(D, \pi; h)\right),$$

thereby proving (3.3) for $\phi = J$ and g. Similarly, $\mathbb{E}_x\big(g^2(X_t; D, \pi, h)\big)$ behaves like t^2 as t approaches infinity, which is dominated by $\mathrm{e}^{-\lambda t}$; thus, (3.3) also holds for $\phi = g^2$. It follows from Lemma 3.1 that J and g are as stated. □

In the following theorem, we present the optimal amount of investment and life insurance that the individual will buy.

Theorem 3.3. *For a given value of the premium rate $h \ge 0$, the optimal constant pre-commitment amount of life insurance equals*

$$D^*(h) = \frac{1}{\gamma h}\left(\gamma c - \frac{\lambda}{h}\left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)\right)_+, \tag{3.13}$$

and the optimal constant pre-commitment amount to invest in the risky asset equals

$$\pi^*(h) = \begin{cases} \dfrac{\lambda}{h}\cdot\dfrac{\mu}{\gamma\sigma^2}, & \text{if } \ \gamma c > \dfrac{\lambda}{h}\left((h-\lambda) - \dfrac{\mu^2}{\sigma^2}\right) \iff D^*(h) > 0, \\[2ex] \dfrac{\mu}{\mu^2 + \lambda\sigma^2}\left(\dfrac{\lambda}{\gamma} - c\right), & \text{if } \ \gamma c \le \dfrac{\lambda}{h}\left((h-\lambda) - \dfrac{\mu^2}{\sigma^2}\right) \iff D^*(h) = 0. \end{cases} \tag{3.14}$$

Proof. First, we maximize $J(x; D, \pi, h)$ (or equivalently, $A(D, \pi; h)$) with respect to $\pi \in \mathbb{R}$ to obtain the maximizer $\pi^*(h; D)$; then, we maximize $A(D, \pi^*(h; D); h)$ with respect to $D \in \mathbb{R}_+$. To that end, define j by:

$$j(D, \pi) = \lambda A(D, \pi; h) = \lambda D + (c - hD + \mu\pi) - \frac{\gamma}{2}\left(\sigma^2\pi^2 + \frac{(c - hD + \mu\pi)^2}{\lambda}\right), \tag{3.15}$$

in which we omit the dependence on h for simplicity.

By differentiating $j(D, \pi)$ in (3.15) with respect to π, we obtain

$$\frac{\partial j(D, \pi)}{\partial \pi} = -\gamma\left(\sigma^2 + \frac{\mu^2}{\lambda}\right)\pi + \mu\left(1 - \frac{\gamma(c - hD)}{\lambda}\right),$$

with the second derivative strictly negative. Thus, the maximizer $\pi^*(h; D)$ equals the critical point, that is,

$$\pi^*(h; D) = \frac{\mu}{\mu^2 + \lambda\sigma^2}\cdot\frac{\lambda - \gamma c + \gamma hD}{\gamma}.$$

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Next, define $\mathcal{k}$ by

$$\mathcal{k}(D) = j(D, \pi^*(h; D)).$$

By differentiating $\mathcal{k}$, we obtain

$$\begin{aligned}\mathcal{k}'(D) &= \frac{\partial j(D, \pi^*(h; D))}{\partial D} + \frac{\partial j(D, \pi^*(h; D))}{\partial \pi} \cdot \frac{\partial \pi^*(h; D)}{\partial D} \\ &= \frac{\partial j(D, \pi^*(h; D))}{\partial D} \\ &= (\lambda - h) + \frac{\gamma h}{\lambda} \left(c - hD + \mu \pi^*(h; D)\right),\end{aligned}$$

and the second derivative is strictly negative. Indeed, the second derivative equals

$$\mathcal{k}''(D) = -\frac{\gamma h^2}{\lambda} + \frac{\gamma h \mu}{\lambda} \cdot \frac{\mu h}{\mu^2 + \lambda \sigma^2} = -\frac{\gamma h^2 \sigma^2}{\mu^2 + \lambda \sigma^2} < 0.$$

Thus, the maximizer $D^*(h)$ equals the critical point truncated to lie in $\mathbb{R}_+$. By substituting for $\pi^*(h; D)$ in $\mathcal{k}'(D) = 0$ and solving for D, we get (3.13). Finally, we compute $\pi^*(h) = \pi^*(h; D^*(h))$ and obtain (3.14). □

Remark 3.1. Compare the two expressions for the time-consistent equilibrium and optimal constant pre-commitment life insurance, respectively, $\hat{D}(h)$ in (2.9) and $D^*(h)$ in (3.13):

$$\hat{D}(h) = \frac{1}{\gamma h} \left(\gamma c - \left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)\right)_+,$$

and

$$D^*(h) = \frac{1}{\gamma h} \left(\gamma c - \frac{\lambda}{h} \left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)\right)_+.$$

Both begin with the term c/h, which we discuss in Remark 2.1, and they both adjust c/h by $\frac{1}{\gamma h}\left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)$, except that D^* applies an additional factor of λ/h to this adjustment term. We generally expect $h > \lambda$; if so, in the pre-commitment setting, the buyer adjusts c/h by a smaller absolute amount than in the time-consistent setting. When $0 < h - \lambda < \frac{\mu^2}{\sigma^2}$, then $\hat{D}(h) > D^*(h)$; in other words, when the premium loading for life insurance is less than the square of the Sharpe ratio, then the buyer purchases more term life insurance than whole life insurance. Also, in this case, the death benefits $\hat{D}(h)$ and $D^*(h)$ are both greater than c/h because life insurance is a good deal, as measured by a relatively small value of the premium loading $h - \lambda$. On the other hand, when the premium loading is greater than the square of the Sharpe ratio, the buyer purchases less term life insurance than whole life insurance, and in this case, both death benefits are less than c/h because life insurance is not as good a deal due to the larger premium loading.

Similarly, compare the investment in the risky asset for the time-consistent equilibrium and for the optimal constant pre-commitment setting, respectively, $\hat{\pi}(h)$ in (2.10) and $\pi^*(h)$ in (3.14):

$$\hat{\pi}(h) = \frac{\mu}{\gamma \sigma^2},$$

and, when $\gamma c > \frac{\lambda}{h}\left((h - \lambda) - \frac{\mu^2}{\sigma^2}\right)$,

$$\pi^*(h) = \frac{\lambda}{h} \cdot \frac{\mu}{\gamma \sigma^2}.$$

Again, in the pre-commitment setting, we see this additional factor of λ/h, and if $h > \lambda$ (as we expect in equilibrium), then $\pi^*(h) < \hat{\pi}(h)$. In other words, when $h > \lambda$, the buyer will invest more in the risky asset when buying term life insurance than when buying whole life insurance. Because investment is risky

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and a life insurance death benefit will be paid upon death with probability 1, under the pre-commitment setting, the individual is less willing to commit to investing a larger amount in the risky asset than under the time-consistent setting.

3.2 Seller's problem

For the seller's problem, recall the computations in Section 2.3, and define K by:

$$K(y) := \sup_{h\geq 0} \mathbb{E}_y(Y_\tau) = y + \sup_{h\geq 0} \left(\frac{h}{\lambda} - 1\right) D^*(h). \tag{3.16}$$

Theorem 3.4. *When $\gamma c < 2\lambda + \frac{\mu^2}{\sigma^2}$, the optimal premium rate equals*

$$h^* = \frac{2\lambda\left(\lambda + \frac{\mu^2}{\sigma^2}\right)}{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}. \tag{3.17}$$

When $\gamma c \geq 2\lambda + \frac{\mu^2}{\sigma^2}$, the supremum in (3.16) is attained at $h^* = \infty$.

Proof. If $D^*(h) = 0$, then $K(y) = y$, and the seller can achieve average wealth strictly greater than y by choosing $h = \lambda + \varepsilon$ (for some $\varepsilon > 0$ small enough) to guarantee $D^*(h) > 0$. Thus, without loss of generality, we assume that h is such that $D^*(h) > 0$. (In Theorem 3.5 below, we confirm that $D^*(h^*) > 0$.)

Because $D^*(h)$ in (3.13) is strictly positive, we have

$$\mathbb{E}_y(Y_\tau) = y + \frac{h-\lambda}{\gamma\lambda h}\left(\gamma c - \frac{\lambda(h-\lambda)}{h} + \frac{\lambda}{h}\cdot\frac{\mu^2}{\sigma^2}\right). \tag{3.18}$$

Let ϖ denote the second term in the above expression modulo a factor of $1/(\gamma\lambda)$, that is,

$$\varpi(h) = \frac{h-\lambda}{h}\left(\gamma c - \frac{\lambda(h-\lambda)}{h} + \frac{\lambda}{h}\cdot\frac{\mu^2}{\sigma^2}\right).$$

By differentiating ϖ, we obtain

$$\begin{aligned}\varpi'(h) &= \frac{\lambda}{h^2}\left(\gamma c - \frac{\lambda(h-\lambda)}{h} + \frac{\lambda}{h}\cdot\frac{\mu^2}{\sigma^2}\right) + \frac{h-\lambda}{h}\left(-\frac{\lambda^2}{h^2} - \frac{\lambda}{h^2}\cdot\frac{\mu^2}{\sigma^2}\right)\\ &\propto \gamma c - 2\,\frac{\lambda(h-\lambda)}{h} - \frac{h-2\lambda}{h}\cdot\frac{\mu^2}{\sigma^2} =: \ell(h),\end{aligned}$$

and

$$\ell'(h) = -2\,\frac{\lambda^2}{h^2} - \frac{2\lambda}{h^2}\cdot\frac{\mu^2}{\sigma^2} < 0,$$

which implies that the critical point in $\mathbb{R}_+$ equals the maximizer, if it exists. The critical point in $\mathbb{R}_+$ exists if and only if $\gamma c < 2\lambda + \frac{\mu^2}{\sigma^2}$. In this case, $\varpi'(h) = 0$ has a solution in $\mathbb{R}_+$, which is given in (3.17).

On the other hand, when $\gamma c \geq 2\lambda + \frac{\mu^2}{\sigma^2}$, the function $\varpi(h)$ is strictly increasing for $h \in \mathbb{R}_+$, so the supremum is achieved at $h^* = \infty$. □

3.3 Stackelberg equilibrium

From Theorems 3.3 and 3.4, we obtain the following theorem, which presents the Stackelberg equilibrium for the whole life insurance game (with investment).

Theorem 3.5. *The Stackelberg equilibrium of the whole life insurance game with investment is given as follows:*

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1. *If* $\gamma c < 2\lambda + \mu^2/\sigma^2$, *then*

   a. *The equilibrium premium rate (per dollar of death benefit) for the whole life insurance equals*

$$h^* = \frac{2\lambda\left(\lambda + \frac{\mu^2}{\sigma^2}\right)}{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}, \tag{3.19}$$

   and the seller's expected wealth at time τ *equals*

$$K(y) = y + \frac{\left(\gamma c + \frac{\mu^2}{\sigma^2}\right)^2}{4\gamma\lambda\left(\lambda + \frac{\mu^2}{\sigma^2}\right)}.$$

   b. *In response to the seller offering the premium rate* h^* *in (3.19), the buyer pre-commits to purchasing the following constant amount of whole life insurance:*

$$D^*(h^*) = \frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{4\gamma\lambda} \cdot \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda + \frac{\mu^2}{\sigma^2}},$$

   and pre-commits to investing the following constant amount in the risky asset:

$$\pi^*(h^*) = \frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{2\left(\lambda + \frac{\mu^2}{\sigma^2}\right)} \cdot \frac{\mu}{\gamma\sigma^2}.$$

   In this case, the buyer's value function equals

$$J(x; D^*(h^*), \pi^*(h^*), h^*) = x + G,$$

   in which G equals

$$\begin{aligned} G &= \frac{c}{h^*} + \frac{1}{2\gamma}\left(\frac{h^* - \lambda}{h^*}\right)^2 + \frac{\lambda}{2\gamma(h^*)^2} \cdot \frac{\mu^2}{\sigma^2} \\ &= \frac{1}{8\gamma\lambda}\left\{4\gamma c \cdot \frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{\lambda + \frac{\mu^2}{\sigma^2}} + \lambda\left(\frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda + \frac{\mu^2}{\sigma^2}}\right)^2 + \left(\frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{\lambda + \frac{\mu^2}{\sigma^2}}\right)^2 \frac{\mu^2}{\sigma^2}\right\}. \end{aligned}$$

2. *If* $\gamma c \geq 2\lambda + \mu^2/\sigma^2$, *then the insurance market collapses as follows:*

   a. *The equilibrium premium rate (per dollar of death benefit) for the whole life insurance is arbitrarily large. Indeed, the supremum in (3.16) occurs at*

$$h^* = \infty,$$

   and, in the limit, the seller's expected wealth at time τ *equals*

$$K(y) = y + \frac{\gamma c - \lambda}{\gamma\lambda}.$$

   b. *In response to the seller offering an arbitrarily large premium rate, the buyer purchases an infinitesimally small amount of whole life insurance and invests an infinitesimally small amount in the risky asset. In the limit, the buyer's value function equals*

$$J(x; D^*(h^*), \pi^*(h^*), h^*) = x + \frac{1}{2\gamma}.$$

We acknowledge that the premium rate of $h^* = \infty$ when $\gamma c \geq 2\lambda + \mu^2/\sigma^2$ is not realistic, and in the following corollary, we offer an ε-optimal strategy.

Corollary 3.6. *Suppose $\gamma c \geq 2\lambda + \mu^2/\sigma^2$, and let $\varepsilon > 0$ be arbitrary. Then, there exists $M(\varepsilon) > 0$ such that, if $h > M(\varepsilon)$, then*

$$\left(\frac{h}{\lambda} - 1\right) D^*(h) + \varepsilon > \frac{\gamma c - \lambda}{\gamma \lambda},$$

that is, the expected wealth of the seller at time τ when offering the premium h is within ε of the optimal expected wealth.

Proof. When $\gamma c \geq 2\lambda + \mu^2/\sigma^2$, the expectation of Y_τ in (3.18) increases with respect to h on $\mathbb{R}_+$ and approaches $y + (\gamma c - \lambda)/\gamma\lambda$ as h approaches infinity. Thus, the statement of the corollary holds when $M(\varepsilon)$ solves

$$\left(\frac{M(\varepsilon)}{\lambda} - 1\right) D^*(M(\varepsilon)) + \varepsilon = \frac{\gamma c - \lambda}{\gamma \lambda},$$

specifically,

$$M(\varepsilon) = \frac{1}{2\gamma\varepsilon}\left[\left(\gamma c - 2\lambda - \frac{\mu^2}{\sigma^2}\right) + \sqrt{\left(\gamma c - 2\lambda - \frac{\mu^2}{\sigma^2}\right)^2 + 4\gamma\lambda\varepsilon\left(\lambda + \frac{\mu^2}{\sigma^2}\right)}\right].$$

□

Remark 3.2. Alternatively, suppose the premium rate h is constrained to lie in $[0, \bar{h}]$, in which $\bar{h}$ is an upper bound for h and $\bar{h} > \lambda$. Then, when $\gamma c \geq 2\lambda + \mu^2/\sigma^2$, the equilibrium premium rate $h^* = \bar{h}$, and the seller's expected wealth at time τ equals

$$K(y) = y + \frac{\bar{h} - \lambda}{\gamma\lambda\bar{h}}\left(\gamma c - \frac{\lambda(\bar{h} - \lambda)}{\bar{h}} + \frac{\lambda}{\bar{h}} \cdot \frac{\mu^2}{\sigma^2}\right).$$

In response, the buyer pre-commits to purchasing the following constant amount of whole life insurance:

$$D^*(h^*) = \frac{1}{\gamma\bar{h}}\left(\gamma c - \frac{\lambda}{\bar{h}}\left((\bar{h} - \lambda) - \frac{\mu^2}{\sigma^2}\right)\right) > 0,$$

and pre-commits to investing the following constant amount in the risky asset:

$$\pi^*(h^*) = \frac{\lambda}{\bar{h}} \cdot \frac{\mu}{\gamma\sigma^2}.$$

In this case, the buyer's value function equals

$$J(x; D^*(h^*), \pi^*(h^*), h^*) = x + \frac{c}{\bar{h}} + \frac{1}{2\gamma}\left(\frac{\bar{h} - \lambda}{\bar{h}}\right)^2 + \frac{\lambda}{2\gamma(\bar{h})^2} \cdot \frac{\mu^2}{\sigma^2}.$$

In the following corollary, we show how the Stackelberg equilibrium varies with some of the parameters. We omit the proof because the properties are straightforward to demonstrate from the expressions in Theorem 3.5.

Corollary 3.7. *Suppose $\gamma c < 2\lambda + \mu^2/\sigma^2$.*

1. *The equilibrium premium rate for whole life insurance h^* increases with respect to γ and c. Moreover, if $\lambda - \gamma c > 0$ (< 0), then h^* increases (decreases) with respect to μ and decreases (increases) with respect to σ. Also, if $\lambda - \gamma c > 0$, then h^* increases with respect to λ, but if γc is close enough to $2\lambda + \mu^2/\sigma^2$, then h^* decreases with respect to λ.*
2. *The equilibrium death benefit of whole life insurance $D^*(h^*)$ increases with respect to μ, and it decreases with respect to γ and σ. Moreover, if $\lambda - \gamma c > 0$ (< 0), then $D^*(h^*)$ increases (decreases) with respect to c.*

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3. *The equilibrium amount invested in the risky asset* $\pi^*(h^*)$ *decreases to 0 as* γc *increases to* $2\lambda + \mu^2/\sigma^2$. *Moreover,* $\pi^*(h^*)$ *increases with respect to* μ, *and it decreases with respect to* γ *and* σ.

As for the Stackelberg equilibrium for term life insurance, the equilibrium premium rate increases with increasing risk aversion, as measured by γ, and with increasing income rate c. However, the behavior of the premium rate with respect to μ/σ depends on whether $\gamma c < \lambda$ or $\gamma c > \lambda$. If γc is small enough, then h^* increases with more favorable financial markets, as measured by increasing values of μ/σ, and *vice versa* if γc is large enough. Also, the behavior of the premium rate with respect to λ also depends on the magnitude of γc; if γc is small enough, then h^* increases with increasing λ.

Similarly to the equilibrium death benefit for term life insurance, $D^*(h^*)$ increases with more favorable financial markets and decreases with increasing risk aversion. As we discuss following Corollary 2.5, an increase in γ allows for an increase in h^*, which leads to a decrease in the death benefit. But, the equilibrium amount invested in the risky asset decreases with increasing risk aversion or decreasing financial markets, which one expects.

In the next section, we perform numerical experiments to compare the time-consistent equilibrium controls (for term life insurance plus investment) with the constant pre-commitment equilibrium controls (for whole life insurance plus investment).

4. Numerical examples

Let $\hat{D} = \hat{D}(\hat{h})$ denote the term life insurance amount for the time-consistent equilibrium, and let $D^* = D^*(h^*)$ denote the whole life insurance amount for the constant pre-commitment equilibrium when $\gamma c < 2\lambda + \mu^2/\sigma^2$, that is,

$$\hat{D} = \frac{1}{\gamma}\left(\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}} - 1\right) \qquad \text{and} \qquad D^* = \frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{4\gamma\lambda} \cdot \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda + \frac{\mu^2}{\sigma^2}}.$$

Also, the corresponding equilibrium premium rates equal

$$\hat{h} = \lambda\sqrt{1 + \frac{\gamma c + \frac{\mu^2}{\sigma^2}}{\lambda}} \qquad \text{and} \qquad h^* = \frac{2\lambda\left(\lambda + \frac{\mu^2}{\sigma^2}\right)}{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c},$$

and the corresponding equilibrium investments in the risky asset equal

$$\hat{\pi} = \frac{\mu}{\gamma\sigma^2} \qquad \text{and} \qquad \pi^* = \frac{2\lambda + \frac{\mu^2}{\sigma^2} - \gamma c}{2\left(\lambda + \frac{\mu^2}{\sigma^2}\right)} \cdot \frac{\mu}{\gamma\sigma^2}.$$

Note that $\pi^* < \hat{\pi}$, but one cannot order either $\hat{D}$ and D^* or $\hat{h}$ and h^*, as we discovered in our numerical work.

For our graphs, we use as a base case the following parameter values: $\lambda = 0.04$, $\mu = 0.08$, $\sigma = 0.20$, $c = 2$, and $\gamma = 0.04$; and we allow γ or λ to vary such that $0 \le \gamma c < 2\lambda + \mu^2/\sigma^2$. In Figure 1, we plot the graph of the premium rates $\hat{h}$ for term life insurance and h^* for whole life insurance over different values of γ. The graph shows that the premium rate h^* is first less than $\hat{h}$ and then greater as γ increases, which implies that as the buyer becomes much more risk-averse, the equilibrium price of term life insurance becomes cheaper than that of whole life insurance. In Figure 2, we plot the premium rates by varying λ between 0 and 0.2. We observe that when λ is small, the equilibrium price of the term life insurance is more expensive than that of the whole life insurance. However, when λ is larger, that is, the buyer is less healthy, the price of the whole life is greater than the price for term life. Note that both h^* and $\hat{h}$ are greater than λ, as expected.

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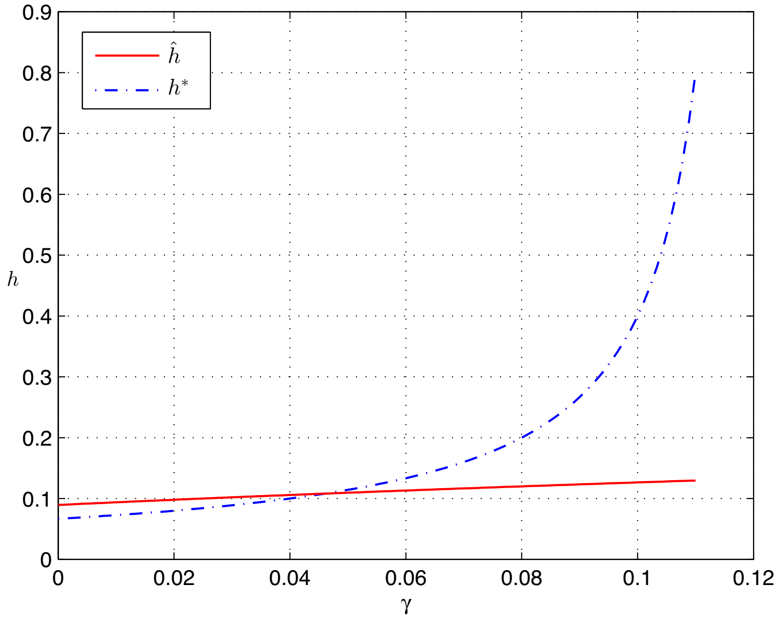

Figure 1. *$\hat{h}$ versus h^* over γ.*

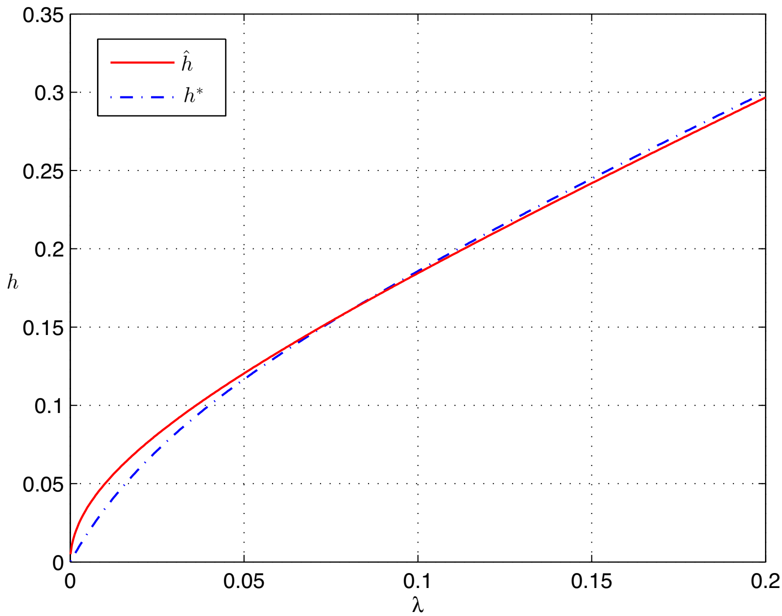

Figure 2. *$\hat{h}$ versus h^* over λ.*

In Figures 3 and 4, we plot the graphs of the term life insurance amount $\hat{D}$ and the whole insurance amount D^* over different values of γ and λ, respectively. The graphs show that the equilibrium life insurance amounts $\hat{D}$ and D^* are both decreasing and convex with respect to γ and λ. We are unable to demonstrate analytically that D^* decreases with respect to λ, but we see that it holds for this numerical example. Under the particular parameter values, from Figure 3, we observe that the term life insurance amount $\hat{D}$ is larger than the whole life insurance amount D^*. However, in Figure 4, the value of D^* is larger than $\hat{D}$ when λ is small, which demonstrates that we cannot order D^* and $\hat{D}$.

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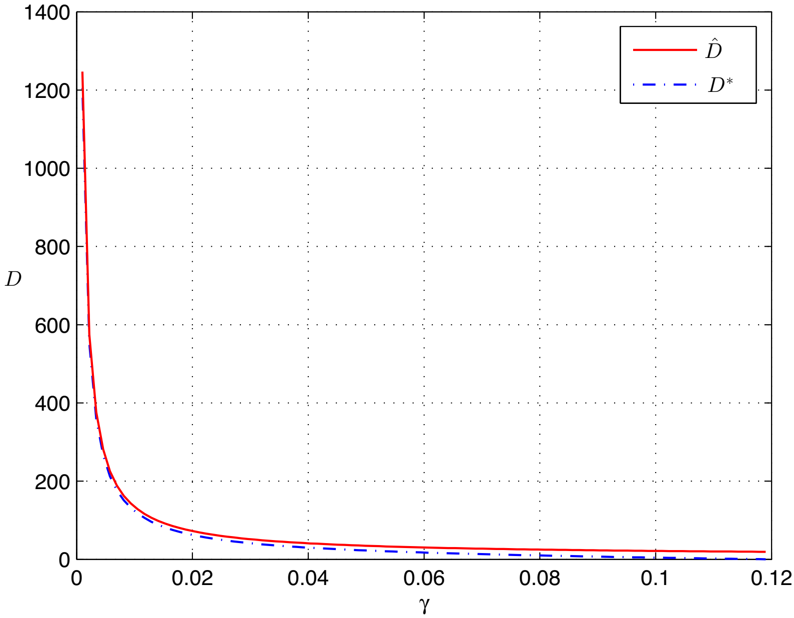

Figure 3. *$\hat{D}$ versus D^* over γ.*

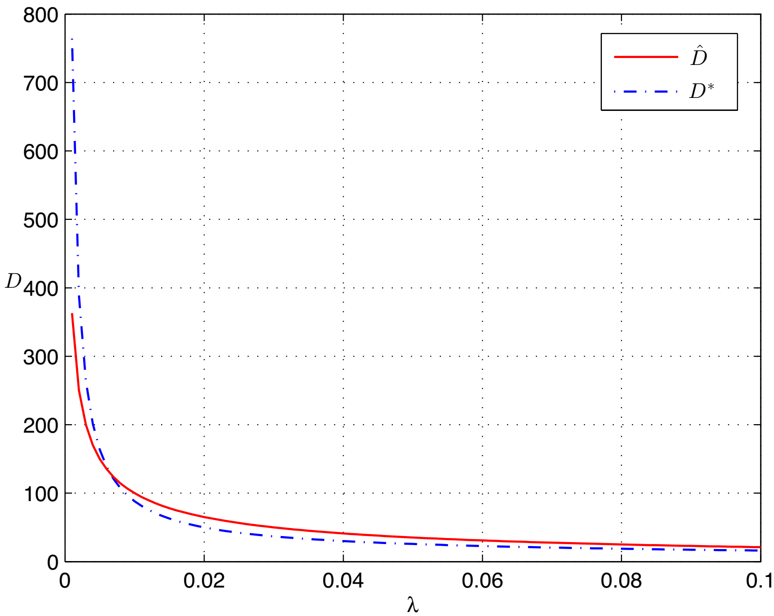

Figure 4. *$\hat{D}$ versus D^* over λ.*

In Figure 5, we plot the equilibrium investment amounts $\hat{\pi}$ and π^* over different values of γ. As expected from Corollaries 2.5 and 3.7, the equilibrium investment amounts decrease as the risk aversion γ increases.

In Figures 6 and 7, we plot the graphs of the seller's value function $U(0)$ for the term life insurance game and $K(0)$ for the whole life insurance game, with initial wealth 0, over γ and λ, respectively. We observe that the expected wealth $U(0)$ for the term life insurance game is always greater than the

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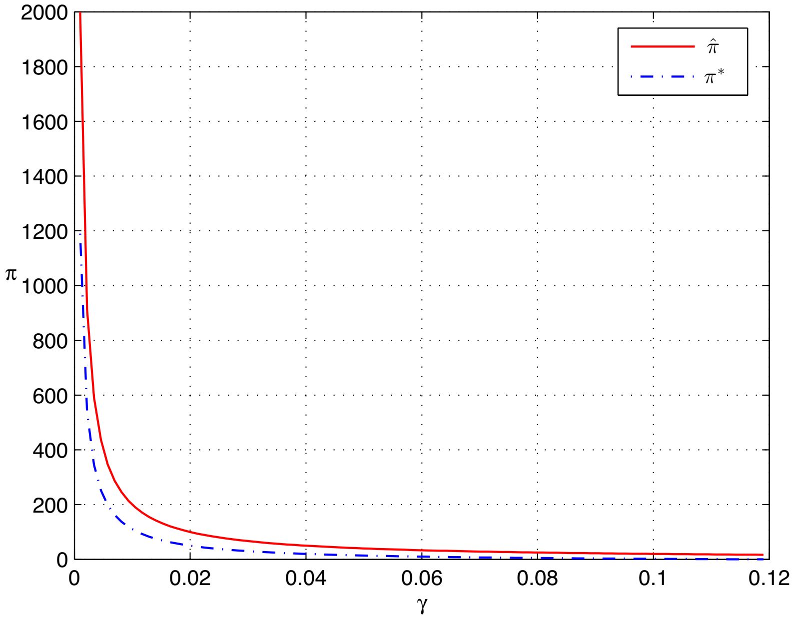

Figure 5. *$\hat{\pi}$ versus π^* over γ.*

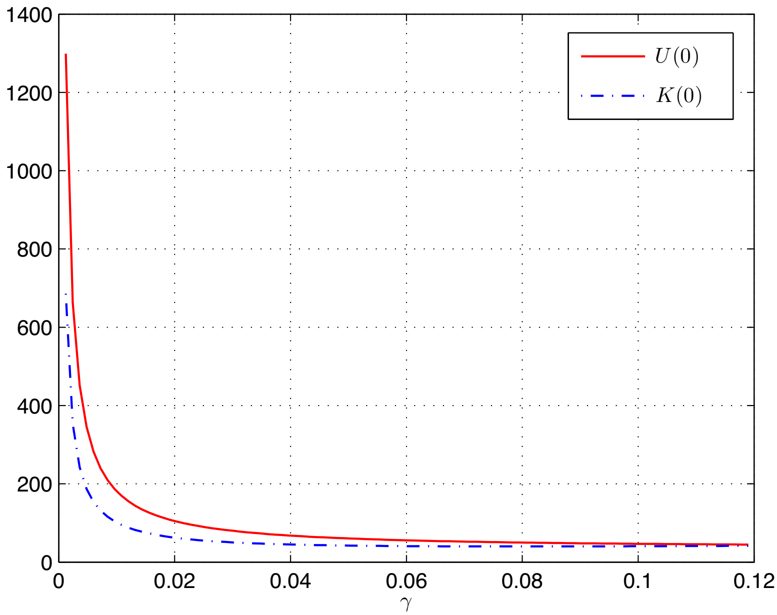

Figure 6. *U(0) versus K(0) over γ.*

expected wealth $K(0)$ for the whole life insurance game. Moreover, both $U(0)$ and $K(0)$ are decreasing with respect to γ and λ, respectively.

In Figures 8 and 9, we plot the graphs of the buyer's value function $V(0)$ for the term life insurance game and $J(0)$ for the whole life insurance game over γ and λ, respectively. In contrast to the seller's scenarios, the buyer's value function $J(0)$ for the whole life insurance game is greater than her value function $V(0)$ for the term life insurance game. For the whole life insurance game, the buyer pre-commits to using the constant strategy and thereby gains value. It is not surprising, then, that the seller loses

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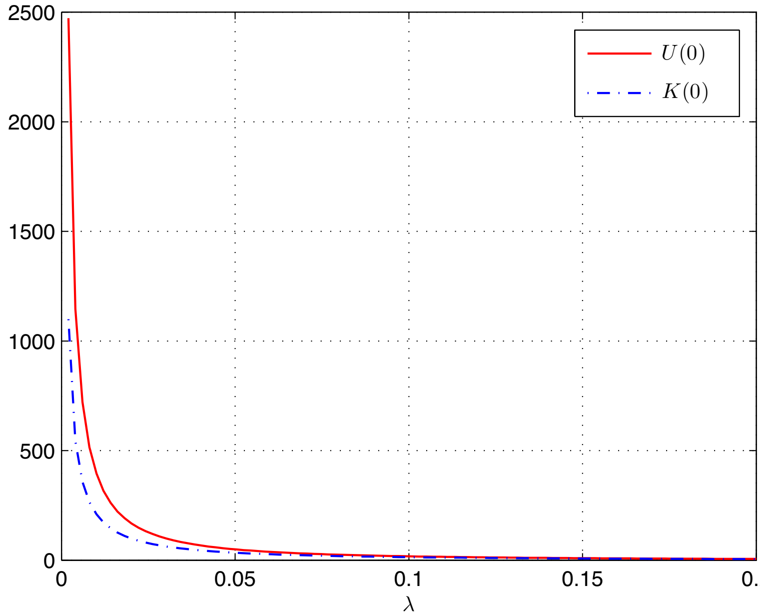

Figure 7. *U(0) versus K(0) over λ.*

value for the whole life insurance game (versus the term life game). The game is not zero-sum because of the additional variance term in the buyer's objective function, but in general a gain in value for the buyer will correspond to some sort of loss for the seller. In Figure 2, for the parameters, we always have $\gamma c = 0.08$ is (strictly) less than $2\lambda + \mu^2/\sigma^2 = 2\lambda + 0.16$, and h^* increases with respect to λ, as we expect from Item 1 in Corollary 3.7. For Figure 10, we set $c = 4.1$ and vary the values of λ from 0.0021 to 0.2. In this case, $\gamma c = 0.1640$ and the minimum value of $2\lambda + \mu^2/\sigma^2$ equals 0.1642; thus, γc is close enough to $2\lambda + \mu^2/\sigma^2$ for h^* to first decrease, and then increase with respect to λ, as we show in Corollary 3.7.

5. Concluding remarks

In this paper, we considered two continuous-time Stackelberg equilibrium life insurance games: one involving term life insurance and the other, whole life insurance. In both games, the buyer receives a fix income, invests in a risky asset, and purchases life insurance. We assumed that the buyer applies the mean-variance criterion to her wealth at her time of death, namely, at time τ. We solved the buyer's problem in the term life insurance game from a time-consistent perspective (as in, e.g., Landriault *et al*. (2018)), that is, the buyer continuously decides how much life insurance to purchase and how much money to invest into the risky asset. By solving the extended HJB equations, we found the explicit equilibrium control strategy and the value functions for the buyer in the term life insurance game.

We showed that the buyer purchases a constant amount of term life insurance and invests a constant amount in the risky asset. In light of this result, for the whole life insurance game, we assumed that the buyer always purchases a constant amount of life insurance and invests a constant amount in the risky asset. To find the optimal constant pre-commitment strategy for the buyer in the whole life insurance game, we modified the extended HJB equations for the term life game to this problem and found an explicit expression for the buyer's objective function J and, then, maximized J over all amounts of whole life death benefit and investment in the risky asset.

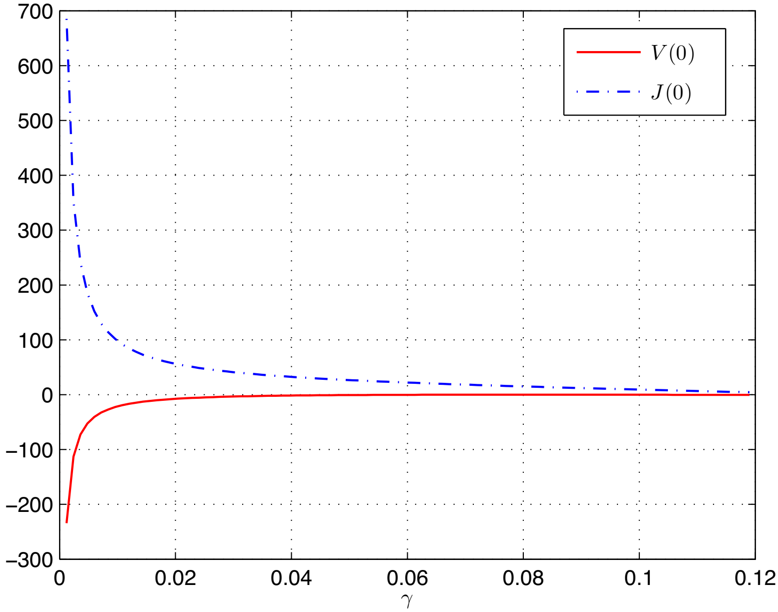

Figure 8. *V(0) versus J(0) over* γ.

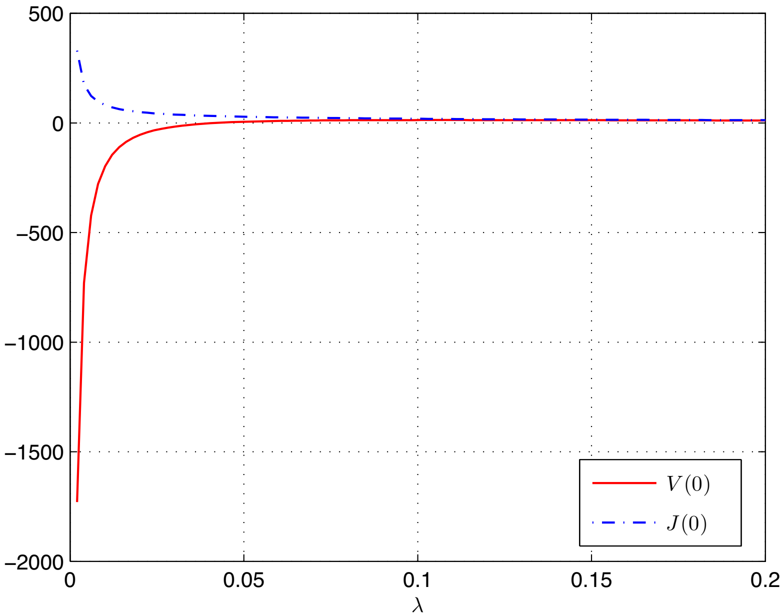

Figure 9. *V(0) versus J(0) over* λ.

Given the strategy of the buyer for either game, the seller chooses the premium rate to maximize its expected terminal wealth at time τ. Then, we used optimal premium rate to obtain explicit expressions of the Stackelberg equilibria of both life insurance games and analyzed the effects of the parameters on the equilibria. Our results showed that the properties of the equilibrium controls for the term life insurance game are more intuitively pleasing than those for the whole life insurance game. For example, $\hat{h}$ is always finite, and we do not need conditions of the relationship between γc and λ to prove monotonicity of $\hat{h}$

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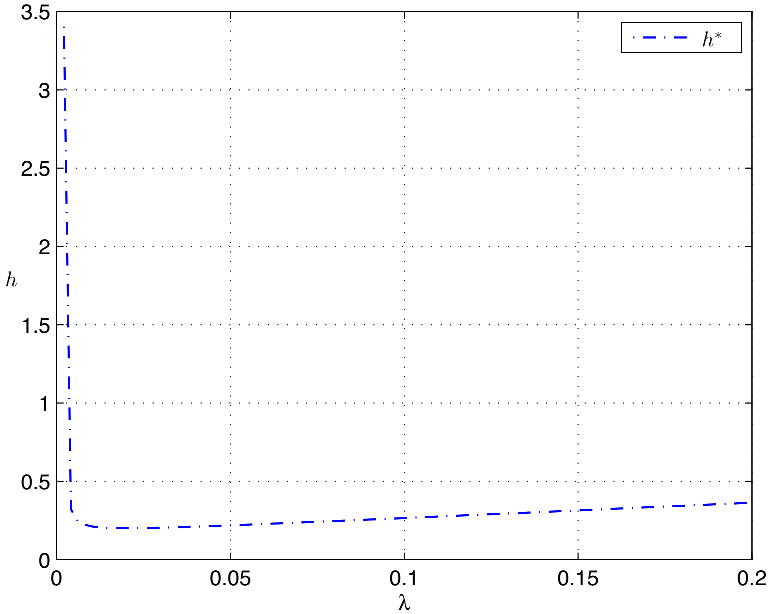

Figure 10. *h^* over λ when γc is close to $2\lambda + \mu^2/\sigma^2$.*

and $\hat{D}$ with respect to some of the parameters, as we do for h^* and D^*. We also found that whole life insurance is more attractable to the buyer, as we expected.

In future research, we will consider the Stackelberg equilibrium games in life annuities under a mean-variance criterion from both the time-consistent and pre-commitment perspectives and will investigate the properties of the equilibrium strategies for the optimization problems. Furthermore, we acknowledge that incorporating time-dependent mortality and insurance premium rates are more realistic than the constant rates we use in this paper. However, allowing for time dependence leads to a time-inhomogeneous model, making the resulting solutions more complex and less explicit. Moreover, a positive interest rate is essential for pricing life insurance products. However, incorporating a risk-free asset into our model leads to equilibrium investment and life insurance strategies for term life insurance that are not constants; indeed, they become linear functions of current wealth x, and the corresponding value function is quadratic. This complexity affects the seller's problem and the Stackelberg equilibrium discussed in Sections 2.3 and 2.4, as well as the optimal problem for the whole life insurance game in Section 3. We plan to address these issues in future work.

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