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Classification of homomorphisms from $C(\Omega)$ to a C^*-algebra

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Abstract. Let Ω be a compact subset of $\mathbb{C}$ and let A be a unital simple, separable C^*-algebra with stable rank one, real rank zero, and strict comparison. We show that, given a Cu-morphism $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ with $\alpha(\langle \mathbb{1}_\Omega \rangle) \le \langle 1_A \rangle$, there exists a homomorphism $\phi : C(\Omega) \to A$ such that $\mathrm{Cu}(\phi) = \alpha$. Moreover, if $K_1(A)$ is trivial, then ϕ is unique up to approximate unitary equivalence. We also give classification results for maps from a large class of C^*-algebras to A in terms of the Cuntz semigroup.

1 Introduction

The Cuntz semigroup is an invariant for C^*-algebras that is intimately related to Elliott's classification program for simple, separable, nuclear C^*-algebras. Its original construction $W(A)$ resembles the semigroup $V(A)$ of Murray–von Neumann equivalence classes of projections, and is a positively ordered, abelian semigroup whose elements are equivalence classes of positive elements in matrix algebras over A [13]. This was modified in [12] by constructing an ordered semigroup, termed $\mathrm{Cu}(A)$, in terms of countably generated Hilbert modules. Moreover, a Cuntz category was described to which the Cuntz semigroup belongs and as a functor into which it preserves inductive limits. The Cuntz semigroup has been successfully used to classify certain classes of C^*-algebras, as well as maps between them. In 2008, Ciuperca and Elliott classified homomorphisms from $C_0((0,1])$ into an arbitrary C^*-algebra of stable rank one in terms of the Cuntz semigroup [10]. Later, the codomain was extended to a larger class in [28]. These results can also be regarded as a classification of positive elements. Subsequently, Robert greatly expanded the domain $C_0((0,1])$ to the class of direct limits of one-dimensional NCCW-complexes with trivial K_1-group [26]. More specifically, he employed a series of techniques to reduce complicated domains to $C[0,1]$ and applied the classification result in [10]. For the more general domain $C(\Omega)$, it is still expected that the Cuntz semigroup can be used in some sense. Further research and investigation are needed to explore the applicability and potential of the Cuntz semigroup in this broader field.

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In this paper, let Ω be a compact subset of $\mathbb{C}$, our primary focus is on the classification of homomorphisms from the algebra of continuous functions $C(\Omega)$ to a unital simple, separable C^*-algebra A with stable rank one, real rank zero, and strict comparison. Using the properties of the Cuntz semigroup, we can lift the Cu-morphism to a homomorphism approximately. Based on spectral information, we associate these homomorphisms to normal elements and use a result of Hu and Lin. Then, we establish a uniqueness result and use this to get a homomorphism exactly. Finally, we classify the homomorphisms from $C(\Omega)$ to A in terms of the Cuntz semigroup. Additionally, we employ the augmented Cuntz semigroup introduced by Robert to classify more general non-unital cases.

2 Preliminaries

Definition 2.1 Let A be a unital C^*-algebra. Recall that A is said to have stable rank one, written $sr(A)=1$, if the set of invertible elements of A is dense, and to have real rank zero, written $rr(A)=0$, if the set of invertible self-adjoint elements is dense in the set A_{sa} of self-adjoint elements of A. If A is not unital, let us denote the minimal unitization of A by $A^{\sim}$. A non-unital C*-algebra is said to have stable rank one (or real rank zero) if its unitization has stable rank one (or real rank zero).

Let p and q be two projections in A. Recall that p is *Murray–von Neumann equivalent* to q in A, written $p\sim q$, if there exists $x\in A$ such that $x^*x=p$ and $xx^*=q$. We will write $p\leq q$ if p is equivalent to some subprojection of q. The class of a projection p in $K_0(A)$ (see [30] for the definition of K_0) will be denoted by $[p]$.

Let us say that A has cancellation of projections if, for any projections $p,q,e,f\in A$ with $pe=0$, $qf=0$, $e\sim f$, and $p+e\sim q+f$, necessarily $p\sim q$. Then A has cancellation of projections if and only if $p\sim q$ implies that there exists a unitary $u\in A^{\sim}$ such that $u^*pu=q$. It is well known that every unital C^*-algebra of stable rank one has cancellation of projections.

Definition 2.2 [3, 4] A (*bounded*) *quasitrace* on a C^*-algebra A is a function $\tau: A\to\mathbb{C}$ such that:

(i) $0\leq\tau(x^*x)=\tau(xx^*)$ for all x in A;
(ii) τ is linear on commutative *-subalgebras of A;
(iii) If $x=a+ib$ with a,b self-adjoint, then $\tau(x)=\tau(a)+i\tau(b)$.

If τ extends to a quasitrace on $M_2(A)$, then τ is called a 2-quasitrace. A linear quasitrace is a trace.

If A is unital and $\tau(1)=1$, then we say τ is *normalized*. Denote by $QT_2(A)$ the space of all the normalized 2-quasitraces on A and by $T(A)$ the space of all the tracial states on A. Note that every 2-quasitrace in $QT_2(A)$ is lower semicontinuous (see [3, Remark 2.27(v)]).

Remark 2.1 It is an open question whether every 2-quasitrace on a C^*-algebra is a trace (asked by Kaplansky). A theorem of Haagerup [20] says that if A is exact and unital then every bounded 2-quasitrace on A is a trace. This theorem can be extended to obtain that every lower semicontinuous 2-quasitrace (not necessarily bounded) on an exact C^*-algebra must be a trace (see [3, Remark 2.29(i)]). Brown and Winter [7] presented a short proof of Haagerup's result in the finite nuclear dimension case. Note that if A is a unital simple C^*-algebra of stable rank one and real rank zero, with strict comparison, then $QT_2(A) = T(A)$ (see [24, Theorem 2.9]).

Definition 2.3 (*Cuntz semigroup*) Denote the cone of positive elements of A by A_+. Let $a, b \in A_+$. One says that a is *Cuntz subequivalent* to b, denoted by $a \lesssim_{\mathrm{Cu}} b$, if there exists a sequence (r_n) in A such that $r_n^* b r_n \to a$. One says that a is *Cuntz equivalent* to b, denoted by $a \sim_{\mathrm{Cu}} b$, if $a \lesssim_{\mathrm{Cu}} b$ and $b \lesssim_{\mathrm{Cu}} a$. The *Cuntz semigroup* of A is defined as $\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+ / \sim_{\mathrm{Cu}}$. We will denote the class of $a \in (A \otimes \mathcal{K})_+$ in $\mathrm{Cu}(A)$ by $\langle a \rangle$. Note that $\mathrm{Cu}(A)$ is a positively ordered abelian semigroup with zero (or monoid) when equipped with the addition: $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$, and the relation:

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \lesssim_{\mathrm{Cu}} b, \quad a, b \in (A \otimes \mathcal{K})_+.$$

The following facts are well known; see [29].

Lemma 2.2 *Let A be a C^*-algebra, let $a, b \in A_+$, and let p, q be projections. Then*

(i) *$a \lesssim_{\mathrm{Cu}} b$ if and only if $(a - \varepsilon)_+ \lesssim_{\mathrm{Cu}} b$ for all $\varepsilon > 0$;*
(ii) *if $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \lesssim_{\mathrm{Cu}} b$;*
(iii) *$p \leq q$ if and only if $p \lesssim_{\mathrm{Cu}} q$.*

Definition 2.4 [[12] (*The category Cu*)] Let $(S, \leq)$ be a positively ordered abelian semigroup with zero (or monoid). For x and y in S, let us say that x is compactly contained in y (or x is way-below y), and denote it by $x \ll y$, if for every increasing sequence (y_n) in S that has a supremum, if $y \leq \sup_{n \in \mathbb{N}} y_n$, then there exists k such that $x \leq y_k$. This is an auxiliary relation on S, called the compact containment relation. If $x \in S$ satisfies $x \ll x$, we say that x is compact.

We say that S is a Cu-semigroup of the Cuntz category *Cu*, if it has a 0 element (so is a monoid) and satisfies the following order-theoretic axioms:

(O1): Every increasing sequence of elements in S has a supremum.
(O2): For any $x \in S$, there exists a $\ll$-increasing sequence $(x_n)_{n \in \mathbb{N}}$ in S such that $\sup_{n \in \mathbb{N}} x_n = x$.
(O3): Addition and the compact containment relation are compatible.
(O4): Addition and suprema of increasing sequences are compatible.

A Cu-morphism between two Cu-semigroups is a positively ordered monoid morphism that preserves the compact containment relation and suprema of increasing sequences.

Definition 2.5 Let S be a Cu-semigroup. S is said to have weak cancellation if, for every $x, y, z, z' \in S$ with $z' \ll z$, we have that $x + z \ll y + z'$ implies $x \leq y$. It was shown in [31, Theorem 4.3] that the Cuntz semigroup of a C^*-algebra with stable rank one has weak cancellation (see also [15]).

The following is a foundation result which establishes the relation between C^*-algebras and the category *Cu*.

Theorem 2.3 [12] *Let A be a C^*-algebra. Then $\mathrm{Cu}(A)$ is a Cu-semigroup. Moreover, if $\varphi : A \to B$ is a $*$-homomorphism between C^*-algebras, then φ naturally induces a* Cu-*morphism* $\mathrm{Cu}(\varphi) : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$.

Definition 2.6 [17] Let A be a C^*-algebra. A *functional* on $\mathrm{Cu}(A)$ is a map $f : \mathrm{Cu}(A) \to [0, \infty]$ which takes 0 into 0 and preserves addition, order, and the suprema of increasing sequences. Denote by $\mathrm{F}(\mathrm{Cu}(A))$ the set of all the functionals on $\mathrm{Cu}(A)$ endowed with the topology in which a net (λ_i) converges to λ if

$$\limsup \lambda_i(x) \leq \lambda(y) \leq \liminf \lambda_i(y)$$

for all $x, y \in \mathrm{Cu}(A)$ such that $x \ll y$.

If A is unital, a functional λ on $\mathrm{Cu}(A)$ is said to be normalized if $\lambda([1]) = 1$. Denote by $\mathrm{F}_{[1]}(\mathrm{Cu}(A))$ the set of all the normalized functionals on $\mathrm{Cu}(A)$.

Definition 2.7 Let $\tau \in QT_2(A)$, we define a map $d_\tau : A \otimes \mathcal{K} \to [0, \infty]$ by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

It has the following properties:

(1) if $a \lesssim_{\mathrm{Cu}} b$, then $d_\tau(a) \leq d_\tau(b)$;
(2) if a and b are mutually orthogonal, then $d_\tau(a + b) = d_\tau(a) + d_\tau(b)$;
(3) $d_\tau((a - \varepsilon)_+) \to d_\tau(a)$ $(\varepsilon \to 0)$.

This map depends only on the Cuntz equivalence class of $a \in A \otimes \mathcal{K}$. Hence, we will also d_τ to denote the induced normalized functional on $\mathrm{Cu}(A)$.

Remark 2.4 Given $\lambda \in \mathrm{F}_{[1]}(\mathrm{Cu}(A))$, the function

$$\tau_\lambda(a) = \int_0^\infty \lambda(\langle (a - t)_+ \rangle) dt$$

defined on the positive cone A_+ can be extended to a normalized lower semicontinuous quasitrace on A. If A is separable, it can be checked that $QT_2(A)$ has a countable basis (see [17, Theorem 3.7]).

The following result is [17, Theorem 4.4] (see also [19, Theorem 6.9]).

Theorem 2.5 *Let A be a unital C^*-algebra. Then the cones $QT_2(A)$ and $\mathrm{F}_{[1]}(\mathrm{Cu}(A))$ are compact and Hausdorff, and the map $\tau \mapsto d_\tau$ is a homeomorphism between them.*

It follows that if A is exact then every functional on $\mathrm{Cu}(A)$ arises from a lower semicontinuous trace.

Combining the above results, we obtain a characterization of strict comparison.

Proposition 2.6 *Suppose that A is simple unital, then the following statements are equivalent:*

(i) *A has strict comparison (of positive elements), i.e., for any non-zero $a, b \in (A \otimes \mathcal{K})_+$, $d_\tau(a) < d_\tau(b)$, $\tau \in QT_2(A)$, implies $a \lesssim_{\mathrm{Cu}} b$.*
(ii) *For any $s, t \in \mathrm{Cu}(A)$, $\lambda(s) < \lambda(t)$, $\lambda \in \mathrm{F}_{[1]}(\mathrm{Cu}(A))$, implies $s \leq t$.*

Let Ω be a compact metric space. Denote by $\overline{\mathbb{N}}$ the set of natural numbers with 0 and ∞ adjoined. By [27], if the covering dimension of Ω is at most two and $\check{H}^2(K) = 0$ (the Čech cohomology with integer coefficients) for any compact subset $K \subset \Omega$, then the Cuntz semigroup of $C(\Omega)$ is isomorphic to the ordered semigroup $\mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$. If Ω is an interval or a graph without loops, the classification results of the present paper were obtained in [10, 11]. Note that if Ω is a compact subset of $\mathbb{C}$, then we have $\mathrm{Cu}(C(\Omega)) \cong \mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$. (This can be deduced from $\check{H}^2(K) = \lim_{i\to\infty} H^2(N(\mathcal{U}_i))$, where $\mathcal{U}_i$ is an open cover of K and $N(\mathcal{U}_i)$ is the nerve of $\mathcal{U}_i$, while $H^2(N(\mathcal{U}_i)) = 0$; see [1, pp. 256–257].)

Definition 2.8 Let $\Omega \subset \mathbb{C}$ be a compact subset and let $O \subset \Omega$ be an open set. For $r > 0$, set $O_r = \{x \in \Omega \mid \mathrm{dist}(x, O) < r\}$. Let f_O denote the positive function corresponding to O as follows:

$$f_O(x) = \begin{cases} \min\{1, \mathrm{dist}(x, \Omega\backslash O)\}, & \text{if } x \in O, \\ 0, & \text{otherwise.} \end{cases}$$

Then $0 \le f_O \le 1$ and $\mathrm{support}(f_O) = O$. We shall use $\mathbb{1}_O$ to denote the class $\langle f_O \rangle$. Let $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ be a Cu-morphism with $\alpha(\mathbb{1}_\Omega) = \langle 1_A \rangle$. For any $\tau \in T(A)$, $d_\tau \circ \alpha$ defines a lower semicontinuous subadditive rank function on $C(\Omega)$. By [4, Proposition I.2.1], this function uniquely corresponds to a countably additive measure on Ω, denoted by $\mu_{\alpha^*\tau}$, i.e., for any open set $O \subset \Omega$, we have

$$\mu_{\alpha^*\tau}(O) := d_\tau(\alpha(\mathbb{1}_O)).$$

The following result combines [6, Corollaries 4.6 and 4.7], together with the fact that if A is separable, unital and has stable rank one then $x \in W(A)$ if $x \in \mathrm{Cu}(A)$ and $x \le \langle 1_A \rangle$ (see [25, 6.2(1)]).

Proposition 2.7 *Let A be a separable, unital C^*-algebra with stable rank one. Suppose that $x \in \mathrm{Cu}(A)$ satisfies $x \le \langle 1_A \rangle$. Then there exists $a \in A_+$ such that $x = \langle a \rangle$. Moreover, if x is compact, then a can be chosen to be a projection.*

Proposition 2.8 *Let A be a separable, unital C^*-algebra with stable rank one and let p be a projection in A. Suppose that $x_1, x_2, \ldots, x_n \in \mathrm{Cu}(A)$ are compact elements and satisfy $x_1 + x_2 + \cdots + x_n \le \langle p \rangle$. Then there exist mutually orthogonal projections $p_1, \ldots, p_n$ such that $\langle p_i \rangle = x_i$ and*

$$p_1 + p_2 + \cdots + p_n \le p.$$

Proof By Proposition 2.7, there exist projections $q_1, q_2, \ldots, q_n$ such that $\langle q_i \rangle = x_i$ for any i. By Lemma 2.2,

$$[q_1] + [q_2] + \cdots + [q_n] \le [p].$$

Since A has cancellation of projections, with $v_1 v_1^* = q_1$ and $v_1^* v_1 \le p$, and setting $p_1 = v_1^* q_1 v_1$, we have

$$[q_2] + \cdots + [q_n] \le [p - p_1].$$

There exists a partial isometry v_2 such that $v_2 v_2^* = q_2$ and $v_2^* v_2 \le p - p_1$. Set $p_2 = v_2^* v_2$ and continue this procedure; we obtain a collection of mutually orthogonal projections $\{p_i\}$ such that

$$\langle p_i \rangle = \langle q_i \rangle = x_i, \quad i = 1, 2, \ldots, n,$$

and

$$p_1 + p_2 + \cdots + p_n \le p. \qquad \blacksquare$$

Theorem 2.9 [12, Corollary 5] *If A is a C^*-algebra with $rr(A) = 0$, then $Cu(A)$ is algebraic ([2, Definition 5.5.1]: every element is the supremum of an increasing sequence of compact elements).*

3 Distances between homomorphisms

Definition 3.1 Let A be a unital C^*-algebra and let Ω be a compact metric space. Denote by $\mathrm{Hom}_1(C(\Omega), A)$ the set of all unital homomorphisms from $C(\Omega)$ into A. Let $\phi, \psi : C(\Omega) \to A$ be two unital homomorphisms. Define the Cuntz distance between ϕ, ψ by

$$d_W(\phi, \psi) = \inf\{\, r > 0 \,|\, \phi(f_O) \lesssim_{\mathrm{Cu}} \psi(f_{O_r}), \psi(f_O) \lesssim_{\mathrm{Cu}} \phi(f_{O_r}),\ O \subset \Omega, \text{ open}\}.$$

Write $\phi \sim \psi$ if $d_W(\phi, \psi) = 0$. It is easy to see that "$\sim$" is an equivalence relation. Put

$$H_{c,1}(C(\Omega), A) = \mathrm{Hom}_1(C(\Omega), A)/\sim.$$

Remark 3.1 The definition of d_W can be regarded as the symmetric version of the distance $D_c(\cdot, \cdot)$ defined in [21]. When A is a unital simple C^*-algebra with stable rank one, $(H_{c,1}(C(\Omega), A), d_W)$ is a metric space; see [21, Proposition 2.15]. There are some works where this distance is considered in special cases (see [10, 11, 16]).

Definition 3.2 Let $\varphi \in \mathrm{Hom}_1(C(\Omega), A)$. Then $\ker \varphi = \{f \in C(\Omega) : f|_X = 0\}$ for some compact subset $X \subset \Omega$. We shall call X the spectrum of φ. We may also use φ_X to denote φ. If $X \subset \mathbb{C}$, every homomorphism $\varphi_X : C(\Omega) \to A$ corresponds to a normal element $x = \varphi_X(\mathrm{id}) \in A$, where $\mathrm{id} : X \to X \subset \mathbb{C}$ is the identity function.

Conversely, suppose that x, y are normal elements in A with $\mathrm{sp}(x) = X$ and $\mathrm{sp}(y) = Y$. We can define $\varphi_X, \varphi_Y : C(X \cup Y) \to A$ to be two homomorphisms with $\varphi_X(f) = f(x)$ and $\varphi_Y(f) = f(y)$ for all $f \in C(X \cup Y)$. Define the Cuntz distance between normal elements as follows:

$$d_W(x, y) := d_W(\varphi_X, \varphi_Y).$$

Definition 3.3 Let Ω be a compact metric space and let $\alpha, \beta : \mathrm{Lsc}(\Omega, \overline{\mathbb{N}}) \to \mathrm{Cu}(A)$ be two Cu-morphisms. Define the Cuntz distance between α, β by

$$d_{\mathrm{Cu}}(\alpha, \beta) := \inf\{\, r > 0 \,|\, \alpha(\mathbb{1}_O) \le \beta(\mathbb{1}_{O_r}),\ \beta(\mathbb{1}_O) \le \alpha(\mathbb{1}_{O_r}),\ \forall\ O \subset \Omega, \text{ open}\}.$$

Denote by $\mathrm{Cu}(C(\Omega), A)$ the set of all Cu-morphisms from $\mathrm{Cu}(C(\Omega))$ to $\mathrm{Cu}(A)$.

Remark 3.2 For any $\alpha, \beta, \gamma \in \mathrm{Cu}(C(\Omega), A)$ and $\phi, \psi \in \mathrm{Hom}_1(C(\Omega), A)$, the following properties hold:

(i) $d_{\mathrm{Cu}}(\alpha, \beta) = d_{\mathrm{Cu}}(\beta, \alpha)$;
(ii) $d_{\mathrm{Cu}}(\alpha, \beta) \le d_{\mathrm{Cu}}(\alpha, \gamma) + d_{\mathrm{Cu}}(\beta, \gamma)$;
(iii) $d_W(\phi, \psi) = d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \mathrm{Cu}(\psi))$.

Proposition 3.3 *Let Ω be a compact subset of $\mathbb{C}$. Then d_{Cu} is a metric on the Cuntz category morphisms from* $\mathrm{Cu}(C(\Omega))$ *to* $\mathrm{Cu}(A)$.

Proof Let us identify $\mathrm{Cu}(C(\Omega))$ with the semigroup of lower semicontinuous functions $\mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$. Suppose that $d_{\mathrm{Cu}}(\alpha, \beta) = 0$. We need only to show that α and β

agree on the functions $\mathbb{1}_O$ for any open set $O \subset \Omega$ (their overall equality is apparent through the additivity and preservation of suprema of increasing sequences).

For any open set $O \subset \Omega$, there exists a sequence of open subsets O_n such that $\sup_n \mathbb{1}_{O_n} = \mathbb{1}_O$ and $\overline{O_n} \subset O_{n+1}$ for any n. Since O_n is bounded, there exists $r_n > 0$ such that $(O_n)_{r_n} \subset O_{n+1}$, and by the definition of d_{Cu}, we have $\alpha(\mathbb{1}_{O_n}) \le \beta(\mathbb{1}_{O_{n+1}})$ and $\beta(\mathbb{1}_{O_n}) \le \alpha(\mathbb{1}_{O_{n+1}})$.

Then we have

$$\alpha(\mathbb{1}_{O_1}) \le \beta(\mathbb{1}_{O_2}) \le \cdots \le \alpha(\mathbb{1}_{O_{2n-1}}) \le \beta(\mathbb{1}_{O_{2n}}) \le \cdots.$$

Note that

$$\sup_n \alpha(\mathbb{1}_{O_{2n-1}}) = \alpha(\mathbb{1}_O), \ \sup_n \beta(\mathbb{1}_{O_{2n}}) = \beta(\mathbb{1}_O),$$

which implies $\alpha(\mathbb{1}_O) = \beta(\mathbb{1}_O)$, as desired. ∎

We will now present a version of the Marriage lemma.

Proposition 3.4 *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathrm{Cu}(C(\Omega), A)$. Then*

$$d_{\mathrm{Cu}}\left(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\right) \le \min_{\sigma \in S_n} \max_{1 \le i \le n} d_{\mathrm{Cu}}(\alpha_i, \beta_{\sigma(i)}),$$

where S_n is the set of all permutations of $(1, 2, \ldots, n)$.

Proof Let $d = \min_{\sigma \in S_n} \max_{1 \le i \le n} d_{\mathrm{Cu}}(\alpha_i, \beta_{\sigma(i)})$. Then for any $\varepsilon > 0$, there exists $\sigma \in S_n$ such that

$$d_{\mathrm{Cu}}(\alpha_i, \beta_{\sigma(i)}) < d + \varepsilon, \quad i = 1, 2, \ldots, n.$$

For any open set $O \subset \Omega$, we get

$$\alpha_i(\mathbb{1}_O) \le \beta_{\sigma(i)}(\mathbb{1}_{O_{d+\varepsilon}}), \ \beta_{\sigma(i)}(\mathbb{1}_O) \le \alpha_i(\mathbb{1}_{O_{d+\varepsilon}}), \quad i = 1, 2, \ldots, n.$$

Then we have

$$\sum_{i=1}^n \alpha_i(\mathbb{1}_O) \le \sum_{i=1}^n \beta_i(\mathbb{1}_{O_{d+\varepsilon}}), \ \sum_{i=1}^n \beta_i(\mathbb{1}_O) \le \sum_{i=1}^n \alpha_i(\mathbb{1}_{O_{d+\varepsilon}}).$$

Hence,

$$d_{\mathrm{Cu}}\left(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\right) \le d + \varepsilon.$$

Since ε is arbitrary, the conclusion follows. ∎

Definition 3.4 Let A be a unital C^*-algebra and Ω be a compact metric space. Let $x, y \in A$ be normal elements and $\phi, \psi : C(\Omega) \to A$ be two homomorphisms. We say ϕ and ψ are approximately unitarily equivalent, written $\phi \sim_{aue} \psi$, if there exists a sequence of unitaries $u_n \in A$ such that $u_n \phi u_n^* \to \psi$ pointwise. Define the distance between unitary orbits of x and y by

$$d_U(x, y) = \inf\{\|uxu^* - y\| \ : u \text{ is a unitary in } A\}.$$

Lemma 3.5 *Let $\{x_n\}$ be a sequence of normal elements in A with limit x. Suppose that Ω is a compact subset of $\mathbb{C}$ such that $\mathrm{sp}(x_n) \subset \Omega$. Then for any finite set $F \subset C(\Omega)$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f(x_n) - f(x)\| < \varepsilon$ for all $f \in F$ and $n \geq N$.*

Proof We may suppose that $\|x_n\| \leq M$ for all n, so that also $\|x\| \leq M$. For any $f \in C(\Omega)$ and $\varepsilon > 0$, by the Stone–Weierstrass theorem, there exists a polynomial $P(z, \bar{z})$ such that

$$\|f - P(z, \bar{z})\| < \frac{\varepsilon}{3}.$$

Note that

$$\begin{aligned}\|(x_n^*)^s x_n^t - (x^*)^s x^t\| \leq & \|(x_n^*)^s x_n^t - x_n (x^*)^{s-1} x^t\| + \|x_n (x^*)^{s-1} x^t - (x^*)^s x^t\| \\ \leq & M \|(x_n^*)^{s-1} x_n^t - (x^*)^{s-1} x^t\| + M^{s+t} \|x_n - x\|.\end{aligned}$$

By induction, we have

$$\|(x_n^*)^s x_n^t - (x^*)^s x^t\| \leq (s+t) M^{s+t} \|x_n - x\|.$$

Therefore, there exists N_f such that if $\|x_n - x\|$ is sufficiently small for all $n \geq N_f$, we will have

$$\|P(x_n, x_n^*) - P(x, x^*)\| < \frac{\varepsilon}{3}.$$

Now we have

$$\begin{aligned}\|f(x_n) - f(x)\| \leq & \|f(x_n) - P(x_n, x_n^*)\| + \|P(x_n, x_n^*) - P(x, x^*)\| \\ & + \|P(x, x^*) - f(x)\| \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since F is finite, $N := \max\{N_f \mid f \in F\}$ is as desired. ∎

Definition 3.5 Let A be a unital C^*-algebra and let x and y be normal elements in A. Let us say that x and y have *the same index*, written $\mathrm{ind}(x) = \mathrm{ind}(y)$, if

$$[\lambda - x] = [\lambda - y] \text{ in } K_1(A)$$

for all $\lambda \notin \mathrm{sp}(x) \cup \mathrm{sp}(y)$. (Note that $\lambda - x, \lambda - y$ are invertible and so give rise to the K_1-classes; see [30].)

The following theorem shows the relation between $d_W(x, y)$ and $d_U(x, y)$; see [21, Corollary 6.4 and Theorem 6.7].

Theorem 3.6 *Let A be a unital simple separable C^*-algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Suppose that x and y are two normal elements in A with $\mathrm{ind}(x) = \mathrm{ind}(y)$. Then*

$$d_U(x, y) \leq 2 d_W(x, y).$$

Theorem 3.7 *Let A be a unital simple separable C^*-algebra with real rank zero, stable rank one, and weakly unperforated $K_0(A)$. Let Ω be a compact subset of $\mathbb{C}$. Suppose that $x_1, \ldots, x_n, x$ are normal elements in A with $\mathrm{sp}(x_i) \subset \Omega$, $1 \leq i \leq n$, $\mathrm{sp}(x) \subset \Omega$, and $\phi, \psi : C(\Omega) \to A$ are two unital homomorphisms. Then*

(1) *if* $d_U(x_n, x) \to 0$, *then* $d_W(x_n, x) \to 0$;
(2) *if* $d_W(\phi, \psi) = 0$ *and* $\mathrm{ind}(\phi(\mathrm{id})) = \mathrm{ind}(\psi(\mathrm{id}))$, *then* $\phi \sim_{aue} \psi$.

Proof (1) Without loss of generality, we may assume that $x_n \to x$. Suppose that $X_n = \mathrm{sp}(x_n)$ and $X = \mathrm{sp}(x)$. We need to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_W(\varphi_{X_n}, \varphi_X) < \varepsilon, \quad n \geq N.$$

Let $\delta = \varepsilon/2$. Since Ω is compact, there is a finite open cover $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ of Ω with diameter$(\Omega_i) \leq \delta$, $i = 1, 2, \ldots, m$. Let $\mathcal{F}$ denote the set of unions of some of the sets $\Omega_1, \Omega_2, \ldots, \Omega_m$. For any $Y \in \mathcal{F}$, define

$$g_Y(z) = \begin{cases} 1 - \mathrm{dist}(z, Y)/\delta, & \text{if } z \in (Y)_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$F = \{g_Y(z) \mid Y \in \mathcal{F}\}.$$

Since F is finite, by Lemma 3.5, there exists $N \in \mathbb{N}$ such that

$$\|g(x_n) - g(x)\| < \delta, \quad g \in F, \ n \geq N.$$

Now for any open set $O \subset \Omega$, let $Y_O = \bigcup_{O \cap \Omega_i \neq \emptyset} \Omega_i$. Then $Y_O \in \mathcal{F}$ and

$$O \subset Y_O \subset O_\delta \subset (Y_O)_\delta \subset O_{2\delta}.$$

Then we have

$$\varphi_{X_n}(f_O) \lesssim_{\mathrm{Cu}} \varphi_{X_n}(f_{Y_O}) \quad \text{and} \quad \varphi_X(f_{Y_O}) \lesssim_{\mathrm{Cu}} \varphi_X(f_{O_{2\delta}}).$$

Note that $g_{Y_O} \in F$, so for all $n \geq N$, we have

$$\|g_{Y_O}(x_n) - g_{Y_O}(x)\| < \delta.$$

It follows from Lemma 2.2(ii) that

$$(g_{Y_O}(x_n) - \delta)_+ \lesssim_{\mathrm{Cu}} g_{Y_O}(x).$$

Note that support$(g_{Y_O}) = (Y_O)_\delta$, so that $f_{Y_O} \lesssim_{\mathrm{Cu}} (g_{Y_O} - \delta)_+$, and we get

$$f_{Y_O}(x_n) \lesssim_{\mathrm{Cu}} (g_{Y_O}(x_n) - \delta)_+ \lesssim_{\mathrm{Cu}} g_{Y_O}(x) \lesssim_{\mathrm{Cu}} f_{(Y_O)_\delta}(x).$$

Therefore,

$$\varphi_{X_n}(f_{Y_O}) = f_{Y_O}(x_n) \lesssim_{\mathrm{Cu}} f_{(Y_O)_\delta}(x) = \varphi_X(f_{(Y_O)_\delta}).$$

Now we have

$$\varphi_{X_n}(f_O) \lesssim_{\mathrm{Cu}} \varphi_{X_n}(f_{Y_O}) \lesssim_{\mathrm{Cu}} \varphi_X(f_{(Y_O)_\delta}) \lesssim_{\mathrm{Cu}} \varphi_X(f_{O_{2\delta}}).$$

Similarly, for any open $O \subset \Omega$, we also have

$$\varphi_X(f_O) \lesssim_{\mathrm{Cu}} \varphi_{X_n}(f_{O_{2\delta}}).$$

Finally, we obtain

$$d_W(\varphi_{X_n},\varphi_X) < 2\delta = \varepsilon.$$

(2) Set $a = \phi(\mathrm{id})$, $b = \psi(\mathrm{id})$. By hypothesis, we have $d_W(a,b) = 0$ and $\mathrm{ind}(a) = \mathrm{ind}(b)$, and so by Theorem 3.6, we get $d_U(a,b) = 0$. This means that there exists a sequence of unitaries $u_n \in A$ such that $u_n^* a u_n \to b$. Then for any finite subset $F \subset C(\Omega)$ and $\varepsilon > 0$, by Lemma 3.5, there exists $N \in \mathbb{N}$ such that

$$\|f(u_n^* a u_n) - f(b)\| < \varepsilon, \quad f \in F,\ n \geq N.$$

From the Stone–Weierstrass theorem, it can be checked that

$$f(u_n^* a u_n) = u_n^* f(a) u_n, \quad f \in F.$$

Now we get

$$\|u_n^* \phi(f) u_n - \psi(f)\| < \varepsilon, \quad f \in F.$$

Since ε is arbitrary, we have $\phi \sim_{aue} \psi$. ■

Remark 3.8 The question whether the metrics d_W and d_U are equivalent relates to the distances between unitary orbits. There are some results for self-adjoint elements and normal elements. Under certain conditions, one can even get $d_W = d_U$; see [16, 21–23, 28] for more details. Distances for Cu-morphisms between general pairs of Cu-semigroups are studied in detail in [9, Section 5]. Some similar results intersecting with this work can be found in [8], which employs a different method.

4 Approximate lifting

In this section, we present an approximate existence result. Given a Cu-morphism with certain properties, we can approximately lift it to a homomorphism between C^*-algebras.

Proposition 4.1 *Let A be a unital, simple, separable C^*-algebra of stable rank one. Then for any $x \in \mathrm{Cu}(A)(x \neq 0)$ with $x \leq \langle 1_A \rangle$, we have $\inf_{\tau \in T(A)} d_\tau(x) > 0$. (Here, d_τ is a normalized functional on $\mathrm{Cu}(A)$.)*

Proof From the definition of $\mathrm{Cu}(A)$ and Lemma 2.7, there exists $a \in A_+$ such that $a \leq 1_A$ and $\langle a \rangle = x$. By the simplicity of A, there exist $a_1, a_2, \ldots, a_k$ in A such that $1_A = \sum_{i=1}^k a_i^* a a_i$. Then for any $\tau \in T(A)$,

$$1 = \tau(1_A) = \sum_{i=1}^k \tau(a_i^* a a_i) = \sum_{i=1}^k \tau(a^{1/2} a_i^* a_i a^{1/2}) \leq \sum_{i=1}^k \|a_i^* a_i\| \cdot \tau(a).$$

Now we get $\tau(a) > 0$, whence from the compactness of $T(A)$ and

$$d_\tau(x) \geq \tau(a),$$

we get $\inf_{\tau \in T(A)} d_\tau(x) > 0$. ■

Suppose that Ω is a compact space, and for any $x \in \Omega$, write $B(x,r) = \{y \in \Omega \mid \mathrm{dist}(y,x) < r\}$ and $R(x,s) = \{y \in \Omega \mid \mathrm{dist}(y,x) = s\}$.

Lemma 4.2 *Let A be a unital, simple, separable C^*-algebra with $QT_2(A) = T(A)$ and Ω be a compact metric space. Let $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ be a Cu-morphism with $\alpha(\mathbb{1}_\Omega) \le \langle 1_A \rangle$. Then for any $x \in \Omega$ and $r, \sigma > 0$, there exist $s \in (r/2, r)$ and $\varepsilon > 0$ such that $s \pm \varepsilon \in (r/2, r)$ and*

$$d_\tau(\alpha(\mathbb{1}_{R(x,s)_\varepsilon})) \le \sigma, \quad \tau \in QT_2(A).$$

Proof For any open set $O \subset \Omega$ and $\tau \in T(A)$, let $\mu_{\alpha^*\tau}$ be the countably additive measure on Ω such that

$$\mu_{\alpha^*\tau}(O) = d_\tau(\alpha(\mathbb{1}_O)).$$

If $\alpha(\mathbb{1}_{B(x,r)}) = 0$, the proof is trivial. In general, we have $\mu_{\alpha^*\tau}(B(x,r)) \le 1$. Since $R(x,s) \cap R(x,s') = \varnothing$, if $s \ne s'$, there are at most finitely many s in $(r/2, r)$ such that

$$\mu_{\alpha^*\tau}(R(x,s)) > \sigma/2.$$

Since we have $QT_2(A) = T(A)$, by Remark 2.4, $QT_2(A)$ is compact metrizable and has a countable basis, and so we may choose a countable dense subset Y of $QT_2(A)$.

For any $\tau \in Y$, we define

$$\mathcal{S}_\tau = \{s \mid \mu_{\alpha^*\tau}(R(x,s)) > \sigma/2\}.$$

Then $\bigcup_{\tau\in Y} \mathcal{S}_\tau$ has at most countably many points and

$$(r/2, r) \backslash \bigcup_{\tau\in Y} \mathcal{S}_\tau \ne \varnothing.$$

Now there exist an $s \in (r/2, r)$ such that $\mu_{\alpha^*\tau}(R(x,s)) \le \sigma/2$, i.e.,

$$\mu_{\alpha^*\tau}(\Omega \backslash R(x,s)) \ge 1 - \sigma/2, \quad \tau \in Y.$$

That is,

$$d_\tau(\alpha(\mathbb{1}_{\Omega\backslash R(x,s)})) \ge 1 - \sigma/2, \quad \tau \in Y.$$

By the density of Y and Theorem 2.5, we have

$$d_\tau(\alpha(\mathbb{1}_{\Omega\backslash R(x,s)})) \ge 1 - \sigma/2, \quad \tau \in QT_2(A).$$

Let $\{\varepsilon_n\}$ be a strictly decreasing sequence such that

$$\varepsilon_n \le \min\{s - r/2, r - s\}, \ n = 1, 2 \dots, \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

The sequence $\{\mathbb{1}_{\Omega\backslash\overline{R(x,s)_{\varepsilon_n}}}\}$ is increasing in $\mathrm{Cu}(C(\Omega))$ with supremum $\mathbb{1}_{\Omega\backslash R(x,s)}$.

Since α and d_τ preserve the suprema of increasing sequences,

$$d_\tau(\alpha(\mathbb{1}_{\Omega\backslash R(x,s)})) = \lim_{n\to\infty} d_\tau(\alpha(\mathbb{1}_{\Omega\backslash\overline{R(x,s)_{\varepsilon_n}}})), \quad \tau \in QT_2(A).$$

For any $\tau \in QT_2(A)$, by [14, Lemma 3.1], there exist an integer $N_\tau \in \mathbb{N}$ and an open neighborhood V_τ of τ such that

$$1 - \sigma \le d_\tau(\alpha(\mathbb{1}_{\Omega\backslash R(x,s)})) - \frac{\sigma}{2} < d_\gamma(\alpha(\mathbb{1}_{\Omega\backslash\overline{R(x,s)_{\varepsilon_n}}})), \quad n > N_\tau, \ \gamma \in V_\tau.$$

Then $\{V_\tau \mid \tau \in QT_2(A)\}$ forms an open cover of $QT_2(A)$, and so from the compactness of $QT_2(A)$, there are finitely many sets $\{V_{\tau_1}, V_{\tau_2}, \ldots, V_{\tau_k}\}$ covering $QT_2(A)$. Now we set

$$N_0 = \max\{N_{\tau_1}, N_{\tau_2}, \ldots, N_{\tau_k}\}.$$

For any $n \geq N_0$, we have

$$d_\tau(\alpha(\mathbb{1}_{\Omega \setminus \overline{R(x,s)_{\varepsilon_n}}})) > 1 - \sigma, \quad \tau \in QT_2(A).$$

Then for any $0 < \varepsilon \leq \varepsilon_{N_0}$, we have $s \pm \varepsilon \in (r/2, r)$ and

$$d_\tau(\alpha(\mathbb{1}_{R(x,s)_\varepsilon})) \leq \sigma, \quad \tau \in QT_2(A). \qquad \blacksquare$$

Definition 4.1 Let Ω be a compact metric space and $\mathcal{F}$ be a finite collection of open subsets of Ω. Let $X, Y \in \mathcal{F}$, we say X and Y are almost connected if there exists a sequence of sets $X = \Omega_1, \Omega_2, \ldots, \Omega_n = Y$ in $\mathcal{F}$ such that for each i, $\Omega_i \in \mathcal{F}$ and $\Omega_i \cap \Omega_{i+1} \neq \emptyset$. Under this relation, $\mathcal{F}$ has finitely many almost connected components.

Definition 4.2 Let $\alpha : \mathrm{Lsc}(\Omega, \overline{\mathbb{N}}) \to \mathrm{Cu}(A)$ be a Cu-morphism and $\alpha(\mathbb{1}_\Omega) \leq \langle 1_A \rangle$. Let $\delta > 0$ and let $\{O_1, O_2, \ldots, O_N\}$ be a collection of mutually disjoint open sets of Ω. Set $U = \bigcup_{i=1}^N O_i$. We say $\{O_1, O_2, \ldots, O_N\}$ is an *almost δ-cover with respect to α* if,

(i) $\mathrm{dist}(x, U) < \delta$ for all $x \in \Omega$;
(ii) $\mathrm{diameter}(O_i) \leq \delta$, for any $i = 1, 2, \ldots, N$;
(iii) $\mathrm{dist}(O_i, O_j) > 0$, for any $i \neq j$, $i, j \in \{1, \ldots, N\}$;
(iv) $\alpha(\mathbb{1}_{\Omega \setminus \overline{U}}) \leq \alpha(\mathbb{1}_{(O_i)_\delta \cap U})$, for any $i = 1, 2, \ldots, N$;
(v) $\{(O_1)_\delta, (O_2)_\delta, \ldots, (O_N)_\delta\}$ has a unique almost connected component.

Lemma 4.3 *Let A be a unital, simple, separable C^*-algebra with stable rank one, strict comparison and $QT_2(A) = T(A)$ and let Ω be a compact metric space. Let $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ be a Cu-morphism and $\delta > 0$. Suppose that Ω has an open cover $\{B(x_1, \delta/4), \ldots, B(x_m, \delta/4)\}$ satisfying*

(1) $\alpha(\mathbb{1}_\Omega) \leq \langle 1_A \rangle$;
(2) $\alpha(\mathbb{1}_{B(x_i, \delta/2)}) \neq 0$, *for any* $i \in \{1, 2, \ldots, m\}$;
(3) $\{B(x_1, \delta/4), \cdots, B(x_m, \delta/4)\}$ *has a unique almost connected component.*

Then Ω has an almost δ-cover with respect to α.

Proof By Proposition 4.1, we set

$$\sigma = \min_{1 \leq i \leq m} \inf_{\tau \in T(A)} \{d_\tau(\alpha(\mathbb{1}_{B(x_i, \delta/2)}))\} > 0.$$

For each $i \in \{1, 2, \ldots, m\}$, by Lemma 4.2 for x_i, $\delta/2$, and $\sigma/(2m+1)$, there exist $s_i \in (\delta/4, \delta/2)$ and ε_i such that $s_i \pm \varepsilon_i \in (\delta/4, \delta/2)$ and

$$\mu_{\alpha^*\tau}(R(x_i, s_i)_{\varepsilon_i}) \leq \frac{\sigma}{2m+1} < \frac{\sigma}{2m}, \quad \tau \in QT_2(A).$$

Set $R = \bigcup_{i=1}^m R(x_i, s_i)$, then

$$\mu_{\alpha^*\tau}(R) \leq \sum_{i=1}^m \mu_{\alpha^*\tau}(R(x_i, s_i)_{\varepsilon_i}) < \frac{\sigma}{2m} \cdot m = \frac{\sigma}{2}.$$

Since $\Omega\backslash R$ is open, there exists a positive function $f_{\Omega\backslash R} \in C(\Omega)$ corresponding to $\Omega\backslash R$ (see 2.8) such that

$$d_\tau(\alpha(\langle f_{\Omega\backslash R}\rangle)) = \mu_{\alpha^*\tau}(\Omega\backslash R) > 1 - \frac{\sigma}{2}, \quad \tau \in QT_2(A).$$

Let $\{\sigma_n\}$ be a strictly decreasing sequence such that

$$\sigma_n \leq \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}, \ n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n\to\infty} \sigma_n = 0.$$

Set

$$W_n = \operatorname{supp}\{(f_{\Omega\backslash R} - \sigma_n)_+\}.$$

Then $\{\mathbb{1}_{W_n}\}$ is an increasing sequence in $\mathrm{Cu}(C(\Omega))$ with supremum $\mathbb{1}_{\Omega\backslash R}$. Since α preserves suprema, we have

$$\alpha(\mathbb{1}_{\Omega\backslash R}) = \sup_{n\in\mathbb{N}} \alpha(\mathbb{1}_{W_n}).$$

Hence,

$$d_\tau(\alpha(\mathbb{1}_{\Omega\backslash R})) = \lim_{n\to\infty} d_\tau(\alpha(\mathbb{1}_{W_n})), \quad \tau \in QT_2(A).$$

Since $QT_2(A)$ is compact, with a similar method of the proof of Lemma 4.2, there exists N_0 such that

$$d_\tau(\alpha(\mathbb{1}_{W_n})) > 1 - \sigma, \quad n > N_0, \ \tau \in QT_2(A).$$

Now for fixed integers $n_0 > n_1 > N_0$, we have $\sigma_{n_0} < \sigma_{n_1}$ and

$$W_{n_0} \cup R_{\sigma_{n_0}} \cup \{x \mid f_{\Omega\backslash R}(x) = \sigma_{n_0}\} = W_{n_1} \cup R_{\sigma_{n_1}} \cup \{x \mid f_{\Omega\backslash R}(x) = \sigma_{n_1}\} = \Omega.$$

As $W_{n_0} \supset W_{n_1}$, we then have

$$\{x \mid f_{\Omega\backslash R}(x) = \sigma_{n_0}\} \subset R_{\sigma_{n_1}} \subset \bigcup_{i=1}^m R(x_i, s_i)_{\varepsilon_i}.$$

Now we set

$$\eta := \sigma_{n_0}, \quad U := W_{n_0}.$$

Note that

$$\eta < \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\} \quad \text{and} \quad d_\tau(\alpha(\mathbb{1}_U)) > 1 - \sigma, \quad \forall\, \tau \in QT_2(A).$$

We also have

$$U \cup R_\eta \cup \{x \mid f_{\Omega\backslash R}(x) = \eta\} = \Omega \subset U \cup \bigcup_{i=1}^m R(x_i, s_i)_{\varepsilon_i}.$$

Define

$$O_1 := U \cap B(x_1, s_1),$$

$$O_2 := (U\backslash O_1) \cap B(x_2, s_2),$$

$$\ldots$$

$$O_m := (U\backslash \cup_{i=1}^{m-1} O_i) \cap B(x_m, s_m).$$

Note that all the O_i are open sets in U. Let us delete the empty sets and rewrite those remaining as $\{O_1, O_2, \ldots, O_N\}$; then $U = \bigcup_{i=1}^{N} O_i$.

Let us now show that $\{O_1, O_2, \ldots, O_N\}$ is an almost δ-cover with respect to α. For any $x \in \Omega$, if $x \in U$, it is trivial that $\operatorname{dist}(x, U) = 0$; if $x \in \bigcup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i}$, there exists i_0 such that $x \in B(x_{i_0}, \delta/2)$. Since

$$\sum_{i=1}^{m} \mu_{\alpha^*\tau}(R(x_i, s_i)_{\varepsilon_i}) < \frac{\sigma}{2} < \mu_{\alpha^*\tau}\left(B\left(x_{i_0}, \frac{\delta}{2}\right)\right),$$

we have $B(x_{i_0}, \delta/2) \backslash \bigcup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i} \neq \varnothing$, and hence, there exists $y \in B(x_{i_0}, \delta/2) \cap U$ such that $\operatorname{dist}(x, y) < \delta$. From the construction of O_i, for any $i \geq 1$, O_i is contained in $B(x_j, s_j)$ for some j, and so diameter$(O_i) \leq \delta$. If $i \neq j$, then O_i and O_j can be separated by R_η, and so $\operatorname{dist}(O_i, O_j) > 0$. Then (i)–(iii) hold.

Now we check (iv). Given any O_i, there exists j such that

$$O_i \subset B(x_j, s_j) \subset B\left(x_j, \frac{\delta}{2}\right) \quad \text{and} \quad \alpha(\mathbb{1}_{B(x_j, \frac{\delta}{2})}) \neq 0.$$

Set

$$Y_1 := B\left(x_j, \frac{\delta}{2}\right) \cap U, \quad Y_2 := B\left(x_j, \frac{\delta}{2}\right) \cap \bigcup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i}.$$

Then we have

$$Y_1 \subset B\left(x_j, \frac{\delta}{2}\right) \subset Y_1 \cup Y_2 \subset (O_i)_\delta.$$

Recall that

$$\sum_{i=1}^{m} d_\tau(\alpha(\mathbb{1}_{R(x_i, s_i)_{\varepsilon_i}})) < \frac{\sigma}{2}, \quad \tau \in QT_2(A).$$

Then

$$d_\tau(\alpha(\mathbb{1}_{Y_2})) < d_\tau(\alpha(\mathbb{1}_{\cup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i}})) < \frac{\sigma}{2},$$

and hence,

$$\sigma \leq d_\tau(\alpha(\mathbb{1}_{B(x_j, \frac{\delta}{2})})) \leq d_\tau(\alpha(\mathbb{1}_{Y_1})) + d_\tau(\alpha(\mathbb{1}_{Y_2})) < d_\tau(\alpha(\mathbb{1}_{Y_1})) + \frac{\sigma}{2}.$$

Now we have

$$d_\tau(\alpha(\mathbb{1}_{Y_1})) > \frac{\sigma}{2}, \quad \tau \in QT_2(A).$$

Since

$$\Omega \backslash \overline{U} \subset \bigcup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i},$$

we have

$$d_\tau(\alpha(\mathbb{1}_{\Omega \backslash \overline{U}})) \leq d_\tau(\alpha(\mathbb{1}_{\cup_{i=1}^{m} R(x_i, s_i)_{\varepsilon_i}})) < \frac{\sigma}{2} < d_\tau(\alpha(\mathbb{1}_{Y_1})), \ \tau \in QT_2(A).$$

Since A has strict comparison, by Proposition 2.6 and the inclusion $Y_1 \subset (O_i)_\delta \cap U$, we have

$$\alpha(\mathbb{1}_{\Omega\setminus\overline{U}}) \le \alpha(\mathbb{1}_{Y_1}) \le \alpha(\mathbb{1}_{(O_i)_\delta\cap U}).$$

Finally, notice that for any i, we have shown that $B(x_i, \delta/2) \cap U \ne \varnothing$, and then $B(x_i, \delta/2) \subset (O_{j_i})_\delta$ for some j_i. Combining this with assumption (3), $\{(O_{j_1})_\delta, (O_{j_2})_\delta, \dots, (O_{j_m})_\delta\}$ is also an open cover of Ω and has a unique almost connected component. Note that $\Omega = \bigcup_{i=1}^m (O_{j_i})_\delta = \bigcup_{k=1}^N O_k$, then for any $k \in \{1, 2, \dots, N\}$, there exists $(O_{j_i})_\delta$ such that $O_k \cap (O_{j_i})_\delta \ne \varnothing$. Thus any two elements in $\{(O_1)_\delta, (O_2)_\delta, \dots, (O_N)_\delta\}$ are almost connected through $\{(O_{j_1})_\delta, (O_{j_2})_\delta, \dots, (O_{j_m})_\delta\}$. In general, $\{(O_1)_\delta, (O_2)_\delta, \dots, (O_N)_\delta\}$ has a unique almost connected component, that is, (v) holds. ■

Lemma 4.4 *Let A be a unital, simple, separable C^*-algebra with stable rank one, real rank zero, strict comparison and let Ω be a compact metric space. Let $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ be a Cu-morphism and p is a projection in A. Suppose that*

(1) $\alpha(\mathbb{1}_\Omega) = \langle p\rangle$;
(2) *Ω has an almost δ-cover with respect to α.*

Then there exists a $*$-homomorphism $\phi : C(\Omega) \to pAp$ with finite dimensional range such that

$$d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) < 9\delta.$$

Proof Suppose that $\{O_1, O_2, \dots, O_N\}$ is an almost δ-cover respect to α.

Let

$$U = \bigcup_{i=1}^N O_i, \quad \rho = \frac{1}{4}\min\{\delta, \mathrm{dist}(O_i, O_j),\ i \ne j,\ 1 \le i, j \le N\}.$$

Then the facts that $\mathbb{1}_{O_i} \ll \mathbb{1}_{(O_i)_\rho}$ and α preserves the compact containment relation imply that

$$\alpha(\mathbb{1}_{O_i}) \ll \alpha(\mathbb{1}_{(O_i)_\rho}) \ll \alpha(\mathbb{1}_{(O_i)_{2\rho}}).$$

Since $\mathrm{Cu}(A)$ is algebraic (see 2.9), for each i, there exists an increasing sequence of compact elements $\{x_i^n\}_n$ with supremum $\alpha(\mathbb{1}_{(O_i)_{2\rho}})$. From the compact containment relation, there exists $n_i \in \mathbb{N}$ such that $\alpha(\mathbb{1}_{(O_i)_\rho}) \le x_i^{n_i}$. For convenience, we use x_i to denote $x_i^{n_i}$; then,

$$\alpha(\mathbb{1}_{(O_i)_\rho}) \le x_i \le \alpha(\mathbb{1}_{(O_i)_{2\rho}}).$$

Now we have

$$x_1 + x_2 + \cdots + x_N \le \alpha(\mathbb{1}_{\cup_{i=1}^N (O_i)_{2\rho}}) \le \langle p\rangle.$$

By Proposition 2.8, there exists a collection of mutually orthogonal projections $\{p_i\}$ such that

$$\langle p_i\rangle = x_i, \quad i = 1, 2, \dots, N$$

and

$$p_1 + p_2 + \cdots + p_N \le p.$$

Set $p_0 = p - \sum_{i=1}^{N} p_i$. Note that

$$\langle p_0 \rangle + \sum_{i=1}^{N} \langle p_i \rangle = \alpha(\mathbb{1}_\Omega) \ll \alpha(\mathbb{1}_{\Omega\setminus\overline{U}}) + \alpha(\mathbb{1}_{U_\rho})$$

and

$$\alpha(\mathbb{1}_{U_\rho}) \ll \alpha(\mathbb{1}_{\cup_{i=1}^{N}(O_i)_{2\rho}}) \ll \sum_{i=1}^{N} \langle p_i \rangle.$$

By weak cancellation in $\mathrm{Cu}(A)$ (Definition 2.5), we have

$$\langle p_0 \rangle \le \alpha(\mathbb{1}_{\Omega\setminus\overline{U}}) \le \alpha(\mathbb{1}_{(O_k)_\delta \cap U}), \quad \forall\, k = 1, 2 \ldots, N.$$

Now choose $z_0 \in \Omega\setminus\overline{U}$ and $z_i \in O_i$ $(1 \le i \le N)$. Define

$$\phi(f) = \sum_{i=0}^{N} f(z_i) p_i, \quad f \in C(\Omega).$$

Then we need to show $d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) < 9\delta$.

For any nonempty open set $V \subset \Omega$, we have $V_\delta \cap U \ne \varnothing$. Now we consider the following two cases.

Case 1: There exists $k \in \{1, 2, \ldots, N\}$ such that $O_k \subset V_{8\delta}\setminus V_{3\delta}$.

Define index sets

$$I_0 = \{i \mid V \cap (O_i)_\rho \ne \varnothing, 1 \le i \le N\},$$

$$I_1 = \{i \mid O_i \cap (O_k)_\delta \ne \varnothing, 1 \le i \le N\}.$$

If $i \in I_1$, then $O_i \cap V_{2\delta} = \varnothing$, we have $I_0 \cap I_1 = \varnothing$. We also note that

$$\bigcup_{i\in I_0} (O_i)_{2\rho} \cup \Big(\bigcup_{i\in I_1} O_i\Big) \subset V_{9\delta}.$$

Then we have

$$\begin{aligned} \mathrm{Cu}(\phi)(\mathbb{1}_V) \le& \langle p_0 \rangle + \sum_{z_i \in V,\, i\ne 0} \langle p_i \rangle \\ \le& \alpha(\mathbb{1}_{(O_k)_\delta \cap U}) + \sum_{i\in I_0} \langle p_i \rangle \\ \le& \sum_{i\in I_1} \alpha(\mathbb{1}_{O_i}) + \sum_{i\in I_0} \alpha(\mathbb{1}_{(O_i)_{2\rho}}) \\ \le& \alpha(\mathbb{1}_{V_{9\delta}}). \end{aligned}$$

Note that

$$V \subset (V \cap \Omega\setminus\overline{U}) \cup (V \cap U_\rho).$$

Now we have

$$\begin{aligned}\alpha(\mathbb{1}_V) \leq& \alpha(\mathbb{1}_{V\cap\Omega\backslash\overline{U}}) + \alpha(\mathbb{1}_{V\cap U_\rho})\\ \leq& \alpha(\mathbb{1}_{(O_k)_\delta\cap U}) + \alpha(\mathbb{1}_{V\cap U_\rho})\\ \leq& \sum_{i\in I_1}\alpha(\mathbb{1}_{O_i}) + \sum_{i\in I_0}\alpha(\mathbb{1}_{(O_i)_\rho})\\ \leq& \sum_{i\in I_1}\langle p_i\rangle + \sum_{i\in I_0}\langle p_i\rangle\\ \leq& \mathrm{Cu}(\phi)(\mathbb{1}_{V_{9\delta}}).\end{aligned}$$

Case 2: There doesn't exist $k \in \{1, 2, \ldots, N\}$ such that $O_k \subset V_{8\delta}\backslash V_{3\delta}$.
In this case, $U \cap (V_{7\delta}\backslash V_{4\delta}) = \varnothing$. Now we define index sets

$$J_0 = \{i \mid O_i \subset V_{4\delta}, 1 \leq i \leq N\},$$

$$J_1 = \{i \mid O_i \subset \Omega\backslash V_{7\delta}, 1 \leq i \leq N\}.$$

Thus, we have $J_0 \cup J_1 = \{1, 2, \ldots, N\}$ and $J_0 \cap J_1 = \varnothing$.

By (i), if $\Omega\backslash V_{6\delta} \neq \varnothing$, then $J_1 \neq \varnothing$. Then for arbitrary $i \in J_0, i' \in J_1$, we have $\mathrm{dist}((O_i)_\delta, (O_{i'})_\delta) > \delta$, this means that $(O_i)_\delta$ and $(O_{i'})_\delta$ can't be almost connected, and this contradicts (v). Then we must have $V_{6\delta} = \Omega$. It is clear that

$$\mathrm{Cu}(\phi)(\mathbb{1}_V) \leq \mathrm{Cu}(\phi)(\mathbb{1}_{V_{6\delta}}) = \alpha(\mathbb{1}_\Omega)$$

and

$$\alpha(\mathbb{1}_V) \leq \alpha(\mathbb{1}_{V_{6\delta}}) = \mathrm{Cu}(\phi)(\mathbb{1}_\Omega).$$

Combining these two cases, we have

$$d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) < 9\delta.$$ ∎

Now we must consider the possibility that certain open sets in the covering may be transformed into zero by the Cu-morphism. In such situations, it is essential to delicately organize the open sets into appropriate groupings.

Theorem 4.5 *Let A be a unital, simple, separable C^*-algebra with stable rank one, real rank zero and strict comparison and let Ω be a compact metric space. Let $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ be a Cu-morphism with $\alpha(\mathbb{1}_\Omega) \leq \langle 1_A\rangle$. Then for any $\varepsilon > 0$, there exists a $*$-homomorphism $\phi : C(\Omega) \to A$ such that*

$$d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) < \varepsilon.$$

Proof Since Ω is compact, for $\delta = \varepsilon/9$, there exist $x_1, x_2, \ldots, x_m \in \Omega$ such that

$$\Omega = \bigcup_{i=1}^{m} B(x_i, \delta/4).$$

Denote

$$\Lambda = \{1, 2, \ldots, m\},$$

$$\mathcal{F} = \{B(x_i, \delta/4) \mid \alpha(\mathbb{1}_{B(x_i,\delta/4)}) \neq 0\}.$$

Then $\mathcal{F}$ has finitely many almost connected (Definition 4.1) components $\mathcal{F}_1, \ldots, \mathcal{F}_l$.

For each $i \in \{1, 2, \ldots, l\}$, we also define

$$\Lambda_i = \{j \mid B(x_j, \delta/4) \in \mathcal{F}_i\}, \ \Lambda_0 = \Lambda \backslash \bigcup_{i=1}^{l} \Lambda_i,$$
$$\Omega_i = \bigcup_{j \in \Lambda_i} B(x_j, \delta/4), \ \Omega_0 = \bigcup_{j \in \Lambda_0} B(x_j, \delta/4).$$

(One may say that $\Omega_1, \ldots, \Omega_l$ are "separated" by Ω_0.)

Since $\Omega_i \cap \Omega_j \neq \varnothing$ for all $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$, we have

$$\alpha(\mathbb{1}_\Omega) \leq \sum_{i=0}^{l} \alpha(\mathbb{1}_{\Omega_i}) = \sum_{i=1}^{l} \alpha(\mathbb{1}_{\Omega_i}) \leq \alpha(\mathbb{1}_\Omega).$$

Thus,

$$\alpha(\mathbb{1}_{\Omega_1}) + \alpha(\mathbb{1}_{\Omega_2}) + \cdots + \alpha(\mathbb{1}_{\Omega_l}) = \alpha(\mathbb{1}_\Omega).$$

Now we will prove that $\alpha(\mathbb{1}_{\Omega_i})$ is compact for each $i \in \{1, 2, \ldots, l\}$.

For each i, let $\{a_{n,i}\}_n$ be a $\ll$-increasing sequence in $\mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$ with supremum $\mathbb{1}_{\Omega_i}$. Set $b_n = a_{n,1} + a_{n,2} + \cdots + a_{n,l}$, then $\{b_n\}_n$ is also a $\ll$-increasing sequence with supremum

$$\sup_n b_n = \sup_n a_{n,1} + \sup_n a_{n,2} + \cdots + \sup_n a_{n,l}.$$

Since α preserves suprema of increasing sequences, then we have

$$\begin{aligned}\sup_n \alpha(b_n) &= \alpha(\sup_n a_{n,1}) + \alpha(\sup_n a_{n,2}) + \cdots + \alpha(\sup_n a_{n,l}) \\ &= \alpha(\mathbb{1}_{\Omega_1}) + \alpha(\mathbb{1}_{\Omega_2}) + \cdots + \alpha(\mathbb{1}_{\Omega_l}) \\ &= \alpha(\mathbb{1}_\Omega).\end{aligned}$$

From the compactness of $\alpha(\mathbb{1}_\Omega)$, there exists $k \in \mathbb{N}$ such that $\alpha(b_k) = \alpha(\mathbb{1}_\Omega)$, i.e.,

$$\alpha(a_{k,1}) + \alpha(a_{k,2}) + \cdots + \alpha(a_{k,l}) = \alpha(\mathbb{1}_{\Omega_1}) + \alpha(\mathbb{1}_{\Omega_2}) + \cdots + \alpha(\mathbb{1}_{\Omega_l}).$$

Since we have $a_{k,m} \ll \mathbb{1}_{\Omega_m}$ (in $\mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$) for any $m = 1, 2, \ldots, l$, then

$$\sum_{m \neq i} \alpha(a_{k,m}) \ll \sum_{m \neq i} \alpha(\mathbb{1}_{\Omega_m}) \quad (\text{in } \mathrm{Cu}(A)).$$

Since $\alpha(\mathbb{1}_\Omega)$ is compact, we also have

$$\alpha(\mathbb{1}_{\Omega_1}) + \alpha(\mathbb{1}_{\Omega_2}) + \cdots + \alpha(\mathbb{1}_{\Omega_l}) \ll \alpha(a_{k,1}) + \alpha(a_{k,2}) + \cdots + \alpha(a_{k,l}).$$

From the weak cancellation of $\mathrm{Cu}(A)$, we have

$$\alpha(\mathbb{1}_{\Omega_i}) \leq \alpha(a_{k,i}) \ll \alpha(\mathbb{1}_{\Omega_i}).$$

This means that $\alpha(\mathbb{1}_{\Omega_i})$ is compact in $\mathrm{Cu}(A)$.

Since we have $\alpha(\mathbb{1}_{\Omega_1}) + \alpha(\mathbb{1}_{\Omega_2}) + \cdots + \alpha(\mathbb{1}_{\Omega_l}) = \alpha(\mathbb{1}_\Omega)$, by Proposition 2.8, there exists a collection of mutually orthogonal projections $\{p_i\}$ such that

$$\langle p_i \rangle = \alpha(\mathbb{1}_{\Omega_i}), \quad i = 1, 2, \ldots, l$$

and

$$p_1 + p_2 + \cdots + p_l \leq 1_A.$$

Let $h(t) \in \mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$. For any open set $V \subset \Omega$, define

$$h|_V(t) = \begin{cases} h(t), & \text{if } t \in V \\ 0, & \text{if } t \notin V. \end{cases}$$

For each $i \in \{1, 2, \ldots, l\}$, define α_i as follows:

$$\alpha_i(h(t)) = \alpha(h|_{\Omega_i}(t)).$$

It can be checked that $\alpha_1, \alpha_2, \ldots, \alpha_l$ are Cu-morphisms from $\mathrm{Lsc}(\Omega, \overline{\mathbb{N}})$ to $\mathrm{Cu}(A)$. We also have

$$\alpha_1 + \alpha_2 + \cdots + \alpha_l = \alpha.$$

For each i, we apply Lemmas 4.3 and 4.4 for Ω_i, δ, p_i and α_i (the key point is that $\alpha(\mathbb{1}_{\Omega_i})$ is compact); this gives $\phi_i : C(\Omega) \to p_i A p_i$ such that

$$d_{\mathrm{Cu}}(\mathrm{Cu}(\phi_i), \alpha_i) < 9\delta.$$

Denote $\phi = \sum_{i=1}^{l} \phi_i$. Since $\phi_1, \phi_2, \ldots, \phi_l$ have mutually orthogonal ranges, we have

$$\mathrm{Cu}(\phi_1) + \mathrm{Cu}(\phi_2) + \cdots + \mathrm{Cu}(\phi_l) = \mathrm{Cu}(\phi).$$

By Proposition 3.4, we obtain

$$d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) < 9\delta = \varepsilon. \qquad \blacksquare$$

Remark 4.6 In most cases, we assume that Ω is a compact space, but we point out that the main point is that $\alpha(\mathbb{1}_\Omega)$ is compact in $\mathrm{Cu}(A)$. In the presence of stable rank one, $\alpha(\mathbb{1}_\Omega)$ can be lifted to a projection p in A, and then we may regard α as a Cu-morphism from $\mathrm{Cu}(C(\Omega))$ to $\mathrm{Cu}(pAp)$. We also note that if A is a unital, simple, separable C^*-algebra with strict comparison, then pAp also has strict comparison and $K_0(A)$ is weakly unperforated; in this case, if A has real rank zero, then A has stable rank one [18, Corollary 9.5].

5 Classification results

Denote by $\mathcal{C}$ the class of all simple, separable C^*-algebras with stable rank one, real rank zero, and strict comparison (see 5.4). In this section, we give classification results for both the unital case and the non-unital case.

Definition 5.1 Let A and B be C^*-algebras such that A has a strictly positive element s_A. Let us say that *the functor* Cu *classifies the pair* (A, B) if for any Cu-morphism

$$\alpha : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$$

such that $\alpha(\langle s_A \rangle) \leq \langle s_B \rangle$, where s_B is a positive element of B, there exists a $*$-homomorphism $\phi : A \to B$, unique up to approximate unitary equivalence, such that $\alpha = \mathrm{Cu}(\phi)$. We shall say *the functor* Cu *classifies* $(A, \mathcal{C})$ if Cu classifies the pair (A, B) for any B in $\mathcal{C}$.

Theorem 5.1 *Let Ω be a compact subset of $\mathbb{C}$ and A be a unital C^*-algebra in $\mathcal{C}$. Suppose that $\alpha : \mathrm{Cu}(C(\Omega)) \to \mathrm{Cu}(A)$ is a Cu-morphism with $\alpha(\mathbb{1}_\Omega) \le \langle 1_A \rangle$. Then there exists a homomorphism $\phi : C(\Omega) \to A$ such that $\mathrm{Cu}(\phi) = \alpha$. In particular, if $K_1(A)$ is trivial,* Cu *classifies the pair $(C(\Omega), A)$.*

Proof From Theorem 4.5, there exists a sequence of homomorphisms ϕ_n with finite dimensional range such that $d_{\mathrm{Cu}}(\mathrm{Cu}(\phi_n), \alpha) \to 0$. Let $x_n = \phi_n(\mathrm{id})$ and $\varepsilon > 0$. As the range of ϕ_n is finite dimensional, we have $[\lambda - x_n] = 0$ in $K_1(A)$ for all $\lambda \notin \mathrm{sp}(x_n)$. By Theorem 3.6 and Remark 2.5, there exists $N_1 > 0$ such that

$$d_U(x_n, x_m) \le 2d_W(x_n, x_m) < \frac{\varepsilon}{2}, \quad n, m \ge N_1.$$

Then for $\varepsilon/2^2$, there exists $N_2 > N_1$ such that

$$d_U(x_n, x_m) < \frac{\varepsilon}{2^2}, \quad n, m \ge N_2.$$

Similarly, for any k, there exists $N_k > N_{k-1}$ such that

$$d_U(x_n, x_m) < \frac{\varepsilon}{2^k}, \quad , n, m \ge N_k.$$

Then for each $k \ge 1$, there exists a unitary $u_k \in A$ such that

$$\|x_{N_k} - u_k^* x_{N_{k+1}} u_k\| < \frac{\varepsilon}{2^k}.$$

Write

$$\begin{aligned}
\widetilde{x}_1 &=: x_{N_1}, \\
\widetilde{x}_2 &=: u_1^* x_{N_2} u_1, \\
&\vdots \\
\widetilde{x}_k &=: (u_{k-1} \cdots u_2 u_1)^* x_{N_k} u_{k-1} \cdots u_2 u_1, \\
&\vdots
\end{aligned}$$

Then $\{\widetilde{x}_k\}$ is a Cauchy sequence. We may assume that $\widetilde{x}_k \to x$. Note that all the $\widetilde{x}_k$ and x are normal and $\sigma(\widetilde{x}_k), \sigma(x) \subset \Omega$.

Define $\phi : C(\Omega) \to A$ by $\phi(f) = f(x)$. By Lemma 3.7(i), we have

$$d_W(\phi_{N_k}, \phi) = d_W(x_{N_k}, x) = d_W(\widetilde{x}_k, x) \to 0.$$

From the properties of d_{Cu} (see 3.3), we have

$$\begin{aligned}
d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) &\le d_{\mathrm{Cu}}(\mathrm{Cu}(\phi_{N_k}), \alpha) + d_{\mathrm{Cu}}(\mathrm{Cu}(\phi_{N_k}), \mathrm{Cu}(\phi)) \\
&= d_{\mathrm{Cu}}(\mathrm{Cu}(\phi_{N_k}), \alpha) + d_W(\phi_{N_k}, \phi) \to 0.
\end{aligned}$$

Then the $*$-homomorphism $\phi : C(\Omega) \to A$ satisfies $d_{\mathrm{Cu}}(\mathrm{Cu}(\phi), \alpha) = 0$, and so by Proposition 3.3, we have $\alpha = \mathrm{Cu}(\phi)$.

Suppose that $\psi : C(\Omega) \to A$ also satisfies $\mathrm{Cu}(\psi) = \alpha$. As $K_1(A)$ is trivial, we obtain $\mathrm{ind}(\phi(\mathrm{id})) = \mathrm{ind}(\psi(\mathrm{id}))$. By Lemma 3.7 (ii), we obtain $\phi \sim_{aue} \psi$. Thus, ϕ is unique up to approximate unitary equivalence. ∎

The following properties are established in [11, Proposition 5.2].

Proposition 5.2 *The following statements hold true:*

(i) *If* Cu *classifies the pair* (A, B) *and* B *has stable rank one, then* Cu *classifies the pair* $(M_n(A), B)$ *for every* $n \in \mathbb{N}$*;*

(ii) *Let* C *be a* C^**-algebra of stable rank one. If* Cu *classifies the pairs* (A, D) *and* (B, D) *for all hereditary subalgebras* D *of* C*, then* Cu *classifies the pair* $(A \oplus B, C)$*;*

(iii) *If* Cu *classifies the pairs* (A_i, B) *for a sequence*

$$A_1 \xrightarrow{\rho_1} A_2 \xrightarrow{\rho_2} \cdots,$$

then Cu *classifies the pair* $(\varinjlim(A_i, \rho_i), B)$*;*

(iv) *Let* A, B *and* C *be* C^**-algebras such that* A *is stably isomorphic to* B*, and* C *has stable rank one. If* Cu *classifies the pair* $(A, C \otimes \mathcal{K})$*, then* Cu *classifies the pair* (B, C)*.*

Combining Theorem 5.1 and Proposition 5.2, we obtain the following result.

Theorem 5.3 *Let* A *be either a matrix algebra over a compact subset of* $\mathbb{C}$ *or a sequential inductive limit of such* C^**-algebras, or a unital* C^**-algebra stably isomorphic to one such inductive limit. Suppose that* B *is unital in* $\mathcal{C}$ *and* $K_1(B)$ *is trivial. Then for every Cu-morphism in the category* **Cu**

$$\alpha : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$$

such that $\alpha(\langle 1_A \rangle) \le \langle 1_B \rangle$*, there exists a homomorphism* $\phi : A \to B$ *such that* $\mathrm{Cu}(\phi) = \alpha$*. Moreover,* ϕ *is unique up to approximate unitary equivalence.*

Remark 5.4 In general, if B is non-unital simple, one needs to have densely-defined, lower semicontinuous 2-quasitraces to formulate strict comparison. But in our setting, B has real rank zero, every non-zero projection is a full projection, and so by [5, Theorem 2.8], we have $pBp \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Then we can say B has strict comparison if pBp has.

Remark 5.5 If B is non-unital and A is unital, then $\alpha(\langle 1_A \rangle)$ is still compact, and there exists a projection p in B (B has stable rank one) such that $\alpha(\langle 1_A \rangle) = \langle p \rangle$. Apply Theorem 5.3, there exists a homomorphism $\phi : A \to pBp$ such that $\mathrm{Cu}(\phi) = \alpha$ and ϕ is unique up to approximate unitary equivalence (see [29, Proposition 2.3.1]). When A is non-unital, we need Robert's augmented Cuntz semigroup to overcome the difficulty.

Definition 5.2 (*Augmented Cuntz semigroup*) Let A be a unital C^*-algebra. Let us define $\mathrm{Cu}^\sim(A)$ as the ordered semigroup of formal differences $\langle a \rangle - n\langle 1 \rangle$, with $\langle a \rangle \in \mathrm{Cu}(A)$ and $n \in \mathbb{N}$. That is, $\mathrm{Cu}^\sim(A)$ is the quotient of the semigroup of pairs $(\langle a \rangle, n)$, with $\langle a \rangle \in \mathrm{Cu}(A)$ and $n \in \mathbb{N}$, by the equivalence relation $(\langle a \rangle, n) \sim (\langle b \rangle, m)$ if

$$\langle a \rangle + m\langle 1 \rangle + k\langle 1 \rangle = \langle b \rangle + n\langle 1 \rangle + k\langle 1 \rangle,$$

for some $k \in \mathbb{N}$. The image of $(\langle a \rangle, n)$ in this quotient will be denoted by $\langle a \rangle - n\langle 1 \rangle$. If A is non-unital, denote by $\pi : A^\sim \to \mathbb{C}$ the quotient map from the unitization of A onto $\mathbb{C}$. Define $\mathrm{Cu}^\sim(A)$ as the subsemigroup of $\mathrm{Cu}^\sim(A^\sim)$ consisting of the elements $\langle a \rangle - n\langle 1 \rangle$, with $\langle a \rangle$ in $\mathrm{Cu}(A^\sim)$ such that $\mathrm{Cu}(\pi)(\langle a \rangle) = n < \infty$. We refer the reader to [26] for more details.

The functor $\mathrm{Cu}^\sim$ can also be used to classify the C^*-pair, with the meaning of "$\mathrm{Cu}^\sim$ classifies the pair" the same as the one defined above for Cu. Note that we will not explore the detailed structure of $\mathrm{Cu}^\sim$, we only need the following facts; see [26, Theorem 3.2.2].

Theorem 5.6 *Let A, B be* C*-*algebras of stable rank one.*

(i) *If A is unital, then the functor* $\mathrm{Cu}^\sim$ *classifies $(A, \mathcal{C})$ if and only if* Cu *classifies $(A, \mathcal{C})$;*
(ii) *The functor* $\mathrm{Cu}^\sim$ *classifies the pair $(A, \mathcal{C})$ if and only if it classifies $(A^\sim, \mathcal{C})$;*
(iii) *Suppose* $\mathrm{Cu}^\sim$ *classifies the sequence of pairs $(A_i, \mathcal{C})$ as in Proposition 5.2 and all the A_i are* C*-*algebras of stable rank one. If $A = \varinjlim A_i$, then* $\mathrm{Cu}^\sim$ *classifies $(A, \mathcal{C})$;*
(iv) *If* $\mathrm{Cu}^\sim$ *classifies $(A, \mathcal{C})$ and $(B, \mathcal{C})$, then* $\mathrm{Cu}^\sim$ *classifies $(A \oplus B, \mathcal{C})$;*
(v) *If* $\mathrm{Cu}^\sim$ *classifies $(A, \mathcal{C})$, then it classifies $(A', \mathcal{C})$ for any A' stably isomorphic to A.*

Theorem 5.7 *Let A be either a matrix algebra over a compact subset of $\mathbb{C}$, or a sequential inductive limit of such* C*-*algebras, or a* C*-*algebra stably isomorphic to one such inductive limit. Let $B \in \mathcal{C}$. Suppose that $K_1(B)$ is trivial. Then for every morphism in the category* ***Cu***

$$\alpha : \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$$

such that $\alpha(\langle s_A \rangle) \leq \langle s_B \rangle$, where $s_A \in A_+$ and $s_B \in B_+$ are strictly positive elements, there exists a homomorphism $\phi : A \to B$ such that $\mathrm{Cu}^\sim(\phi) = \alpha$. *Moreover, ϕ is unique up to approximate unitary equivalence.*

With a combination of Theorems 5.3 and 5.7, we present the following classification result of a class of C^*-algebras.

Corollary 5.8 *Let A, B be sequential inductive limits of finite direct sums of matrix algebras over compact subsets of $\mathbb{C}$. Suppose that $A, B \in \mathcal{C}$ and $K_1(A)$, $K_1(B)$ are trivial. Then*

(1) *$A \cong B$ if and only if $(\mathrm{Cu}^\sim(A), \langle s_A \rangle) \cong (\mathrm{Cu}^\sim(B), \langle s_B \rangle)$, where $s_A \in A_+$ and $s_B \in B_+$ are strictly positive elements;*
(2) *if A, B are unital, $A \cong B$ if and only if $(\mathrm{Cu}(A), \langle 1_A \rangle) \cong (\mathrm{Cu}(B), \langle 1_B \rangle)$.*

Proof See the proof of [11, Corollary 1.2]. ∎

We also have the following result communicated to us by H. Thiel.

Corollary 5.9 *Let A be a sequential inductive limit of finite direct sums of matrix algebras over compact subsets of $\mathbb{C}$. Suppose that A is unital in $\mathcal{C}$ and $K_1(A)$ is trivial. Then A is an AF algebra.*

Proof Since A has real rank zero and stable rank one, $\mathrm{Cu}(A)$ is algebraic (Theorem 2.9) and has weak cancellation (Definition 2.5). Moreover, $\mathrm{Cu}(A)$ is the limit of Cu-semigroups of the form $\mathrm{Lsc}(X, \overline{\mathbb{N}})$, and so $\mathrm{Cu}(A)$ is unperforated (because each $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ is). By [2, Corollary 5.5.13], there exists an AF algebra B such that $\mathrm{Cu}(A)\cong\mathrm{Cu}(B)$. Then by Corollary 5.8, this lifts to an isomorphism $A\cong B$. ∎

Remark 5.10 In [26], Robert defined an equivalence relation $\leftrightsquigarrow$ to reduce every 1-NCCW complex with trivial K_1 to $C[0,1]$. One may expect that any 1-NCCW complex with torsion-free K_1 can be reduced to continuous functions over finite graphs. However, this is not true in general, as the following example shows.

Let $F_1=\mathbb{C}\oplus\mathbb{C}$ and $F_2=\mathbb{C}\oplus M_2(\mathbb{C})$. Let A be the pullback of the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & C([0,1],F_2) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_0\oplus\mathrm{ev}_1} \\ F_1 & \xrightarrow{\ \phi\ } & F_2\oplus F_2\ , \end{array}$$

where

$$\phi(\lambda\oplus\mu)=\left(\lambda\oplus\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\right)\oplus\left(\mu\oplus\begin{pmatrix}\mu & \\ & \mu\end{pmatrix}\right).$$

Then $K_0(A)=\mathbb{Z}, K_1(A)=\mathbb{Z}$. But A has a quotient whose K_1 is $\mathbb{Z}_2$, and this phenomenon will not happen for $C(\mathbb{T})$ (or $C(X)$ where X is any finite graph). We remark that if $A\leftrightsquigarrow B$, then for any quotient algebra A' of A, there exists a quotient algebra B' of B such that $K_1(A')=K_1(B')$. Then A can't be reduced to $C(\mathbb{T})$ (or $C(X)$ where X is any finite graph) via Robert's equivalence relation.

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