Proceedings of the Edinburgh Mathematical Society (2010) **53**, 271–291 ©
DOI:10.1017/S001309150800103X Printed in the United Kingdom

STURM–LIOUVILLE PROBLEMS FOR THE p-LAPLACIAN ON A HALF-LINE

PAUL BINDING[1], PATRICK J. BROWNE[2] AND ILLYA M. KARABASH[1]

[1]*Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta T2N 1N4, Canada*

[2]*Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 5E6, Canada*

(Received 13 October 2008)

Abstract The nonlinear eigenvalue problem

$$-\left(\left|\frac{y'(x)}{s(x)}\right|^{p-1}\operatorname{sgn} y'(x)\right)' = (p-1)(\lambda - q(x))|y(x)|^{p-1}\operatorname{sgn} y(x)$$

for $0 \leqslant x < \infty$, fixed $p \in (1,\infty)$, and with $y'(0)/y(0)$ specified is studied under various conditions on the coefficients s and q, leading to either oscillatory or non-oscillatory situations.

Keywords: Prüfer angle; p-Laplacian; eigenvalue; oscillatory solution

2010 *Mathematics subject classification:* Primary 34B15; 34B40

1. Introduction

Suppose that q is a continuous real-valued function on $[0,\infty)$ and that $q(x)$ tends to $+\infty$ with x. Then it is well known that the Sturm–Liouville equation

$$-y'' + qy = \lambda y \quad \text{on } [0,\infty) \tag{1.1}$$

is of limit point type at $+\infty$, and, given an initial condition of the form

$$y'(0)\sin\alpha = y(0)\cos\alpha \tag{1.2}$$

with $0 \leqslant \alpha < \pi$, the resulting spectrum σ is discrete. Indeed, σ consists of simple eigenvalues $\lambda_0, \lambda_1, \dots$ accumulating at $+\infty$; moreover, well-known oscillation theory guarantees that, for each $k \geqslant 0$, any eigenfunction y_k corresponding to λ_k vanishes precisely k times in $(0,\infty)$. For such results we refer the reader to [**10**, Chapter XIII], for example. From this it follows that the Prüfer angle $\theta(\lambda, x)$ (which may be defined as the continuous branch of $\cot^{-1}(y'(x)/y(x))$ for any solution y of (1.1), (1.2), given the initial condition $\theta(\lambda, 0) = \alpha$) has certain asymptotic properties in x. Indeed, since it is well known that $\theta(\lambda, x)$ must increase through values which are multiples of π, it follows that

$$k\pi < \theta(\lambda, x) < (k+1)\pi \quad \text{for sufficiently large } x \tag{1.3}$$

whenever $\lambda_{k-1} < \lambda < \lambda_k$, for any $k \geqslant 0$ (if we define $\lambda_{-1} = -\infty$).

In [**8**], under additional conditions on q, Crandall and Reno improved (1.3) to

$$\theta(\lambda, x) \to k\pi+ \text{ (respectively, } k\pi-) \text{ if } \lambda_{k-1} < \lambda < \lambda_k \text{ (respectively, } \lambda = \lambda_{k-1}) \text{ for } k > 0 \tag{1.4}$$

as $x \to \infty$, and we shall call this the '$k\pi$ property'. (More precisely, [**8**] contains a combination of statements, proofs and computer results equivalent to (1.4) for a related angle, but it follows from the results cited below that (1.4) also holds for θ as defined above.) Apparently unaware of [**8**], Brown and Reichel [**7**] established (1.4) under different additional conditions on q, which were removed in [**6**]. We remark that computational aspects are stressed in [**7**, **8**] and it is clear that (1.4) is much better suited to eigenvalue computation than (1.3), particularly if regions of attraction to multiples of π (as $x \to \infty$) can be found for $\theta(\lambda, x)$. More general situations with locally integrable q were studied in [**2**] (under Molčanov's conditions [**12**] for discrete spectrum, allowing $-\infty < \liminf q < \limsup q = +\infty$) and in [**3**] (under modifications of Brinck's [**5**] and Molčanov's conditions allowing $\liminf q = -\infty$ as well; see § 2 for details).

Some of the above works (notably, [**3**, **6**, **7**]) dealt with equations involving the p-Laplacian for fixed $p \in (1, \infty)$, and we shall now briefly discuss this extension. Eigenvalue problems for such equations (actually with $q = 0$ on a compact interval) were to our knowledge first studied by Elbert [**11**] via a generalized Prüfer angle depending on a certain function $\sin_p$, which generalizes the usual sine function and has first positive zero at

$$\pi_p = \frac{2\pi}{p \sin(\pi/p)}.$$

With $\cot_p = \sin_p' / \sin_p$, one can define the (Elbert–)Prüfer angle as above but with cot replaced by $\cot_p$ (see § 3). This generalized angle has (perhaps in equivalent form) been used to study numerous problems (see [**4**, **9**] and the references therein) and allows us to reverse some of the ideas of the previous paragraph as follows. If (1.3) is satisfied, then we are in the so-called 'discrete' case, and we can then define the '$k\pi_p$ property' via (1.4) with π replaced by π_p; this extends the previous definition since $\pi_2 = \pi$. We remark that [**3**, **6**, **7**] also discussed related issues like variational principles, the radial p-Laplacian and analogues of limit-circle behaviour, but these will not be considered here.

We shall instead consider 'non-discrete' cases where (at least for $p = 2$) there is an essential spectrum, with a finite minimum λ_{e}, say. Then any eigenvalues $\lambda_k < \lambda_{\mathrm{e}}$ again have eigenfunctions which vanish precisely k times in $(0, \infty)$, so we can apply the philosophy of the previous paragraph to approach the problem for $p \in (1, \infty)$. Specifically, we can define λ_{e} so that the Elbert–Prüfer angle $\theta(\lambda, x)$ remains bounded for all x and $\lambda < \lambda_{\mathrm{e}}$, but is unbounded in x for each $\lambda > \lambda_{\mathrm{e}}$. It is clear that (1.3) holds for $\lambda_k < \lambda_{\mathrm{e}}$ but simple examples (even with $p = 2$ and q piecewise constant and periodic) show that $\theta(\lambda, x)$ may have no limit as $x \to \infty$, and, in particular, (1.4) may fail. Nevertheless, we shall show for a wide class of q that the $k\pi_p$ property does hold for a modified angle φ satisfying $\cot_p \varphi = f \cot_p \theta$ for a suitable function f.

To be specific, in § 2 we discuss a differential inequality satisfied by functions like the φ we seek, and this leads, at least in principle, to regions of attraction for $\varphi(\lambda, x)$ near multiples of π_p, for large x and $\lambda < \lambda_{\mathrm{e}}$. Section 3 is devoted to sets defined via limiting properties of φ, forming a partition of the real line, and with eigenvalues at their endpoints. In § 4 we show how to construct a suitable function f so that φ satisfies (1.4). It may be noted that, in the case when $\liminf q > -\infty$, the simple construction $f(x) = x+1$ suffices. Finally, in § 5 we consider situations where instead $\theta(\lambda, x)$ (or equivalently $\varphi(\lambda, x)$) is unbounded as $x \to \infty$. In this way we obtain (with the work of the previous sections) conditions allowing the precise location of λ_{e}, and also conditions guaranteeing an infinite sequence of eigenvalues converging to λ_{e} from below.

2. A differential inequality

In this section we give a number of preparatory results, which extend [**3**, § 2] to the differential inequality

$$u'(x) \leqslant D + b(x) - g(x)h(u(x)), \quad 0 \leqslant x < \infty. \tag{2.1}$$

Here $D > 0$; the function $b(x) \in L^1_{\mathrm{loc}}[0, +\infty)$ is non-negative and satisfies

$$\int_x^{x+1} b \to 0 \quad \text{as } x \to \infty; \tag{2.2}$$

h is continuous and less than or equal to 1 on $[0, \Omega]$, $h(u) > 0$ for $0 < u < \Omega$ and

$$h(\varepsilon) = o(\varepsilon), \quad h(\Omega - \varepsilon) = o(\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{2.3}$$

from which it follows that

$$h(0) = h(\Omega) = 0.$$

We write

$$g(x) = g^+(x) - g^-(x), \quad \text{where } g^+(x) := \max(g(x), 0),$$

and we assume that g satisfies the conditions:

$$\exists C > 0 : \int_J g^- < C \quad \text{for all intervals } J \text{ of length } |J| \leqslant 1 \tag{B$^-$}$$

and

$$\forall \varepsilon > 0, \quad \lim_{x\to\infty} \int_x^{x+\varepsilon} g^+ = \infty,$$
$$\text{i.e. } \forall \varepsilon > 0,\ \forall A > 0,\ \exists X_{\varepsilon,A} : x > X_{\varepsilon,A} \quad \Longrightarrow \quad \int_x^{x+\varepsilon} g^+ > A. \tag{M$^+$}$$

Note that (B$^-$) and (M$^+$) were employed by Brinck [**5**] and Molčanov [**12**], respectively, but with g instead of $g^{\pm}$, in their studies of conditions for discreteness of spectra when $p = 2$.

Lemma 2.1. *Let u be a solution of (2.1). Given $0 < \gamma < \delta < \Omega$ and $\eta > 0$, there exists $X_{\gamma,\delta,\eta}$ so that*

$$x > X_{\gamma,\delta,\eta}, \quad u(x) \in (\gamma, \delta], \quad u(y) \leqslant \delta \quad \text{for all } y \in [x, x+\eta] \tag{2.4}$$

implies that

$$\text{there exists } \varepsilon \in (0, \eta) \quad \text{such that } u(x+\varepsilon) = \gamma. \tag{2.5}$$

Proof. Let $B = \min\{h(u) : u \in [\gamma, \delta]\}$. Then $0 < B \leqslant 1$. By virtue of (M^+) and (2.2) we select $X_{\gamma,\delta,\eta}$ so that

$$x > X_{\gamma,\delta,\eta} \quad \Longrightarrow \quad \int_x^{x+\eta} g^+ > \biggl(\delta - \gamma + D\eta + \int_x^{x+\eta} b(t)\,\mathrm{d}t + ([\eta]+1)C\biggr)\bigg/B.$$

Suppose that $x > X_{\gamma,\delta,\eta}$ satisfies (2.4) but that no $\varepsilon \in (0, \eta)$ can be found to satisfy (2.5). Then $u(y) \in (\gamma, \delta)$ for all $y \in [x, x+\eta]$ and we have

$$\begin{aligned} u(x+\eta) &\leqslant u(x) + D\eta + \int_x^{x+\eta} b(t)\,\mathrm{d}t - \int_x^{x+\eta} g^+(t)h(u(t))\,\mathrm{d}t + \int_x^{x+\eta} g^-(t)h(u(t))\,\mathrm{d}t \\ &\leqslant \delta + D\eta + \int_x^{x+\eta} b(t)\,\mathrm{d}t - B\int_x^{x+\eta} g^+(t)\,\mathrm{d}t + ([\eta]+1)C < \gamma \end{aligned}$$

by choice of $X_{\gamma,\delta,\eta}$. This contradiction establishes the result. □

Lemma 2.2. *Given $0 < \gamma < \delta < \Omega$ such that*

$$\delta - \gamma - Cm > 0, \tag{2.6}$$

where $m = \max\{h(u) : \gamma \leqslant u \leqslant \delta\}$, there is $Y_{\gamma,\delta}$ such that for any solution of (2.1)

$$x > Y_{\gamma,\delta}, \quad u(x) \leqslant \gamma \quad \Longrightarrow \quad u(x+t) < \delta \quad \text{for all } t > 0.$$

Proof. Put

$$M_X := \max_{x \geqslant X} \int_x^{x+1} b(t)\,\mathrm{d}t. \tag{2.7}$$

By (2.2), we can choose $Z_{\gamma,\delta}$ such that $M := M_{Z_{\gamma,\delta}} < \delta - \gamma - Cm$. Set

$$\eta = \frac{\delta - \gamma - Cm - M}{D + M + Cm}.$$

Then $\eta > 0$. We can take $Y_{\gamma,\delta} = \max\{Z_{\gamma,\delta}, X_{\gamma,\delta,\eta}\}$, where $X_{\gamma,\delta,\eta}$ is defined as in Lemma 2.1. Suppose that $x > Y_{\gamma,\delta}$ has $u(x) \leqslant \gamma$ and that $z > x$ has $u(z) = \delta$. Without loss of generality we take z to be the minimum of all points $r > x$ with $u(r) = \delta$. Now take $y \in [x, z]$ so that $u(y) = \gamma$ and $u(w) \in (\gamma, \delta)$ for all $w \in (y, z)$. Then

$$\begin{aligned} \delta - \gamma &= u(z) - u(y) \\ &= \int_y^z u'(t)\,\mathrm{d}t \\ &\leqslant D(z-y) + \int_y^z b(t)\,\mathrm{d}t + m\int_y^z g^-(t)\,\mathrm{d}t \\ &< D(z-y) + ([z-y]+1)(M+Cm) \\ &\leqslant (z-y)(D+M+Cm) + M + Cm. \end{aligned}$$

Thus, $z - y > \eta$ and so we may apply Lemma 2.1 over $(z-\eta, z)$ to obtain a point $w \in (z-\eta, z)$ with $u(w) = \gamma$. This contradiction establishes the result. □

We now extend the definition of h to all of $\mathbb{R}$ by requiring that it be periodic of period Ω and we shall continue to write h for this extended function. This now raises the possibility that a solution to (2.1) (with this periodic h) need not be bounded, but we can demonstrate the following.

Lemma 2.3. *Let u be a solution of (2.1) with h extended by Ω-periodicity to $\mathbb{R}$. Then u is bounded above on $\mathbb{R}^+$.*

Proof. Let γ and δ satisfy $0 < \gamma < \delta < \Omega$ and (2.6), which by (2.3) can be achieved by taking $\gamma = \frac{1}{2}\delta$ sufficiently close to 0. If the conclusion of the lemma fails, then there exist finite

$$x_n = \min\{x : u(x) = n\Omega + \gamma\}, \quad n \geqslant 1.$$

Recalling M_X from (2.7), we have

$$\begin{aligned} n\Omega + \gamma &= u(0) + \int_0^{x_n} u'(t)\,\mathrm{d}t \\ &\leqslant u(0) + Dx_n + \int_0^{x_n} b(t)\,\mathrm{d}t + \int_0^{x_n} g^-(t)h(u(t))\,\mathrm{d}t \\ &\leqslant u(0) + Dx_n + ([x_n]+1)(M_0 + C) \end{aligned}$$

so $x_n \to \infty$. Now we use Lemma 2.2 to find $Y_{\gamma,\delta}$ and fix N so that $x_N > Y_{\gamma,\delta}$. Note that $v(x) = u(x) - N\Omega$ satisfies the differential inequality (2.1) and, further, that $v(x_N) = \gamma$. Lemma 2.2 then shows that $v(x) < \delta$ for all $x > x_N$, a contradiction. □

Lemma 2.4. *Let u be a solution of (2.1). If $\liminf_{x\to\infty} u(x) < \Omega$, then*

$$\limsup_{x\to\infty} u(x) \leqslant 0.$$

Proof. By assumption there exist $\delta \in (\Omega/2, \Omega)$ and a sequence $x_n \to \infty$ such that $u(x_n) < 2\delta - \Omega$ for each $n = 1, 2, \dots$. Suppose $\limsup_{x\to\infty} u(x) > 0$, so there exist $\gamma \in (0, \delta - \frac{1}{2}\Omega)$, and $y_n \to \infty$ such that

$$u(y_n) > 2\gamma. \tag{2.8}$$

Now $2\delta - \Omega < \delta$ and $\delta - (2\delta - \Omega) - C \max\{h(u) : 2\delta - \Omega \leqslant u \leqslant \delta\}$ with $\delta = \Omega - \varepsilon$ becomes

$$\varepsilon - \max\{h(u) : \Omega - 2\varepsilon \leqslant u \leqslant \Omega - \varepsilon\},$$

which is positive for small $\varepsilon > 0$ by (2.3). Note that we can take δ as close to Ω as we wish. Thus, we can apply Lemma 2.2 with $(2\gamma - \Omega)$ playing the role of γ to claim the existence of N_1 so that $u(x) < \delta$ for all $x > x_{N_1}$. Furthermore, $2\gamma < \delta$ and so, by Lemma 2.1 with $\eta = 1$, say, there are N_2 and $z_n > y_n$ so that $u(z_n) = \gamma$ for all $n > N_2$. Note that γ can be chosen as close to 0 as we wish so that the conditions of Lemma 2.2 with 2γ playing the role of δ are satisfied. Again (2.3) is used here. Then Lemma 2.2 shows that $u(x) < 2\gamma$ for x large enough, contradicting (2.8). □

Lemma 2.5. *With D, b, g and h as above, suppose that $u(x, \mu)$ satisfies $u'(x, \mu) \leqslant D + b(x) - (g(x) - \mu)h(u(x, \mu))$ on $[0, \infty)$ for $\mu \in [0, \mu_0)$, where $\mu_0 > 0$ is a constant. Assume also that $u(x, \mu)$ is continuous in $\mu \in [0, \mu_0]$ for any $x \geqslant 0$. If $u(x, 0) \to 0$ as $x \to \infty$, then there is $\nu \in (0, \mu_0)$ so that*

$$0 < \mu < \nu \quad \Longrightarrow \quad \limsup_{x\to\infty} u(x, \mu) < \Omega.$$

Proof. Note that (B^-) holds for all $\mu \in [0, \mu_0)$ with g replaced by $g - \mu$ and C replaced by $C + \mu_0$. Note also that the number $X_{\varepsilon,A}$ can be chosen such that (M^+) holds for all $g - \mu$, $\mu \in [0, \mu_0)$, instead of g. In like manner, Lemmas 2.1 and 2.2 also hold with the quantities $X_{\gamma,\delta,\eta}$, $Y_{\gamma,\delta}$ chosen independent of μ.

Now choose $\gamma \in (0, \frac{1}{2}\pi)$ and x_0 so that

$$u(x, 0) < \gamma \quad \text{for all } x \geqslant x_0. \tag{2.9}$$

Since under our hypotheses $u(x, \mu)$ is continuous in μ, we see that

$$u(x_0, \mu) < \gamma \quad \text{for all small enough } \mu > 0.$$

Suppose that for each $n \in \mathbb{N}$ there is $\mu_n < \mu_0/n$ for which

$$\limsup_{x\to\infty} u(x, \mu_n) \geqslant \pi. \tag{2.10}$$

Then there is $z_n > x_0$ so that $u(z_n, \mu_n) = \gamma$, $n \in \mathbb{N}$.

Assume that z_n accumulate at a finite number z_0 as $n \to \infty$. Then, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \gamma - u(z_0, 0) &= (u(z_n, \mu_n) - u(z_0, \mu_n)) + (u(z_0, \mu_n) - u(z_0, 0)) \\ &\leqslant D|z_n - z_0| + \left| \int_{z_0}^{z_n} b(x)\,\mathrm{d}x \right| + \Omega \left| \int_{z_0}^{z_n} |g(x)|\,\mathrm{d}x \right| + \varepsilon \end{aligned}$$

holds for n large enough. This implies $u(z_0, 0) \geqslant \gamma$, contradicting (2.9).

Thus, $z_n \to \infty$ as $n \to \infty$. Let γ be such that $\gamma - C \max\{h(u) : \gamma < u < 2\gamma\} > 0$. Take $Y_{\gamma,2\gamma}$ from Lemma 2.2 and n such that $z_n > Y_{\gamma,2\gamma}$. Since $u(z_n, \mu_n) = \gamma$, Lemma 2.2 yields that $u(x, \mu_n) < 2\gamma$ for all $x > z_n$, contradicting (2.10). □

3. Modified Prüfer angles

From now on, for $p > 1$, we shall adopt the notation

$$^{\circ}|t|^{p-1} = |t|^{p-1}\operatorname{sgn} t$$

for the odd extension of the $(p-1)$th power, and we shall consider the differential equation

$$-(^{\circ}|y'/s|^{p-1})' = (p-1)(\lambda - q)^{\circ}|y|^{p-1}, \tag{3.1}$$

where $q, s \in L_1^{\mathrm{loc}}(0,\infty)$ with $s > 0$ a.e. Additional properties on the coefficients will be assumed subsequently.

Definition 3.1. If $y \in L_p(0,\infty)$ satisfies (3.1) and the initial condition

$$\left(\frac{y'}{sy}\right)(0) = \cot_p \alpha \quad \text{for } \alpha \in (0,\pi_p), \qquad y(0) = 0 \quad \text{for } \alpha = 0, \tag{3.2}$$

then y and λ will be called an eigenfunction and eigenvalue, respectively.

Here $\cot_p$ is defined via Elbert's modified trigonometric functions (see § 1 and [**3**] for further details). Note that the initial condition (3.2) makes sense since $^{\circ}|y'/s|^{p-1} \in AC$ (cf. [**3**]).

For a solution y of initial-value problem (IVP) (3.1), (3.2) the f-modified Elbert–Prüfer angle φ was introduced in [**4**] via

$$y(x,\lambda) = \rho(x,\lambda)\sin_p \varphi(x,\lambda), \qquad f(x)y'(x,\lambda) = s(x)f(x)\rho(x,\lambda)\sin_p' \varphi(x,\lambda).$$

This leads to

$$\left(\frac{fy'}{sy}\right)(x,\lambda) = \cot_p \varphi(x,\lambda),$$

where we require f to be positive and locally absolutely continuous on $[0,\infty)$. In terms of the usual (unmodified, i.e. $f \equiv 1$) Elbert–Prüfer angle $\theta(x,\lambda)$ we have

$$\cot_p \varphi = f \cot_p \theta,$$

where

$$\theta' = s|\sin_p' \theta|^p - (q-\lambda)|\sin_p \theta|^p, \quad \theta(0) = \alpha. \tag{3.3}$$

We shall specify $\varphi(0)$ to lie in the range $[0,\pi_p)$. The positivity of f and well-known properties of θ immediately show the following (see, for example, [**4**]).

Lemma 3.2.

(i) *The angle φ increases through multiples of π_p.*

(ii) $\varphi \in [\frac{1}{2}N\pi_p, \frac{1}{2}(N+1)\pi_p] \iff \theta \in [\frac{1}{2}N\pi_p, \frac{1}{2}(N+1)\pi_p]$ *for any integer $N \geqslant 0$.*

Lemma 3.3. *The modified angle φ satisfies the first-order IVP*

$$\varphi' = -\frac{f'}{f}(\sin_p' \varphi)^{p-1} \sin_p \varphi + \frac{s}{f}|\sin_p' \varphi|^p - f^{p-1}(q-\lambda)|\sin_p \varphi|^p, \tag{3.4}$$

$$\varphi(0) = \cot_p^{-1}(f(0)\cot_p \alpha) \in [0, \pi_p), \tag{3.5}$$

whence

$$\lambda \geqslant \mu \quad \Longrightarrow \quad \varphi(x,\lambda) \geqslant \varphi(x,\mu). \tag{3.6}$$

Definition 3.4. For any $\lambda \in \mathbb{R}$, $n(\lambda)$ is the smallest integer (or $+\infty$ if there is none) such that $\varphi(x,\lambda) < (n+1)\pi_p$ for all $x \in \mathbb{R}_+$.

Remark 3.5. From Lemma 3.2 and (3.6), $n(\lambda)$ is the number of zeros in $\mathbb{R}_+$ of any solution of (3.1), (3.2), and, moreover, θ may be used instead of φ in the above definition.

Our next result appeared as [**4**, Lemma 2.4], but since part of the proof there may be misleading, we shall provide another argument for completeness.

Lemma 3.6. *For any $x_0 \in (0,\infty)$, $\varphi(x_0,\lambda) \to 0$ as $\lambda \to -\infty$.*

Proof. Choose $\delta \in (0, \pi_p - \alpha)$. Since $\alpha \geqslant 0$, Lemma 3.2 (i) shows that $\varphi(x,\lambda) > 0$ for all $\lambda \in \mathbb{R}$ and $x > 0$.

We claim that there exist $\xi_0 \in (0, x_0)$ and $\lambda_0 < 0$ such that $\varphi(\xi_0, \lambda_0) < \delta$. Indeed, assume the converse, i.e. that $\varphi(x,\lambda) \geqslant \delta$ for all $\lambda < 0$ and $x \in (0, x_0)$. Since $\alpha < \pi_p$, there exists ξ_0 such that $\varphi(x,\lambda) < \pi_p - \delta$ for all $x \in (0, \xi_0]$ and $\lambda < 0$. Then, making λ in (3.4) more negative, we can ensure that $\varphi(x,\lambda) < \delta$ for some $x \in (0, \xi_0]$, a contradiction.

Since the function $G := |f'/f| + |s/f| + |f^{p-1}q|$ is integrable on any finite interval, there exists $\varepsilon = (x_0 - \xi_0)/N$ for some $N > 2$, such that

$$\int_x^{x+\varepsilon} G(x)\,\mathrm{d}x < \delta \quad \text{for any } x \in [0, x_0].$$

Then (3.4) yields

$$|\varphi(x,\lambda) - \varphi(x+\varepsilon_1,\lambda)| < \delta \quad \text{for } \varepsilon_1 \in (0,\varepsilon] \text{ and } x \in [0, x_0]. \tag{3.7}$$

In particular, $\varphi(\xi_0 + \varepsilon, \lambda_0) < 2\delta$.

For sufficiently negative λ, say $\lambda \leqslant \lambda_1 \leqslant \lambda_0$, we can argue as for our claim above to ensure that $\varphi(\xi_0 + 2\varepsilon, \lambda) < 2\delta$. Continuing this process for N such steps, we reach $\varphi(x_0,\lambda) < 2\delta$ for λ sufficiently negative. Since δ can be chosen arbitrarily small, this completes the proof. □

We shall now assume the following:

$$\text{there exists a constant } D > 0 \text{ such that } \frac{|f'|}{f} < D, \tag{3.8}$$

$$(2.2) \text{ is satisfied by } b = \frac{s}{f}, \tag{3.9}$$

$$f^{p-1}(q-\lambda) \text{ satisfies } (\mathrm{B}^-) \text{ and } (M^+) \quad \text{for each } \lambda \in \mathbb{R}_- = (-\infty, 0). \tag{3.10}$$

https://doi.org/10.1017/S001309150800103X Published online by Cambridge University Press

Of course, $\mathbb{R}_-$ can be replaced by the interval $(-\infty, \lambda_e)$ after a shift of the eigenparameter provided λ_e is finite. (In the discrete case discussed in § 1, $\lambda_e = \infty$, so $\mathbb{R}_-$ could be replaced by $(-\infty, \lambda_*)$ for arbitrarily large λ_*, but this case has already been analysed in the references cited earlier.) The above assumptions lead to the fact that, for $\lambda \in \mathbb{R}_-$, $\varphi(x, \lambda)$ satisfies a first-order differential inequality of the type considered in § 2, where we take $\Omega = \pi_p$ and $h(u) = |\sin_p u|^p$. By Lemma 2.3, Definition 3.4 and Remark 3.5, we come to the following.

Lemma 3.7. *For each $\lambda \in \mathbb{R}_-$, $n = n(\lambda) \geqslant 0$ is finite, so, for all sufficiently large x,*

$$n\pi_p < \varphi(x, \lambda) < (n+1)\pi_p$$

and any solution $y(x, \lambda)$ of (3.1), (3.2) has n zeros in $(0, \infty)$.

Definition 3.8. For each $n \geqslant 0$, we define

$$\begin{aligned} \Lambda_n &= \{\lambda \in \mathbb{R}_- : n\pi < \varphi(x, \lambda) < (n+1)\pi \text{ for all } x \text{ sufficiently large}\}, \\ \Lambda_n^+ &= \{\lambda \in \Lambda_n : \varphi(x, \lambda) \to (n+1)\pi \text{ as } x \to \infty\}, \\ \Lambda_n^- &= \{\lambda \in \Lambda_n : \varphi(x, \lambda) \to n\pi \text{ as } x \to \infty\}. \end{aligned}$$

Lemma 3.9. $\Lambda_n = \Lambda_n^+ \cup \Lambda_n^-$.

Proof. Suppose that $\lambda \in \Lambda_n \setminus \Lambda_n^+$ and apply Lemma 2.4 to $\varphi(x, \lambda) - n\pi_p$. Since $\liminf(\varphi(x, \lambda) - n\pi_p) < \pi_p$ we see that $\limsup(\varphi(x, \lambda) - n\pi_p) \leqslant 0$. On the other hand, $\varphi(x, \lambda) - n\pi_p > 0$ for x sufficiently large and the result follows readily. □

Note that, since φ is monotonic in λ, each of the sets Λ_n, $\Lambda_n^\pm$ is convex and is therefore an interval or empty.

Lemma 3.10. *If $\lambda \in \Lambda_n^-$, then λ is not an eigenvalue of (3.1), (3.2).*

Proof. Suppose that y is a solution of (3.1), (3.2) with $\lambda \in \Lambda_n^-$. Then, for x sufficiently large, $fy'/sy > 1$. Thus, y and y' have the same sign, which without loss of generality we take to be positive. It follows that y is positive and increasing and hence bounded away from 0 as $x \to \infty$. Hence, $y \notin L_p(0, \infty)$ and so λ is not an eigenvalue of (3.1), (3.2). □

Lemma 3.11. $\Lambda_0^- \neq \varnothing$.

Proof. For $\lambda < 0$, $\varphi(x, \lambda)$ satisfies the inequality

$$\varphi'(x, \lambda) \leqslant D + b(x) - f(x)q(x)|\sin_p \varphi(x, \lambda)|^p \tag{3.11}$$

with fq satisfying the conditions of § 2. We take $\gamma = \frac{1}{2}\delta$ with δ chosen small enough to ensure $\frac{1}{2}\delta - \sin^2\delta > 0$ and we apply Lemma 2.2 to find $Y_{\gamma,\delta}$ so that if $\varphi(x, \lambda)$ is a solution of (3.11), then

$$x > Y_{\gamma,\delta} \implies \varphi(x, \lambda) < \tfrac{1}{2}\delta \implies \varphi(x+t, \lambda) < \delta \quad \text{for all } t \geqslant 0. \tag{3.12}$$

Note that $Y_{\gamma,\delta}$ does not depend on the choice of $\lambda < 0$.

Now, by Lemma 3.6, there exists $\lambda < 0$ such that $\varphi(Y_{\gamma,\delta}, \lambda) < \frac{1}{2}\delta$ and so the conclusion of (3.12) holds. Lemma 2.4 completes the proof. □

https://doi.org/10.1017/S001309150800103X Published online by Cambridge University Press

From Definition 3.4 and monotonicity of φ in λ,

$$N_\mu := \lim_{\lambda \nearrow \mu} n(\lambda) \tag{3.13}$$

exists (finite or infinite) for each $\mu \in \mathbb{R}$. The main result of this section is the following.

Theorem 3.12. *Assume that (3.8)–(3.10) are satisfied, and that*

$$\text{each set } \Lambda_n^+ \text{ consists of at most one point.} \tag{3.14}$$

Then $\mathbb{R}_- = \bigcup_{n=0}^{N_0} \Lambda_n$ *and, in the case* $N_0 > 0$, *there exists a sequence* $\{\lambda_n\}_{n=-1}^{N_0-1} \subset \mathbb{R}_-$ *such that* $\lambda_{-1} = -\infty$ *and*

$$\Lambda_n^- = (\lambda_{n-1}, \lambda_n), \quad \Lambda_n^+ = \{\lambda_n\}, \quad \Lambda_n = (\lambda_{n-1}, \lambda_n] \quad \text{whenever } 0 \leqslant n < N_0.$$

Moreover, if $N_0 < \infty$, *then* $\Lambda_{N_0} = \Lambda_{N_0}^- = (\lambda_{N_0-1}, 0)$.

Proof. First, note that $\varphi(x, \lambda)$ increases monotonically in λ for any x and thus the sets Λ_n^-, Λ_n^+ and Λ_n are intervals. Now Lemmas 2.5 and 3.2 (i) and Equation (3.6) can be used to prove that Λ_n^- is open for each n.

Consider the case $N_0 < \infty$. Let us show that $\Lambda_{N_0}^+ = \varnothing$. Indeed, if $\lambda_* \in \Lambda_{N_0}^+$, then (3.14) implies $\sup_{x \in \mathbb{R}_+} \varphi(x, \lambda) > (N_0 + 1)\pi_p$ for $\lambda \in (\lambda_*, 0)$ and this contradicts the definition of N_0.

By Lemma 3.11, $\Lambda_0^- = \mathbb{R}_-$ if $N_0 = 0$ and $\Lambda_0^- = (-\infty, \lambda_0)$ if $N_0 > 0$, where $\lambda_0 < 0$. In the latter case, Lemmas 2.4 and 2.2 imply that $\lim_{x \to \infty} \varphi(x, \lambda_0) = \pi_p$. Since Λ_1^- is open, we see that $\lambda_0 \in \Lambda_0^+$. Now (3.14) shows that $\Lambda_0^+ = \{\lambda_0\}$.

Applying Lemmas 2.4 and 2.2 again, we see that $\Lambda_1^- \neq \varnothing$. Finally, the proof may be completed by induction on n (cf. [**2**, Theorem 4.2]).

The case $N_0 = \infty$ is similar, but simpler. □

In what follows, f will be chosen in accordance with various assumed properties of s and q in order to show that the eigenvalues of (3.1), (3.2) are exactly the points in the sets Λ_n^+.

4. Non-oscillatory cases

We return to the IVP (3.1), (3.2) and consider the following conditions

$$\text{there exists } \bar{c} > 0 \text{ so that } \int_x^{x+1} s(t)\,\mathrm{d}t < \bar{c} \quad \text{for every } x > 0 \tag{4.1}$$

and

$$\lim_{x \to \infty} \int_x^{x+1} q^-(t)\,\mathrm{d}t = 0. \tag{4.2}$$

Similar conditions were used in [**14**, Theorem 15.1 (a)] to prove $\lambda_{\mathrm{e}} \geqslant 0$ for $p = 2$.

4.1. We start by defining

$$I(x) = \int_x^{x+1} q^-(t)\,\mathrm{d}t,$$

which is absolutely continuous in x, and we write

$$\tilde{I}(x) = \max\{I(t) : t \geqslant x\} \quad \text{for } x \geqslant 1.$$

Next we define

$$\tilde{I}(x) = a2^{-x} \quad \text{for } 0 \leqslant x < 1,$$

where $a > 0$ is chosen so that

$$I(x) \leqslant a2^{-x} \quad \text{for } 0 \leqslant x \leqslant 1$$

and

$$\tilde{I}(1) \leqslant a2^{-1}.$$

Then $\tilde{I}(x)$ is defined for all x, is non-increasing, and $\tilde{I}(x) \to 0$ as $x \to \infty$. Now we set

$$\hat{I}(x) = \begin{cases} \tilde{I}(x), & 0 \leqslant x < 1, \\ \max\{\frac{1}{2}\hat{I}(x-1), \tilde{I}(x)\}, & x \geqslant 1, \end{cases}$$

thereby defining $\hat{I}(x)$ inductively for all x. It is easy to see (again, for example, inductively) that $\hat{I}$ is positive and non-increasing and thus has a limit $L \geqslant 0$ as $x \to \infty$. Moreover, note that, for $x \geqslant 1$,

$$\hat{I}(x) \leqslant \tfrac{1}{2}\hat{I}(x-1) + \tilde{I}(x)$$

from which it follows that $L \leqslant \frac{1}{2}L$ and so $L = 0$. We further note that, for $x \geqslant 1$,

$$\frac{\hat{I}(x)}{\hat{I}(x-1)} \geqslant \frac{1}{2}.$$

Now, defining

$$J(x) = \begin{cases} \displaystyle\int_0^1 \hat{I}(t)\,\mathrm{d}t, & 0 \leqslant x \leqslant 1, \\ \displaystyle\int_{x-1}^{x} \hat{I}(t)\,\mathrm{d}t, & x \geqslant 1, \end{cases}$$

we immediately see that $J'(x) \leqslant 0$ for $x \geqslant 1$, so J is non-increasing. Moreover, $\hat{I}(x-1) \leqslant J(x) \leqslant \hat{I}(x)$ so $J \to 0$ as $x \to \infty$.

Finally, we put

$$f(x) = \left(\frac{1}{J(x)}\right)^{1/(p-1)} \tag{4.3}$$

so

$$f \text{ is non-decreasing and tends to } \infty \text{ as } x \to \infty. \tag{4.4}$$

https://doi.org/10.1017/S001309150800103X Published online by Cambridge University Press

Theorem 4.1. *As defined above, f satisfies (3.8)–(3.10).*

Proof. We first note that

$$\frac{f'(x)}{f(x)} = -\frac{J'(x)}{(p-1)J(x)} = \frac{\hat{I}(x-1)-\hat{I}(x)}{(p-1)J(x)} \leqslant \frac{\hat{I}(x-1)-\hat{I}(x)}{(p-1)\hat{I}(x)} \leqslant \frac{2}{p-1} - \frac{1}{p-1} \leqslant \frac{1}{p-1},$$

so f satisfies (3.8). In addition we have

$$\int_x^{x+1} \frac{s(t)}{f(t)}\,\mathrm{d}t \leqslant \frac{1}{f(x)}\int_x^{x+1} s(t)\,\mathrm{d}t \leqslant \frac{\bar{c}}{f(x)},$$

which tends to zero as $x \to \infty$ by (4.4), thus verifying (3.9).

Next,

$$\begin{aligned}\int_x^{x+1} f(t)^{p-1} q^-(t)\,\mathrm{d}t &\leqslant f(x+1)^{p-1} I(x) \\ &\leqslant f(x+1)^{p-1}\tilde{I}(x) \\ &\leqslant f(x+1)^{p-1}\hat{I}(x) \\ &= \frac{\hat{I}(x)}{J(x+1)} \leqslant \frac{\hat{I}(x)}{\hat{I}(x)} \leqslant 1. \end{aligned} \tag{4.5}$$

Hence, for $\lambda < 0$,

$$\int_x^{x+1} f(t)^{p-1}(q(t)-\lambda)^-\,\mathrm{d}t \leqslant \int_x^{x+1} f(t)^{p-1} q^-(t)\,\mathrm{d}t \leqslant 1, \tag{4.6}$$

so (B^-) is satisfied. Moreover, for any $0 < \varepsilon \leqslant 1$ and $\lambda < 0$,

$$\int_x^{x+\varepsilon} f(t)^{p-1}(q(t)-\lambda)^+\,\mathrm{d}t \to \infty \quad \text{as } x \to \infty.$$

Note that

$$\begin{aligned}&\int_x^{x+\varepsilon} f(t)^{p-1}(q(t)-\lambda)^+\,\mathrm{d}t \\ &\quad = \int_x^{x+\varepsilon} f(t)^{p-1}(q(t)-\lambda)\,\mathrm{d}t + \int_x^{x+\varepsilon} f(t)^{p-1}(q(t)-\lambda)^-\,\mathrm{d}t \\ &\quad \geqslant |\lambda|\int_x^{x+\varepsilon} f(t)^{p-1}\,\mathrm{d}t + \int_x^{x+\varepsilon} f(t)^{p-1}q^+(t)\,\mathrm{d}t - \int_x^{x+\varepsilon} f(t)^{p-1}q^-(t)\,\mathrm{d}t \\ &\quad \geqslant |\lambda| f(x)^{p-1}\varepsilon - 1\end{aligned}$$

by (4.5). Now (M^+) follows from (4.4). □

4.2. We now give some results to prepare for Theorem 4.5, which is the main result of this section.

Lemma 4.2. *Suppose $\lambda < 0$ and $\lambda \in \Lambda_n^+$ for some n. Then $\liminf_{x\to\infty} \theta(x,\lambda) > (n+\frac{1}{2})\pi_p$, where $\theta(x,\lambda)$ is the unmodified Prüfer angle defined by (3.3).*

Proof. Since $\varphi(x,\lambda) \to (n+1)\pi_p$ from below as $x \to \infty$, we see that

$$\theta(x,\lambda) \in ((n+\tfrac{1}{2})\pi_p, (n+1)\pi_p) \quad \text{for } x > X, \tag{4.7}$$

where X is sufficiently large.

Assume that $\liminf_{x\to\infty} \theta(x,\lambda) = (n+\frac{1}{2})\pi_p$. Then, for any $\eta > 0$ small enough, there exists a sequence $\{x_n\}_1^\infty$ such that $\lim_{n\to\infty} x_n = \infty$ and

$$(n+\tfrac{1}{2})\pi_p < \theta(x_n,\lambda) < (n+\tfrac{1}{2})\pi_p + \eta.$$

Let

$$\delta_n := \sup\{\delta \in (0,1] : \theta(x_n+\varepsilon,\lambda) < (n+\tfrac{1}{2})\pi_p + 3\eta \text{ for all } \varepsilon < \delta\}.$$

In particular,

$$\text{if } \theta(x_n+\delta_n,\lambda) < (n+\tfrac{1}{2})\pi_p + 3\eta, \quad \text{then } \delta_n = 1. \tag{4.8}$$

It follows from (4.2) that there exists $X' > X$ such that

$$\int_x^{x+1} q^-(t)\,\mathrm{d}t < \eta \quad \text{if } x \geqslant X'.$$

Assume that $x_n > X'$. We see from (3.3) and (4.1) that

$$\theta(x_n+\delta_n,\lambda) < (n+\tfrac{1}{2})\pi_p + \eta + \bar{c}|\sin_p'(\tfrac{1}{2}\pi_p + 3\eta)|^p + \eta < (n+\tfrac{1}{2})\pi_p + 3\eta$$

whenever

$$|\sin_p'(\tfrac{1}{2}\pi_p + 3\eta)|^p < \eta/\bar{c}. \tag{4.9}$$

Let us show that (4.9) is fulfilled for $\eta > 0$ small enough. Indeed, using properties of the function $\sin_p$ (see, for example, [**4**, § 1]), one finds, for $0 < \nu < \frac{1}{2}\pi_p$,

$$\begin{aligned}
|\sin_p'(\tfrac{1}{2}\pi_p + \nu)|^p &= 1 - |\sin_p(\tfrac{1}{2}\pi_p + \nu)|^p \\
&= 1 - \sin_p^p(\tfrac{1}{2}\pi_p - \nu) \\
&= \int_{\frac{1}{2}\pi_p-\nu}^{\pi_p/2} (\sin_p^p(t))'\,\mathrm{d}t \\
&= p\int_{\pi_p/2-\nu}^{\pi_p/2} \sin_p^{p-1}(t)\sin_p'(t)\,\mathrm{d}t \\
&< p\nu\sin_p'(\tfrac{1}{2}\pi_p - \nu).
\end{aligned}$$

Since $\sin_p'(\frac{1}{2}\pi_p) = 0$, (4.9) holds if $\sin_p'(\frac{1}{2}\pi_p - 3\eta) < 1/3p\bar{c}$.

https://doi.org/10.1017/S001309150800103X Published online by Cambridge University Press

Thus, by (4.8), there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$(n + \tfrac{1}{2})\pi_p < \theta(x_n + \delta, \lambda) < (n + \tfrac{1}{2})\pi_p + 3\eta \quad \text{for all } \delta \in (0, 1].$$

Now with (4.9) we see that if $3\eta/|\sin_p(\frac{1}{2}\pi_p + 3\eta)|^p < -\lambda$, then

$$\theta(x_n + 1, \lambda) < (n + \tfrac{1}{2})\pi_p + 3\eta + \lambda|\sin_p(\tfrac{1}{2}\pi_p + 3\eta)|^p < (n + \tfrac{1}{2})\pi_p, \tag{4.10}$$

so (4.10) holds for η small enough and contradicts (4.7). □

At this point we shall make an assumption complementary to (4.1):

$$\text{there exists } \underline{c} > 0 \quad \text{so that } \underline{c} < \int_x^{x+1} s(t)\,\mathrm{d}t \text{ for every } x > 0. \tag{4.11}$$

Lemma 4.3. *Suppose $0 > \lambda \in \Lambda_n^+$ for some n. If y satisfies (3.1) and (3.2), then*

(i) *y'/s is bounded on $[0, \infty)$,*

(ii) *$|y(x)| < A\mathrm{e}^{-kx}$, $x > 0$, for certain constants $A, k > 0$,*

(iii) *$q^- y^p \in L_1$.*

Proof. (i) Since $\varphi(x, \lambda) \to (n + 1)\pi_p$ from below, we can assume without loss that

$$\theta(x, \lambda) \in ((n + \tfrac{1}{2})\pi_p, (n + 1)\pi_p), \quad y(x) > 0 \quad \text{and} \quad y'(x) < 0$$

for $x \geqslant X$, where X is sufficiently large. Thus, $0 < y(x) \leqslant y(X)$ for $x \geqslant X$.

Assume that for a sequence $x_j \to \infty$ we have $y'(x_j)/s(x_j) \to -\infty$. Then, for $x > X + 1$ and $0 \leqslant t \leqslant 1$,

$$({}^{\circ}|y'/s|^{p-1})' = (p - 1)(q - \lambda)^{\circ}|y|^{p-1} \geqslant -(p - 1)(q - \lambda)^- y^{p-1},$$
$$\int_{x-t}^{x} ({}^{\circ}|y'/s|^{p-1})' \geqslant -(p - 1) \int_{x-t}^{x} (q - \lambda)^- y^{p-1} \geqslant -(p - 1)(y(X))^{p-1}(C_1 + |\lambda|),$$

where

$$C_1 = \max_{x \geqslant X} \int_x^{x+1} q^- \,\mathrm{d}t < \infty.$$

Hence,

$${}^{\circ}|y'(x)/s(x)|^{p-1} - {}^{\circ}|y'(x - t)/s(x - t)|^{p-1} \geqslant -(p - 1)(y(X))^{p-1}(C_1 + |\lambda|),$$
$$|y'(x)/s(x)|^{p-1} - |y'(x - t)/s(x - t)|^{p-1} \leqslant (p - 1)(y(X))^{p-1}(C_1 + |\lambda|).$$

Now, choosing j large enough to ensure

$$|y'(x_j)/s(x_j)|^{p-1} > (p - 1)(y(X))^{p-1}(C_1 + |\lambda|) + \left(\frac{2}{\underline{c}} y(X)\right)^{p-1},$$

we see that

$$\begin{aligned}|y'(x_j - t)/s(x_j - t)| &\geqslant \frac{2}{\underline{c}} y(X),\\ y'(x_j - t) &\leqslant -\frac{2}{\underline{c}} y(X) s(x_j - t),\\ \int_0^1 y'(x_j - t)\,\mathrm{d}t &\leqslant -\frac{2}{\underline{c}} y(X) \int_0^1 s(x_j - t)\,\mathrm{d}t \leqslant -\frac{2}{\underline{c}} y(X)\underline{c},\\ y(x_j) - y(x_j - 1) &\leqslant -2y(X).\end{aligned}$$

Hence,

$$y(x_j) \leqslant -2y(X) + y(x_j - 1) \leqslant -y(X) < 0.$$

This contradiction establishes statement (i).

(ii) By Lemma 4.2, we can assume that

$$\frac{y'(x)}{sy(x)}(x) < -C_2 \quad \text{for } x \geqslant X_1,$$

where C_2 and X_1 are certain positive constants. Using (4.1), we obtain, for $x \geqslant X_1$,

$$\ln y(x) - \ln y(X_1) = \int_{X_1}^{x} \frac{y'(t)}{y(t)}\,\mathrm{d}t < -C_2 \int_{X_1}^{x} s(t)\,\mathrm{d}t < -C_2\underline{c}(x - X_1 - 1)$$

and $y(x) < y(X_1)\mathrm{e}^{C_2\underline{c}(X_1+1)}\mathrm{e}^{-C_2\underline{c}x}$.

(iii) This follows from [**3**, Lemma 3.2]. □

4.3. We are now ready to establish the remaining assumption of Theorem 3.12.

Theorem 4.4. *Each set Λ_n^+ contains at most one point.*

Proof. Suppose λ and μ both belong to Λ_n^+ and $\lambda < \mu$, so

$$\theta(x, \lambda) < \theta(x, \mu) < (n + 1)\pi_p$$

for all x. Suppose y and z are non-trivial solutions of (3.1), (3.2) corresponding to λ and μ, respectively. We define x_0 by

$$\begin{aligned}\theta(x_0, \lambda) &= n\pi_p \quad \text{when } n \geqslant 1,\\ x_0 &= 0 \quad \text{when } n = 0\end{aligned}$$

and we take v to be the solution of the IVP consisting of the differential equation (3.1) on $[x_0, \infty)$ with μ in place of λ and subject to the initial condition $v(x_0) = 0$ when $n \geqslant 1$ or $n = \alpha = 0$, and $v'(x_0)/s(x_0)v(x_0) = \cot_p(\alpha)$ when $n = 0 \neq \alpha$. Note that, for $n = 0$, $v = z$ and, furthermore, we can assume that y, v are of one sign, which we take to be positive on (x_0, ∞).

If we define an angle θ_v on $[x_0,\infty)$ via $v'/sv=\cot_p\theta_v$, then

$$\theta(x,\lambda)-n\pi_p<\theta_v<\theta(x,\mu)-n\pi_p,$$

so $\liminf_{x\to\infty}\theta_v(x)>\frac{1}{2}\pi_p$ follows from Lemma 4.2. As in the proof of Lemma 4.3 (ii), we now have

$$v(x)<A_v\mathrm{e}^{-k_vx}\ \text{for}\ x>x_0\quad\text{and}\quad v'/s\ \text{remains bounded as}\ x\to\infty. \tag{4.12}$$

For small $\varepsilon>0$ we use

$$w=\frac{y^p}{(v+\varepsilon)^{p-1}}$$

so

$$w'=\frac{py^{p-1}y'}{(v+\varepsilon)^{p-1}}-\frac{(p-1)y^pv'}{(v+\varepsilon)^p}.$$

Now the p-Laplacian version of Picone's identity [**1**, Theorem 1.1] shows that

$$R=R(y,v,\varepsilon):=|y'|^p-w'|v'|^{p-2}v'\geqslant 0\quad\text{for a.a.}\ x>x_0,$$

and hence, for any $b>x_0$,

$$\begin{aligned}
0&\leqslant\int_{x_0}^b\frac{R}{s^{p-1}}=\int_{x_0}^b{}^{\circ}|y'/s|^{p-1}y'-\int_{x_0}^b{}^{\circ}|v'/s|^{p-1}w'\\
&=(p-1)\int_{x_0}^b(\lambda-q)y^p-(p-1)\int_{x_0}^b(\mu-q)y^p\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}+B|_{x_0}^b\\
&=(p-1)\int_{x_0}^b y^p\biggl(\lambda-\mu\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)-(p-1)\int_{x_0}^b qy^p\biggl(1-\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)+B|_{x_0}^b\\
&\leqslant(p-1)\int_{x_0}^b y^p\biggl(\lambda-\mu\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)+(p-1)\int_{x_0}^b q^-y^p\biggl(1-\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)+B|_{x_0}^b,
\end{aligned}$$

where

$$B={}^{\circ}|y'/s|^{p-1}y-{}^{\circ}|v'/s|^{p-1}w.$$

Let $b\to\infty$ and note, by Lemma 4.3 (i), (ii) and (4.12), that $B(b)\to 0$. This gives

$$\begin{aligned}
0\leqslant(p-1)\int_{x_0}^\infty y^p\biggl(\lambda-\mu\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)+(p-1)\int_{x_0}^\infty q^-y^p\biggl(1-\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr)&\\
-{}^{\circ}|\cot_p\alpha|^{p-1}(y(x_0))^p\biggl(1-\biggl(\frac{v(x_0)}{v(x_0)+\varepsilon}\biggr)^{p-1}\biggr)&,
\end{aligned}$$

where the last term is to be taken as 0 unless $n=0\neq\alpha$. Now let $\varepsilon\to 0$ and, noting Lemma 4.3 (iii), use Lebesgue's Dominated Convergence Theorem to obtain

$$0\leqslant\int_{x_0}^\infty y^p(\lambda-\mu)<0.$$

This contradiction establishes the result. □

4.4. Taking Theorem 3.12 into account, we can summarize the results of this section as follows.

Theorem 4.5. *Under conditions (4.1), (4.2) and (4.11), the conclusions of Theorem 3.12 hold, and the λ_n therein are precisely the negative eigenvalues of problem (3.1), (3.2). For any eigenfunction y_n (associated with λ_n) we have $y_n \mathrm{e}^{kx} \in L_\infty$ for some $k > 0$, $y_n' s^{-1} \in L_\infty$ and $y_n' s^{1-1/p} \in L_1$.*

Proof. This follows from Theorems 3.12, 4.1 and 4.4, Lemma 4.3 and from suitable amendment to the proof of [**3**, Theorem 4.1]. □

Note that one can now easily obtain Sturmian comparison properties for eigenvalues as in [**3**, Theorem 4.3] and the fact that y_n has precisely n zeros in $(0, +\infty)$ as in [**3**, Corollary 4.2].

4.5. A special case

We conclude this section with the situation where s satisfies (4.1) and (4.11), and $\liminf q(x)$ is finite (say 0 after a shift of eigenparameter) as $x \to \infty$. Then one may replace (4.3) by the simpler formula $f(x) = x + 1$. Indeed, (3.8) and (3.9) are obvious, as is (B^-) in (3.10). To establish (M^+), we note that, for $\lambda < 0$ and x sufficiently large, $q(x) > \frac{1}{2}\lambda > \lambda$. For such x we have

$$\begin{aligned}\int_x^{x+1} (f(t)(q(t) - \lambda))^+ \, \mathrm{d}t &= \int_x^{x+1} (t+1)(q(t) - \lambda) \, \mathrm{d}t \\ &\geqslant \int_x^{x+1} (t+1)(-\tfrac{1}{2}\lambda) \, \mathrm{d}t \to \infty \quad \text{as } x \to \infty.\end{aligned}$$

Thus, this simple modification φ of θ also has the $k\pi_p$ property. As special cases one could consider certain situations of [**10**, Chapter XIII] where $p = 2$ and $s(x)$ and $q(x)$ are both continuous in x and have limits as $x \to \infty$. We note that θ need not have the $k\pi_p$ property, and, for example, when $s = 1$ and q is continuous with $q(x) \to 0$ as $x \to \infty$, Brown and Eastham [**6**] have shown (for $\lambda < 0$) that $\theta(x, \lambda)$ has a limit which is not a multiple of π_p as $x \to \infty$.

5. Oscillatory cases

We shall call the problem (3.1), (3.2) oscillatory at λ if some (and hence every) solution y has infinitely many zeros on $\mathbb{R}^+$. The converse property was the subject of the previous sections. Since the angle $\theta(x, \lambda)$ (or its modified version $\varphi(x, \lambda)$) increases through multiples of π_p, oscillatory behaviour is equivalent to unboundedness of such angles as $x \to \infty$. In this section we shall examine some oscillatory situations, leading to conditions for location of λ_e and for existence of infinitely many eigenvalues below λ_e.

Theorem 5.1. *Assume that there exist sequences* $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$, $\{c_n\}_1^\infty$ *such that*

(i) $0 \leqslant x_n < y_n$ *and* $0 < c_n$ *for all* $n \in \mathbb{N}$, *and* c_n *are bounded,*

(ii) $c_n(y_n - x_n) \to +\infty$ *as* $n \to \infty$,

(iii) $\dfrac{1}{c_n(y_n - x_n)} \displaystyle\int_{x_n}^{y_n} ((c_n - s(t))^+ + q^+(t))\,\mathrm{d}t \to 0$ *as* $n \to \infty$.

Then the problem (3.1), (3.2) is oscillatory at any $\lambda > 0$*; in particular,* $\lambda_\mathrm{e} \leqslant 0$.

Proof. From (3.3),

$$\begin{aligned}\theta(y_n,\lambda) &\geqslant \theta(y_n,\lambda) - \theta(x_n,\lambda)\\ &= \int_{x_n}^{y_n} \theta'(t,\lambda)\,\mathrm{d}t\\ &= \int_{x_n}^{y_n} (s|\sin_p' \theta|^p - (q-\lambda)|\sin_p \theta|^p)\\ &\geqslant \int_{x_n}^{y_n} (c_n|\sin_p' \theta|^p + \lambda|\sin_p \theta|^p) - \int_{x_n}^{y_n} ((c_n - s)|\sin_p' \theta|^p + q|\sin_p \theta|^p)\\ &\geqslant \min\{c_n,\lambda\}(y_n - x_n) - \int_{x_n}^{y_n} ((c_n - s)^+ + q^+)\,\mathrm{d}t.\end{aligned}$$

If $c_n > c > 0$ for all n, then, with $C = \min\{c,\lambda\} > 0$, we have

$$\theta(y_n,\lambda) \geqslant (y_n - x_n)\biggl(C - \frac{1}{y_n - x_n}\int_{x_n}^{y_n} ((c_n - s)^+ + q^+)\,\mathrm{d}t\biggr) \to +\infty$$

since we can replace $c_n(y_n - x_n)$ by $y_n - x_n$ in (ii) and (iii).

If $\{c_n\}$ is not bounded away from 0, we can assume (for simplicity of notation) that $\lim_{n\to\infty} c_n = 0$. Then, for n large enough,

$$\theta(y_n,\lambda) \geqslant c_n(y_n - x_n)\biggl(1 - \frac{1}{c_n(y_n - x_n)}\int_{x_n}^{y_n} ((c_n - s)^+ + q^+)\,\mathrm{d}t\biggr),$$

and the right-hand side tends to $+\infty$ by (ii) and (iii). □

Taking $x_n = 0$, $y_n = x$ and $c_n = c$, we obtain the following.

Corollary 5.2. *Assume that there exists a constant* $c > 0$ *such that*

$$\liminf_{x\to\infty} \frac{1}{x}\int_0^x ((c - s(t))^+ + q^+(t))\,\mathrm{d}t = 0. \tag{5.1}$$

Then the conclusions of Theorem 5.1 hold.

A stronger condition was used in [**14**, Theorem 15.1 (b)] when $p = 2$ for a stronger conclusion.

If we combine Corollary 5.2 with the work of § 4, then we obtain the following.

Corollary 5.3. *If (4.1), (4.2), (4.11) and (5.1) hold, then the $k\pi_p$ property holds for every eigenvalue below $0 = \lambda_e$.*

We turn now to the number of eigenvalues below λ_e, which is also related to oscillatory behaviour. The connection depends on the following result, which we express in terms of N_μ introduced in (3.13).

Theorem 5.4. *Let $\mu \in \mathbb{R}$. Then the problem (3.1), (3.2) is oscillatory at μ if and only N_μ is infinite.*

Proof. Since 'only if' is evident, suppose that IVP (3.1), (3.2) is oscillatory at μ, but that $k = N_\mu$ is finite. Since $\theta(x, \mu) \to +\infty$ with x, we can choose x_k to ensure that

$$\theta(x_k, \mu) > k\pi_p. \tag{5.2}$$

On the other hand, $\theta(x, \lambda)$ cannot decrease through multiples of π_p as x increases, so $\theta(x_k, \lambda) < k\pi_p$ for all $\lambda < \mu$. Since the right-hand side of (3.3) is continuous in λ, obeys Carathéodory's conditions in (x, θ) and is Lipschitz in θ, $\theta(x_k, \lambda)$ is continuous in λ at μ. Letting $\lambda \nearrow \mu$, we obtain $\theta(x_k, \mu) \leqslant k\pi_p$, contradicting (5.2). □

Thus, the distinction between whether there are infinitely or finitely many λ_n in Theorem 3.12 depends on whether IVP (3.1), (3.2) is oscillatory or not at 0. Indeed, from Theorems 4.5 and 5.4, we have the following.

Corollary 5.5. *Assume that (4.1), (4.2) and (4.11) hold. Then each negative eigenvalue satisfies the $k\pi_p$ property. If, in addition,*

$$\textit{IVP (3.1), (3.2) is oscillatory at } 0, \tag{5.3}$$

then there are infinitely many negative eigenvalues converging to $0 = \lambda_e$. Similarly, if (5.3) fails, then there are only finitely many negative eigenvalues.

The oscillatory condition (5.3) is connected with the Elbert–Prüfer angle via Theorem 5.4 and Definition 3.4. The following result gives a corresponding analogue of Theorem 5.1.

Theorem 5.6. *Assume that there exist sequences $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ such that $0 \leqslant x_n < y_n$ and*

$$\lim_{n\to+\infty} \int_{x_n}^{y_n} \min\{-q(t), s(t)\}\, \mathrm{d}t = +\infty. \tag{5.4}$$

Then $N_0 = +\infty$, so IVP (3.1), (3.2) is oscillatory at 0.

Proof. Let us show that

$$\text{for any } N \in \mathbb{N} \text{ there exist } \lambda_0 < 0 \text{ and } n \in \mathbb{N} \text{ such that } \theta(y_n, \lambda_0) > N\pi_p. \tag{5.5}$$

https://doi.org/10.1017/S001309150800103X Published online by Cambridge University Press

Indeed, for negative λ, we have

$$\begin{aligned}\theta(y_n,\lambda)-\theta(x_n,\lambda)&\\ &=\int_{x_n}^{y_n}(s(t)|\sin_p'\theta(t,\lambda)|^p-(q(t)-\lambda)|\sin_p\theta(t,\lambda)|^p)\,\mathrm{d}t\\ &\geqslant\int_{x_n}^{y_n}(\min\{-q(t),s(t)\}|\sin_p'\theta(t,\lambda)|^p+\min\{-q(t),s(t)\}|\sin_p\theta(t,\lambda)|^p)\,\mathrm{d}t\\ &\quad+\lambda\int_{x_n}^{y_n}|\sin_p\theta(t,\lambda)|^p\,\mathrm{d}t\\ &\geqslant\int_{x_n}^{y_n}\min\{-q(t),s(t)\}\,\mathrm{d}t-|\lambda|(y_n-x_n).\end{aligned}\tag{5.6}$$

It follows from (5.4) that

$$\int_{x_n}^{y_n}\min\{-q(t),s(t)\}\,\mathrm{d}t>N\pi_p+1\quad\text{for some } n.$$

Taking $\lambda_0\in(-(y_n-x_n)^{-1},0)$, we see that (5.6) implies $\theta(y_n,\lambda_0)-\theta(x_n,\lambda_0)>N\pi_p$ and so (5.5) is satisfied.

Thus, $n(\lambda_0)\geqslant N$, whence $N_0\geqslant N$ and, since N is arbitrarily large, the proof is complete. □

There is a substantial literature on oscillation conditions for $p=2$ (cf. [**10**, **13**, **15**]), and even for $1<p<\infty$ (cf. [**9**]). We shall give two comparisons with our work. The first concerns the Leighton–Wintner conditions, which were generalized to $1<p<\infty$ in [**9**, Theorem 1.2.9] in a form equivalent to

$$\int^{\infty}s^{1/(p-1)}=+\infty\tag{5.7}$$

and

$$\int^{\infty}q=-\infty.\tag{5.8}$$

Hölder's inequality shows that (4.11) implies (5.7) for $1<p\leqslant 2$, so, for such p, (5.3) may be replaced by (5.8) in Corollary 5.5. This result may be compared with [**13**, Theorem 2.19], which (for $p=2$) uses (5.8) and various extra conditions on q and s to obtain an infinite number of eigenvalues below λ_{e}, but with no conclusion about the $k\pi$ property. Also, if we take $x_n=0$, $y_n=x$ in Theorem 5.6, then we see that

$$\int_0^{\infty}\min\{-q(t),s(t)\}\,\mathrm{d}t=+\infty\tag{5.9}$$

can be used instead of (5.4) in Theorem 5.6. When $p=2$, the Leighton–Wintner conditions are implied by (5.9). On the other hand, (5.7) is not implied by (5.4) for $p=2$, or by the special case (5.9) for any $p\neq 2$.

Our second comparison concerns Kneser's condition, for which we assume $s = 1$. Then

$$\limsup_{x\to\infty} x^p q(x) < -(1-p^{-1})^p \tag{5.10}$$

suffices for (5.3) (see [**9**, Theorem 1.4.5] for an equivalent version). Thus, (5.10) may be used instead of (5.3) in Corollary 5.5. Moreover, [**9**, Theorem 1.4.5] also shows that

$$\liminf_{x\to\infty} x^p q(x) > -(1-p^{-1})^p$$

suffices for (3.1), (3.2) to be non-oscillatory at 0, so, by the final sentence of Corollary 5.5, only finitely many negative eigenvalues exist; this result was recently proved directly (for continuous q) in [**6**, Theorem 3.2].

Acknowledgements. The research of P.B. was supported in part by grants from the NSERC of Canada. I.M.K. was supported by a Postdoctoral Fellowship at the University of Calgary.

References

1. W. Allegretto and Y. Huang, A Picone's identity for the p-Laplacian and applications, *Nonlin. Analysis* **32** (1998), 819–830.
2. P. Binding, L. Boulton and P. J. Browne, A Prüfer angle approach to singular Sturm–Liouville problems with Molčanov potentials, *J. Computat. Appl. Math.* **208** (2007), 226–234.
3. P. Binding and P. J. Browne, Generalized Prüfer angle and variational methods for p-Laplacian eigenvalue problems on the half-line, *Proc. Edinb. Math. Soc.* **51** (2008), 565–579.
4. P. Binding and P. Drábek, Sturm–Liouville theory for the p-Laplacian, *Studia Sci. Math. Hungar.* **40** (2003), 375–396.
5. I. Brinck, Self-adjointness and spectra of Sturm–Liouville operators, *Math. Scand.* **7** (1959), 219–239.
6. M. Brown and M. Eastham, Eigenvalues of the radial p-Laplacian with a potential on $(0,\infty)$, *J. Computat. Appl. Math.* **208** (2007), 111–119.
7. M. Brown and W. Reichel, Eigenvalues of the radially symmetric p-Laplacian, *J. Lond. Math. Soc.* **69** (2004), 657–675.
8. R. E. Crandall and M. Reno, Ground state energy bounds for potentials $|x|^\nu$, *J. Math. Phys.* **23** (1982), 64–70.
9. O. Došly and P. Řehák, *Half-linear differential equations*, North-Holland Mathematics Studies, Volume 202 (Elsevier, 2005).
10. N. Dunford and J. T. Schwartz, *Linear operators*, Volume II (Wiley Interscience, 1964).
11. A. Elbert, A half-linear second-order differential equation, *Colloq. Math. Soc. Janos Bolyai* **30** (1979), 153–179.
12. A. Molčanov, Conditions for the discreteness of the spectrum of self-adjoint second-order differential equations, *Trudy Mosk. Mat. Obshc.* **2** (1953), 169–200 (in Russian).
13. E. Müller-Pfeiffer, *Spectral theory of ordinary differential operators* (Ellis Horwood, New York, 1981).
14. J. Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, Volume 1258 (Springer, 1987).
15. A. Zettl, *Sturm–Liouville theory*, Mathematical Surveys and Monographs, Volume 121 (American Mathematical Society, Providence, RI, 2005).