Proceedings of the Edinburgh Mathematical Society (1985) **28**, 73–90 ©

l-ADIC AND Z/l^∞-ALGEBRAIC AND TOPOLOGICAL K-THEORY

by VICTOR SNAITH†

(Received 12th March 1984)

0. Introduction

Let l be an odd prime and let A be a commutative ring containing $1/l$. Let $K_*(A; Z/l^v)$ denote the mod l^v algebraic K-theory of A [**3**]. As explained in [**4**] there exists a "Bott element" $\beta_v \in K_{2l^{v-1}(l-1)}(Z[1/l]; Z/l^v)$ and, using the K-theory product we may, following [**16**, Part IV], form

$$\mathscr{K}_i(A; Z/l) = K_i(A; Z/l^v)[1/\beta_v] \tag{0.1}$$

which is defined as the direct limit of iterated multiplication by β_v. There is a canonical localisation map

$$\rho_v: K_i(A; Z/l^v) \to \mathscr{K}_i(A; Z/l^v). \tag{0.2}$$

As explained in [**15**], the Lichtenbaum–Quillen conjecture for a regular ring A (or regular scheme X), having suitably nice étale cohomological properties, reduces to the study of the kernel of (0.2) when $i=2$. In [**15**] I characterised the kernel of ρ in dimension two when $v=1$. For simplicity suppose that A is a $Z[1/l, \xi_{l^\infty}]$-algebra. In [**15**, §4.1] a commutative diagram is constructed of the following form, *when* $v=1$.

$$\begin{array}{ccc} K_2(A; Z/l^v) & \xrightarrow{\rho_v} & \mathscr{K}_2(A; Z/l^v) \\ & {}_{\hat{H}_v}\searrow \quad \swarrow_{I_v} & \\ & KU_0(BGLA; Z/l^v) & \end{array} \tag{0.3}$$

It is shown that $\hat{H}_1(x) = H_K(x) - H_K(x)^l$ where H_K is the $KU_*(\_; Z/l)$-Hurewicz map. Since I_v is one-one, this shows that, in dimension two,

$$\ker \rho_1 = \{x \in K_2(A; Z/l) \,|\, H_K(x) = H_K(x)^l\}.$$

†Research partially supported by NSERC Grant A4633.

D

In [15] it is also shown that the Lichtenbaum–Quillen conjecture can be reduced to showing that $\ker \rho_1 = \ker H_K$.

In this paper we generalise these results to establish (0.3) for all $v \geqq 1$ and we verify that these diagrams respect inverse and direct limits over v. The precise statement of our main result is given in Section 3.12. In addition, an implicit formula for $\hat{H}_v$ is given, as a polynomial in the Dyer–Lashof operations of [10] applied to $H_K(x)$. This is given in Section 3.11(b).

Following the ideas of [15, Section 3], we use a description of $\mathcal{K}_2(A; Z/l^v)$ in terms of Adams maps between Moore spaces to define I_v and to define a map $\rho'_v: K_2(A; Z/l^v) \to \mathcal{K}_2(A; Z/l^v)$ for which we can evaluate $I_v\rho'_v = \hat{H}_v$, then we show that the formula for $\hat{H}_v$ and injectivity of I_v implies $\rho'_v = \rho_v$. These results are proved in Section 3. Sections 1, 2 contain preliminary $KU_*(\_; Z/l^v)$-theory calculations which are used to evaluate the adjoints of the Adams maps in KU_*-theory and thence to evaluate $\hat{H}_v = I_v\rho'_v$.

1. $PK_0(QP^2(v); Z/l^v)$

Let $K_*(\_; Z/l^v)$ denote mod 2 graded unitary K-homology with coefficients mod l^v. For our purposes l will be an *odd* prime (although most of this section is valid when $l=2$, see Section 1.8) and $v \geqq 1$ will be an integer. Let $P^n(v) = S^{n-1} \bigcup_{l^v} e^n$ for $n \geqq 2$ and, as usual,

$$QX = \varinjlim_n \Omega^n \Sigma^n X.$$

Since QX is an infinite loopspace [12] its mod K-theory admits Dyer–Lashof operations as introduced in [6; 14]. More generally we have the Dyer–Lashof operations of McClure [10] (see also [11])

$$Q: K_i(QX; Z/l^v) \to K_i(QX; Z/l^{v-1}). \tag{1.1}$$

Using (1.1) [10] describes $K_*(QP^2(v); Z/l)$ from which we will evaluate the primitives, $PK_0(QP^2(v); Z/l^a)$, for $a \leqq v$.

Recall that the inclusion, $i: Y \vee Y \to Y \times Y$, induces an injection, for any space Y,

$$i_*: (K_*(Y; Z/l^a) \otimes 1) \oplus (1 \otimes K_*(Y; Z/l^a)) \rightarrowtail K_*(Y \times Y; Z/l^a).$$

The primitives are defined by

$$PK_*(Y; Z/l^a) = \{z \in K_*(Y; Z/l^a) \mid d_*(z) = i_*(z \otimes 1 + 1 \otimes z)\} \tag{1.2}$$

where $d: Y \to Y \times Y$ is the diagonal map.

Following [10] we have evident natural maps

$$l^s_*: K_\alpha(Y; Z/l^r) \to K_\alpha(Y; Z/l^{r+s}) \quad \text{if} \quad s \geqq 1$$

$$\pi: K_\alpha(Y; Z/l^r) \to K_\alpha(Y; Z/l^t) \quad \text{if} \quad 1 \leqq t \leqq r$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

and

$$\beta_r: K_\alpha(Y; Z/l^r) \to K_{\alpha-1}(Y; Z/l^r).$$

Also the reduced K-groups of $P^2(v)$ are given by

$$\tilde{K}_0(P^2(v); Z/l^a) \cong Z/l^a \cong \tilde{K}_1(P^2(v); Z/l^a)$$

for $1 \leqq a \leqq v$. Let z generate $\tilde{K}_0(P^2(v); Z/l^v)$ and set $u = \pi(z) \in \tilde{K}_0(P^2(v); Z/l)$ then the v-th Bockstein of u is a generator, v, of $\tilde{K}_1(P^2(v); Z/l)$.

From [**10**, Theorem 5] we obtain the following calculation.

Proposition 1.3. *Let l be an odd prime then, as an algebra,*

$$K_*(QP^2(v); Z/l) \cong Z/l[u_1, u_2, \ldots, u_v] \otimes E[v_1, \ldots, v_v)$$

where

$$u_i = \pi Q^{i-1}(z) \quad \text{and}$$

$$v_i = \pi \beta_{v-i+1}(Q^{i-1}(z)).$$

there $Q^j = Q(Q(\ldots(Q(\_))\ldots))$, the j-th iterate of Q and $E(\_)$ denotes an exterior algebra.

Proposition 1.4. *A basis for $PK_0(QP^2(v); Z/l)$ is given by $\{u_1^{l^\alpha}; \alpha \geqq 0\}$ and for $PK_1(QP^2(v); Z/l)$ by $\{v_i; q \leqq i \leqq v\}$.*

Proof. Firstly the operation, Q, is linear in odd degree so that each v_i is primitive because v_1 is. Here we have used the fact that *in odd dimensions* McClure's operation, Q, covers the (linear) operation $Q': K_1(X; Z/l) \to K_1(X; Z/l)$ of [**14**] for any infinite loopspace, X. This means that $v_i = (Q')^{i-1}(v_1)$ and is therefore primitive as claimed.

However in even dimensions Q is not additive [**14**, p. 190] but satisfies instead the formula [**10**, Theorem 1(ii)]

$$Q(x+y) = Q(x) + Q(y) - \pi\left[\sum_{i=1}^{p-1} (\tbinom{l}{i}/l) x^i y^{l-i}\right].$$

Hence $d_*(u_j) \equiv u_j \otimes 1 + 1 \otimes u_j - \sum_i ((\tbinom{l}{i})/l) u_{j-1}^i \otimes u_{j-1}^{l-i} \pmod{u_1, \ldots, u_{j-2}}$ when u_1 is primitive.

Since $A = K_*(QP^2(v); Z/l)$ is finitely generated we have an exact Milnor–Moore sequence $P(A^l) \rightarrowtail P(A) \xrightarrow{\lambda} Q(A)$, where A^l denotes the subalgebra of l-th powers. From the foregoing discussion it is clear that $\mathrm{im}(\lambda)$ is generated by $\{u_1, v_1, v_2, \ldots, v_v\}$ and the result follows from the Milnor–Moore sequence since $P(A^l) = P(A)^l$.

Now let X be an infinite loopspace and suppose $u \in K_0(X; Z/l^a)$ for $a \geqq 2$. Define $x(u) \in K_0(X; Z/l^a)$ by the formula

$$x(u) = u^l + l_* Q(u). \tag{1.5}$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

Lemma 1.6. *If $u \in PK_0(X; Z/l^a)$ in (1.5) then $x(u) \in PK_0(X; Z/l^a)$ also.*

Proof. Firstly

$$d_*(u^l) = (u \otimes 1 + 1 \otimes u)^l$$
$$= u^l \otimes 1 + 1 \otimes u^l + l\left\{\sum_{i=1}^{l-1} (\tbinom{l}{i}/l) u^i \otimes u^{l-i}\right\}.$$

Also, as mentioned above,

$$d_* Q(u) = Q(u) \otimes 1 + 1 \otimes Q(u) - \pi\left\{\sum_{i=1}^{l} (\tbinom{l}{i}/l) u^i \otimes u^{l-i}\right\}$$

so that $x(u)$ is primitive since $l_*\pi$ is multiplication by l.

We can now state the main result of this section.

Theorem 1.7. *Let l be an odd prime. For $2 \leqq a \leqq v$, $PK_0(QP^2(v); Z/l^a)$ is generated by $\{\pi x^\alpha(x) \mid \alpha \geqq 0\}$, where $x^\alpha(z) = x(x(\ldots(x(z))\ldots))$ is the α-th iterate of $x(\_)$ and z generates $\tilde{K}_0(P^2(v); Z/l^v)$.*

Remark 1.8.

(a) This result is true when $P^2(v)$ is replaced by $P^{2m}(v) = S^{2m-1} \bigcup_{l^v} e^{2m}$ for any $m \geqq 1$.

(b) The result is probably true, by an elaboration of the proof which is to follow, when $l = 2$. The difficulty at the prime two lies in the non-commutative structure of $K_*(QX; Z/2)$ and in the fact that [**10**, Theorem 5] may not give the algebra structure when $l = 2$.

1.9 In the proof of Theorem 1.7 we need the following facts concerning the behaviour of the Bockstein spectral sequence $\{E_r^*(X); r \geqq 1\}$ for the space $X = QP^2(v)$. This behaviour follows the well-known pattern—some call it Henselian—of the Bockstein spectral sequences of [**9**]. In fact the proofs of the following assertions follow from the properties of the operations in [**10**, Theorems 1 and 4] in a manner analogous to that in which one deduces the results of [**9**] from the properties of the Pontrjagin squaring operation.

From Section 1.3, if $X = QP^2(v)$,

$$E_1^* = Z/l[u_1, \ldots, u_v] \otimes E(v_1, \ldots, v_v)$$

with $\beta_1(u_v) = d_1(u_v) = v_v$. Also d_1 is zero on the other generators. Hence

$$E_2^* = Z/l[u_1, \ldots, u_{v-1}, u_v^l] \otimes E(v_1, \ldots, v_{v-1}, v_v^{l-1} v_v).$$

More generally if $1 \leqq s \leqq a \leqq v$ set $\hat{u}_j = u_j^{l^{j+s-v-1}}$, $\hat{v}_j = u_j^{l^{j+s-v-1}-1} v_j$ for $v+1-s<j \leqq v$ then

$$\left.\begin{aligned} &E_s^* = Z/l[u_1, \ldots, u_{v-s+1}, \hat{u}_{v-s+2}, \ldots, \hat{u}_v] \otimes E(v_1, \ldots, v_{v-s+1}, \hat{v}_{v-s+2}, \ldots, \hat{v}_v) \\ &\text{with} \\ &\qquad d_s(\hat{u}_j) = \hat{v}_j \qquad (j \geqq v-s+2) \\ &\qquad d_s(u_{v-s+1}) = v_{v-s+1} \end{aligned}\right\} \tag{1.10}$$

and $0 = d_s(u_j) = d_s(v_j)$ otherwise. Also, by the argument of Section 1.4, we have (for $s \leqq v-1$) primitives

$$\begin{aligned} PE_s^0 &= \langle u_1^{l^\alpha} \mid \alpha \geqq 0 \rangle \\ PE_s^1 &= \langle v_1, v_2, \ldots, v_{v-s+1}, \hat{v}_{v-s+2}, \ldots, \hat{v}_v \rangle. \end{aligned} \tag{1.11}$$

Proof of Theorem 1.7. Let $w \in PK_0(X; Z/l^a)$ and consider the following exact sequence

$$K_1(X; Z/l^{a-1}) \xrightarrow{\beta} K_0(X; Z/l) \xrightarrow{(l^{a-1})_*} K_0(X; Z/l^a) \xrightarrow{\pi} K_0(X; Z/l^{a-1}). \tag{1.12}$$

By induction on a, starting with Section 1.4, $\pi(w) = \sum_\alpha \lambda_\alpha \pi x^\alpha(z)$ so $y = w - \sum_\alpha \lambda_\alpha \pi x^\alpha(z) = (l^{a-1})_*(r)$ is a primitive ($r \in K_0(X; Z/l)$). Furthermore there exists $t \in K_1(X \times X; Z/l^{a-1})$ such that the diagonal satisfies $d_*(r) = r \otimes 1 + 1 \otimes r + \beta(t)$. Since $d_1\beta(t) = 0$, $d_1(r) \in PE_1^1 \cap \operatorname{im}(d_1)$. By (1.10) and (1.11)

$$d_1(r) = \lambda v_v = d_1(\lambda u_v)$$

so that $x = r - \lambda u_v$ is a d_1-cycle with diagonal given by

$$d_*(x) = x \otimes 1 + 1 \otimes x + \beta(t) + \lambda \sum_{i=1}^{l-1} (\tbinom{l}{i}/l) u_{v-1}^i \otimes u_{v-1}^{l-i}.$$

Now $\beta(t)$ is d_{a-1}-boundary. Hence if $a=2$ the class of x in E_2^* satisfies

$$d_*[x] = [x] \otimes 1 + 1 \otimes [x] + \lambda \sum_{i=1}^{l-1} (\tbinom{l}{i}/l) u_{v-1}^i \otimes u_{v-1}^{l-i}.$$

However, the reduced diagonal of any canonical generator in E_2^* is a polynomial in $u_1, u_2, \ldots, u_{v-2}$ and u_{v-1}^l, from which it is straightforward to see that $\lambda = 0$. Thus $[x] \in PE_2^0$ so that, by (1.11) $x = \sum \mu_\alpha u_1^{l^\alpha} (\operatorname{mod} \operatorname{im} \beta_1)$ whence (since $l_* \beta_1 = 0$)

$$\begin{aligned} l_*(x) &= \sum \mu_\alpha l_*(u_1^{l^\alpha}) \\ &= \sum \mu_\alpha l_*(\pi x^\alpha(z)) = \sum \mu_\alpha l^{a-1} x^\alpha(z) \end{aligned}$$

as required. Here we have used that if $q \in K_0(X; Z/l^2)$

$$l_*\pi(x(q)) = lx(q) = lq^l + ll_*Q(q)$$

$$= lq^l$$

since $lQ(q) = 0$.

Now suppose $a \geqq 3$. Since $\beta(t)$ is a d_{a-1}-boundary it is a d_2-cycle. Let $s = d_2(x)$ then, in $E_2^*(X \times X)$,

$$d_*(s) = s \otimes 1 + 1 \otimes s$$

$$+ \lambda \sum_{i=1}^{l-1} (\tbinom{l}{i}/l) i u_{v-1}^{i-1} v_{v-1} \otimes u_{v-1}^{l-i}$$

$$+ \lambda \sum_{i=1}^{l-1} (\tbinom{l}{i}/l) u_{v-1}^{i} \otimes (l-i) u_{v-1}^{l-i-1} v_{v-1}.$$

As in the case $a = 2$ it is straightforward to show that the last two terms in the above expression can only be the reduced diagonal of an element of E_2^1 if $\lambda = 0$. Hence $x = r$ and $d_2(r) \in PE_2^1$ whence by (1.10) and (1.11)

$$d_2(r) = \lambda_1 v_{v-1} + \lambda_2 \hat{v} = d_2(\lambda_1 u_{v-1} + \lambda \hat{u}_v).$$

Now write

$$x_2 = r - \lambda_1 u_{v-1} - \lambda_2 u_v^l \in K_0(X; Z/l)$$

so that x_1 is a d_2-cycle with diagonal in $E_2^*(X \times X)$ is given by

$$d_*(x_2) = x_2 \otimes 1 + 1 \otimes x_2 + \beta(t)$$

$$+ \lambda_1 \left(\sum_{i=1}^{l-1} (\tbinom{l}{i}/l) u_{v-2}^{i} \otimes u_{v-2}^{l-i} \right)$$

$$+ \lambda_2 \left(\sum_{i=1}^{l-1} (\tbinom{l}{i}/l) \hat{u}_{v-1}^{i} \otimes \hat{u}_{v-1}^{l-i} \right).$$

Proceeding thus suppose we have constructed a d_s-cycle ($s \leqq a-2$)

$$x_s = r - \lambda_1 u_{v-s+1} - \lambda_2 u_{v-s+2}^{l} - \dots - \lambda_s u_v^{l^{s-1}} \in K_0(X; Z/l)$$

whose diagonal in $E_s^*(X \times X)$ is given by

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

$$d_*(x_s)=x_s\otimes 1+1\otimes x_s+\beta(t)$$

$$+\lambda_1\left(\sum_{i=1}^{l-1}(\tbinom{l}{i}/l)u_{\nu-s}^i\otimes u_{\nu-s}^{l-i}\right)$$

$$+\dots$$

$$+\lambda_s\left(\sum_{i=1}^{l-1}(\tbinom{l}{i}/l)\hat{u}_{\nu-1}^i\otimes \hat{u}_{\nu-1}^{l-i}\right).$$

As above, applying d_s, and observing $d_s\beta(t)=0$,

$$d_sd_*(x_s)=d_*(d_sx_s)$$

$$=d_sx_s\otimes 1+1\otimes d_sx_s+\dots$$

$$+\lambda_s\left(\sum_{i=1}^{l-1}(\tbinom{l}{i}/l)i\hat{u}_{\nu-1}^{i-1}\hat{v}_{\nu-1}\otimes \hat{u}_{\nu-1}^{l-i}\right)$$

$$+\lambda_s\left(\sum_{i=1}^{l-1}(\tbinom{l}{i}/l)\hat{u}_{\nu-1}^i\otimes (l-i)\hat{u}_{\nu-1}^{l-i-1}\hat{v}_{\nu-1}\right)$$

and again the only manner in which the last $2s$ terms in the above expression to appear in the reduced diagonal of an element of $E_2^*(X)$ is for $\lambda_1=\dots=\lambda_s=0$.

By induction we see that r $(=x_{a-1})$ is a d_{a-1}-cycle which represents a primitive in $E_a^0(X)$. Hence, by (1.11), $r=\sum\gamma_\alpha u_1^{l^\alpha}(\operatorname{mod}\operatorname{im}\beta)$ so (since $(l^{a-1})_*\beta=0$)

$$(l^{a-1})_*(r)=(l^{a-1})_*(\Sigma\gamma_\alpha u_1^{l^\alpha})$$

$$=\Sigma\gamma_\alpha(l^{a-1})_*\pi(x^\alpha(z))$$

$$=\Sigma\gamma_\alpha l^{a-1}x^\alpha(z),$$

which completes the proof. In this last step we have used the fact that if $q\in K_0(X;Z/l^\nu)$ then

$$(l^{a-1})_*\pi(x(q))=l_*^{a-1}\pi(q^l)+l_*^{a-1}\pi l_*Q(q)$$

$$=l^{a-1}\pi(q)^l+(l^{a-1})_*\pi'(lQ(q))$$

(where π' is reduction mod l from $K_0(X;Z/l^{\nu-1})$)

$$=l^{a-1}\pi(q)^l.$$

Let BU denote the classifying space for unitary K-theory [**2**, Part III]. Let

$f: P^2(v) \to BU$ represent a generator of $\pi_2(BU; Z/l^v)$ and let $F: QP^2(v) \to BU$ denote the infinite loop map (unique up to homotopy) which gives f upon restriction to $P^2(v)$.

Corollary 1.13. *Let l be an odd prime and $v \geqq 1$ then*

$$F_*: PK_0(QP^2(v); Z/l) \to PK_0(BU; Z/l^v)$$

is injevtive.

Proof. The Q-operation of [**10**, Theorem 1] induces an endomorphism of $K_0(BU; Z/l)/\mathcal{D}$, where $\mathcal{D}$ denotes the decomposables in the algebra structure induced by Whitney sum of bundles. From [**10**, p. 3] this endomorphism, also denoted by Q, coincides with the operation constructed in [**14**] and computed for BU in [**14**, Section 6]. Hence the image of $x^\alpha(z) \in PK_0(QP^2(v); Z/l^v)$ in $K_0(BU; Z/l)/\mathcal{D}$ is $Q^\alpha(v_1)$, when $KU_0(BU; Z/l) = Z/l[v_1, v_2, \ldots]$ in the notation of [**14**, Section 6]. By [**14**, Section 6] the images $F_\alpha(z), F(x(z)), F_*(x^2(z)), \ldots$ are linearly independent mod l from which the result follows easily, since $K_0(BU; Z/l^v) = Z/l^v[v_1, v_2, \ldots]$.

2. The effect of Adams' maps in K-theory

Let l be an odd prime.

In [**1**, Section 12] Adams showed that there exist the following interesting maps between Moore spaces,

$$A_v: p^{q+2l^{v-1}(l-1)}(v) \to p^q(v) \tag{2.1}$$

for q sufficiently large. As in Section 1, $P^m(v)$ denotes $S^{m-1} U_{l^v} e^m$ ($m \geqq 2$, $v \geqq 1$). There exist homotopy commutative diagrams

$$\begin{array}{ccc} P^{q+2l^{v-1}(l-1)}(v) & \xrightarrow{A_v} & P^q(v) \\ \downarrow i & \searrow \alpha_v & \downarrow j \\ S^{q+2l^{v-1}(l-1)-1} & \xrightarrow{\alpha_v} & S^q \end{array} \tag{2.2}$$

in which i and j are the canonical inclusion and collapsing maps, respectively. The maps, A_v, are partially characterised by the following (equivalent) conditions

[**1**, Section 12.3]. Set $m = q + 2l^{v-1}(l-1)$.

$(A_v)_*: K_*(P^m(v)) \to K_*(P^q(v))$ is an isomorphism. (2.3)(a)

the (unitary) K-theory e-invariant of α_v is $(-1/l^v)$. (2.3)(b)

In fact a_ν and α_ν determine elements in the stable homotopy groups

$$\Pi^S_{2l^{\nu-1}(l-1)}(S^0; Z/l^\nu) \quad \text{and} \quad \Pi^S_{2l^{\nu-1}(l-1)-1}(S^0)$$

respectively. Each of these groups has a direct summand [**1**]—the image of the J-homomorphism—which is cyclic of order l^ν. (2.3) suffices to determine the J-component of a_ν and α_ν.

We will see below that the effect of the adjoint of A_ν in KU_*-theory is determined by (2.3).

Let A^s_ν denote the s-th iterate of the map A_ν of (2.1), considered as an S-map. Also denote by A^s_ν the adjoint map

$$A^s_\nu: P^{2l^{\nu-1}(l-1)s+2}(\nu) \to QP^2(\nu). \tag{2.4}$$

As in Section 1,

$$QX = \varinjlim_n \Omega^n \Sigma^n X.$$

Let $k: P^m(\nu) \to P^m(\nu+1)$ and $n: P^m(\nu+1) \to P^m(\nu)$ be maps induced by (a choice of) an inclusion $Z/l^\nu \to Z/l^{\nu+1}$ and a surjection $Z/l^{\nu+1} \to Z/l^\nu$ respectively.

Theorem 2.5. *Let l be an odd prime. Let s, a and ν be integers ($1 \leqq s$, $1 \leqq a \leqq \nu$). The following diagrams commute up to multiplication by an l-adic unit.*

(a) *Let $m = 2l^{\nu-1}(l-1)$.*

$$\begin{array}{ccc} KU_0(P^{sm+2}(\nu); Z/l^a) & \xrightarrow{(A^{sl}_\nu)_*} & KU_0(QP^2(\nu); Z/l^a) \\ \downarrow k_* & & \downarrow (Qk)_* \\ KU_0(P^{ms+2}(\nu+1); Z/l^a) & \xrightarrow{(A^s_{\nu+1})_*} & KU_0(QP^2(\nu+1); Z/l^a) \end{array}$$

(b) *Let $t = 2l^\nu(\nu-1)$.*

$$\begin{array}{ccc} KU_0(P^{ts+2}(\nu+1); Z/l^a) & \xrightarrow{(A^s_{\nu+1})_*} & KU_0(QP^2(\nu+1); Z/l^a) \\ \downarrow n_* & & \downarrow (Qn)_* \\ KU_0(P^{ts+2}(\nu); Z/l^a) & \xrightarrow{(A^{sl}_\nu)_*} & KU_0(QP^2(\nu); Z/l^a) \end{array}$$

Proof. (a) To see whether or not such a diagram commutes it suffices, by Section 1.13, to compare the homomorphisms induced by $FQ(k)A^{sl}_\nu$ and by $FA^s_{\nu+1}k$ on $KU_0(\_; Z/l^\nu)$. Here, as in Section 1.13, $F: QP^2(\nu+1) \to BU$ is the Ω^∞-map extension of

$P^2(v+1) \xrightarrow{j} S^2 \xrightarrow{\beta} BU$ which generates $\pi_2(BU; Z/l^{v+1}) \cong Z/l^{v+1}$. This is because the generator of $KU_0(P^{sm+2}(v); Zl^a)$ is equal to the image of the (primitive) generator of $KU_0(P^{sm+2}(v); Z/l^v)$. From [**1**, Section 12] one sees that the S-maps $P^{sm+2}(v) \to S^2$ given by the adjoints of $j \cdot A^s_{v+1} \cdot k$ and $j \cdot kA^{sl}_v$ both have e-invariant, $-1/l^v$. This means that the J-components of the maps $Q(j)A^s_{v+1}k$ and $Q(jk)A^{sl}_v: P^{sm+2}(v) \to QS^2$ are equal (up to multiplication by an l-adic unit, possibly). However there is an infinite loopmap $Q_0S^0 \to Z \times \operatorname{im} J$ which deloops to give $QS^2 \to B^2(\operatorname{im} J)$, a map which is a $KU_*(\_; Z/l^v)$-isomorphism [7]. The space, $B^2(\operatorname{im} J)$, detects precisely the J-component of $\pi_*(QS^2; Z/l^v)$ so the result follows from the factorisation

$$\begin{array}{ccc} QS^2 & \xrightarrow{F} & BU \\ & \searrow \quad \swarrow & \\ & B^2(\operatorname{im} J) & \end{array}$$

(b) The proof of (b) is similar to that of (a).

3. The l-adic and Z/l^∞-diagrams

Let l be an *odd* prime and let A be $Z[1/l, \xi_{l^\infty}]$-algebra where ξ_{l^n} is a primitive l^n-th root of unity and

$$Z[1/l, \xi_{l^\infty}] = \varinjlim_n Z[1/l, \xi_{l^n}].$$

Let $v \geqq 1$ be an integer and let $\mathscr{K}_*(A; Z/l^v)$ denote Bott periodic algebraic K-theory (mod l^v) as defined in the introduction. Hence, by construction, $\mathscr{K}_*(A; Z/l^v)$ satisfies Bott periodicity with period $2l^{v-1}(l-1) = d_v$, say (i.e. $\mathscr{K}_i \cong \mathscr{K}_{i+d_v}$). Let

$$\rho_v: K_i(A; Z/l^v) \to \mathscr{K}_i(A; Z/l^v) = \varinjlim_n K_{i+nd_v}(A; Z/l^v) \tag{3.1}$$

denote the canonical localisation map.

There is an injective homomorphism [**15**, Section 3]

$$I_v: \mathscr{K}_i(A; Z/l^v) \rightarrowtail KU_i(BGLA; Z/l^v). \tag{3.2}$$

In this section we shall write KU_* for unitary (topological) K-theory—to distinguish it from algebraic K-theory.

The object of this section is to evaluate the compositions $\{I_v \cdot \rho_v; v \geqq 1\}$ and to verify that they respect direct and inverse limits over v.

In order to construct I_v one appeals to the results of [**15**, Section 3]. Suppose that

$$A_v: P^{q+d_v}(v) \to P^q(v) \qquad (d_v = 2l^{v-1}(l-1))$$

is, as in Section 2, one of Adams' maps between Moore spaces. Since, for $i \geqq 2$,

$$K_i(A; Z/l^v) = [P^i(v), BGLA^+]$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

we may form the direct limit

$$\varinjlim \left(K_i(A; Z/l^\nu) \xrightarrow{(\Sigma^{i-q}A_\nu)^*} K_{i+d_\nu}(A; Z/l^\nu) \xrightarrow{(\Sigma^{i+d_\nu-q}A_\nu)^*} \cdots \right). \tag{3.3}$$

If $i \geqq q$ the direct limit of (3.3) makes sense and, in [**15**, Section 3], it is shown to be isomorphic to $\mathscr{K}_i(A; Z/l^\nu)$. In addition this isomorphism identifies ρ of (3.1) with the map which sends $K_i(A; Z/l^\nu)$ in at the left of (3.3) by

$$\varinjlim_n (\Sigma^{i+nd_\nu-q}A_\nu)^*.$$

We may choose generators $z_{m,\alpha}$ of $KU_\alpha(P^m(\nu); Z/l^\nu) \cong Z/l^\nu$ in such a manner that

$$(\Sigma^{m-q}A_\nu)^*(z_{m,\alpha}) = z_{m-d_\nu,\alpha}.$$

If we make such choices then it is clear that the Hurewicz map induces a map from the direct limit of (3.3) to

$$KU_i(BGLA^+; Z/l^\nu) \cong KU_i(BGLA; Z/l^\nu)$$

(note that $KU_i \cong KU_{i+2m}$ for all m) defined by sending

$$f \in K_{i+nd_\nu}(A; Z/l^\nu) = [P^{i+nd_\nu}(\nu), BGLA^+]$$

to $f_*(z_{i+nd_\nu, i+nd_\nu})$. This defines I_ν in (3.2).

Now we will construct maps

$$\rho'_{s,\nu}: K_2(A; Z/l^\nu) \to \mathscr{K}_{2+sd_\nu}(A; Z/l^\nu). \tag{3.4}$$

If we identify $\mathscr{K}_2(A; Z/l^\nu)$ and $\mathscr{K}_{2+sd_\nu}(A; Z/l^\nu)$—by Bott periodicity—then, up to an l-adic unit, $\rho_{s,\nu}$ will be independent of s, if s is large. The $\{\rho'_{s,\nu}\}$ are designed so that we can easily evaluate $I_\nu \cdot \rho'_{s,\nu}$. However, we shall show later that, for large s,

$$\rho_\nu = \rho'_{\nu,s}: K_2(A; Z/l^\nu) \to \mathscr{K}_{2+sd_\nu}(A; Z/l^\nu) \cong \mathscr{K}_2(A; Z/l^\nu).$$

3.5. Construction of $\rho'_{\nu,s}$

Let $A_\nu: P^{q+d_\nu}(\nu) \to P^q(\nu)$ denote the Adams map of (2.2), where q is chosen to be minimal. By adjointing the s-th composite of A_ν (A^s_ν considered as an S-map) we obtain, as in (2.4),

$$A^s_\nu: P^{2+sd_\nu}(\nu) \to QP^2(\nu).$$

If $f: P^2(\nu) \to BGLA^+$ represents $u = [f] \in K_2(A; Z/l^\nu)$ we may form the composite

$$P^{2+sd_\nu}(\nu) \xrightarrow{A^s_\nu} QP^2(\nu) \xrightarrow{Q(f)} Q(BGLA^+) \xrightarrow{D} BGLA^+ \tag{3.6}$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

where D is the structure map [**12**] of the infinite loopspace structure on $BGLA^+$ which comes from the (direct sum) permutative category of finitely generated projective A-modules [**13**].

If $q<2+sd_v$ (3.6) gives a map (of sets)

$$\rho'_{s,v}: K_2(A; Z/l^v) \to K_{2+sd_v}(A; Z/l^v) \xrightarrow{\rho_v} \mathscr{K}_{2+sd_v}(A; Z/l^v). \tag{3.7}$$

In (3.7) ρ_v is given, as mentioned above, by representing $\mathscr{K}_{2+sd_v}(A; Z/l^v)$ as the limit of (3.3).

Let $\hat{H}_v: K_2(A; Z/l^v) \to KU_0(BGLA; Z/l^v) \cong KU_0(BGLA^+; Z/l^v)$ be defined by

$$\hat{H}_v[f] = D_* Q(f)_* (A^s_v)_* (z_{2+sd_{v,0}}).$$

By definition of I_v, in (3.2), the following diagram commutes, up to multiplication by an l-adic unit.

$$\begin{array}{ccc} K_2(A; Z/l^v) & \xrightarrow{\rho'_{s,v}} & \mathscr{K}_2(A; Z/l^v) \\ & \searrow^{\hat{H}_v} & \downarrow I_v \\ & & KU_0(BGLA; Z/l^v) \end{array} \tag{3.8}$$

Since I_v is one-one in (3.8) Lemma 3.10 will imply that, up to an l-adic unit, $\rho'_{s,v}$ is independent of s when $2+sd_v>q$. Hence we define, for $v \geqq 1$,

$$\rho'_v = \rho'_{s,v}: K_2(A; Z/l^v) \to \mathscr{K}_2(A; Z/l^v) \tag{3.9}$$

for some choice of s such that $2+sd_v>q$. Thus ρ'_v is well-defined up to multiplication by an l-adic unit. In [**15**] it is shown that

$$\hat{H}_1(y) = H_K(y) - H_K(y)^l$$

where H_K is the $KU_*(\_; Z/l)$-Hurewicz map.

Lemma 3.10. *For $2+sd_v>q$ the element*

$$(A^s_v)_*(z_{2+sd_{v,0}}) \in KU_0(QP^2(v); Z/l^v)$$

is independent of s, up to multiplication by an l-adic unit.

Proof. As in the proof of Section 2.5, it suffices, by Section 1.13, to compute

$$F_*(A^s_v)_*(z_{2+sd_{v,0}}) \in PKU_0(BU; Z/l^v).$$

However $F \cdot A^s_v$ generates $\pi_{2+sd_v}(BU; Z/l^v) \cong Z/l^v$ so that, up to l-adic units,

$$FA^{s+1}_v = (FA^s_v) \cdot (\Sigma^{2+sd_v-q} A_v)$$

and the result follows since

$$(\Sigma^{2+sd_v-q}A_v)_*: KU_0(P^{2+(s+1)d_v}(v); Z/l^v) \to KU_0(P^{2+sd_v}(v); Z/l^v)$$

is an isomorphism.

Recall [2, p. 47] that $KU_0(\mathbb{C}P^\infty; Z/l^v)$ has a basis $\beta_1, \beta_2, \ldots$ and that $KU_0(BU; Z/l^v) \simeq Z/l^v[\beta_1, \beta_2, \ldots]$. Also, being an infinite loopspace [13] (with the $+$-structure) $KU_*(BU; Z/l^v)$ admits the action of Dyer–Lashof operations [10].

The following result gives the form of $\hat{H}_v$ in (3.8).

Proposition 3.11.

(a) *Let* $b_{v,s}: P^{2+sd_v}(v) \to BU$ *generate* $\pi_{2+sd_v}(BU; Z/l^v)$ $(\cong Z/l^v)$. *There exist* $\{a_{v_j} \in Z/l^v; j = 1, 2, \ldots\}$ *such that*

$$(b_{v,s})_*(z_{2+sd_v,0}) = u\left(\beta_1 + \sum_{j=1}^{N(v)} a_{v,j} X^j(\beta_1)\right).$$

Here u is an l-adic unit and, as in Section 1.5, $X(w) = w^l + l_* Q(w)$.

(b) *Up to an l-adic unit, in* (3.8)

$$\hat{H}_v(y) = H_K(y) + \sum_{j=1}^{N(v)} a_{v,j} X^j(H_K(y))$$

where H_K *is the* $KU_*(\_; Z/l^v)$*-Hurewicz map.*

Proof. Part (a) follows from Section 1.7, together with the fact that $b_{v,s}$ factorises as

$$P^{2+sd_v}(v) \xrightarrow{A_v^s} QP^2(v) \xrightarrow{F} BU.$$

Since F is an infinite loopmap $F_* X(y) = X F_*(y)$. By Section 1.13 F_* is one-one on $PKU_0(QP^2(v); Z/l^v)$ so that

$$(A_v^s)_*(z_{2+sd_v,0}) = u\left(z_{2,0} + \sum_{j=1}^{N(v)} a_{v,j} X^j(z_{2,0})\right).$$

Therefore part (b) follows from

$$\begin{aligned}\hat{H}_v[f] &= D_* Q(f)_* (A_v^s)_*(z_{2+sd_v,0}) \\ &= u D_* Q(f)_* \left(z_{2,0} + \sum_j a_{v,j} X^j(z_{2,0})\right) \\ &= u\left(D_* Q(f)_*(z_{2,0}) + \sum_j a_{v,j} X^j(D_* Q(f)_*(z_{2,0}))\right).\end{aligned}$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

since D and $Q(f)$ are infinite loopmaps,

$$=u\left(H_K[f]+\sum_j a_{\nu,j}X^j(H_K[f])\right)$$

since

$$D_*Q(f)_*(z_{2,0})=f_*(z_{2,0})=H_K[f]\in KU_0(BGLA^+;Z/l^\nu).$$

Now we can state and prove our main result.

Theorem 3.12. *Let l be an odd prime and let A be a commutative $Z[1/l,\xi_{l^\infty}]$-algebra. Then, up to l-adic units, the homomorphisms in (3.8) commute with direct and inverse limits over ν. In addition ρ'_ν of (3.9) may be identified with the natural localisation map, ρ_ν. Consequently, we have the following commutative diagrams.*

$$\begin{array}{ccc}
\varprojlim_\nu K_2(A;Z/l^\nu) & \xrightarrow{\varprojlim_\nu \rho_\nu} & \varprojlim_\nu \mathscr{K}_2(A;Z/l^\nu) \\
& \searrow^{\varprojlim_\nu \hat{H}_\nu} & \downarrow{\scriptstyle \varprojlim_\nu I_\nu} \\
& & \varprojlim_\nu KU_0(BGLA;Z/l^\nu)
\end{array}$$

$$\begin{array}{ccc}
K_2(A;Z/l^\infty) & \xrightarrow{\varinjlim_\nu \rho_\nu} & \mathscr{K}_2(A;Z/l^\infty) \\
& \searrow^{\varinjlim_\nu \hat{H}_\nu} & \downarrow{\scriptstyle \varinjlim_\nu I_\nu} \\
& & KU_0(BGLA;Z/l^\infty)
\end{array}$$

Proof. First it is clear that I_ν, being induced by the KU-Hurewicz map applied to the direct limit of (3.3), commutes with the coefficient homomorphisms induced by $Z/l^\nu \rightarrowtail Z/l^{\nu+1}$ and $Z/l^{\nu+1} \twoheadrightarrow Z/l^\nu$. Hence both

$$\varprojlim_\nu I_\nu \quad \text{and} \quad \varinjlim_\nu I_\nu$$

exist and are injective.

Let us consider the $\left(\varprojlim_\nu\right)$-case. We have a choice of routes. We could show that $\varprojlim_\nu \rho'_\nu$ (and thence $\varprojlim_\nu \hat{H}_\nu$) exists by showing that $\rho'_\nu=\rho_\nu$ and then appealing to properties of the latter. Instead we will show independently that $\varprojlim_\nu \hat{H}_\nu$ (and, by injectivity of I_ν, also $\varprojlim_\nu \rho'_\nu$) exists. The $\left(\varinjlim_n\right)$-case is proved in a similar manner.

Let $X = BGLA^+$ and let $d_v = 2l^{v-1}(l-1)$.
Consider the following diagram.

$$\begin{array}{ccc} \pi_2(X; Z/l^{v+1}) & \xrightarrow{\hat{H}_{v+1}} & KU_0(X; Z/l^{v+1}) \\ \downarrow \pi & & \downarrow \pi \\ \pi_2(X; Z/l^v) & \xrightarrow{\hat{H}_v} & KU_0(X; Z/l^v) \end{array} \tag{3.13}$$

By definition, if $f: P^2(v+1) \to X$ represents a class $[f] \in K_2(A; Z/l^{v+1})$,

$$\begin{aligned} \pi\hat{H}_{v+1}[f] &= \pi D_* Q(f)_*(A^s_{v+1})_*(z_{2+sd_{v+1},0}) \\ &= D_* Q(f)_*(A^s_{v+1})_*(z_{2+sd_{v+1},0}) \\ &= D_* Q(f)_*(A^s_{v+1})_*(z_{2+sld_{v},0}) \end{aligned}$$

since $\pi: KU_0(P^{2m}(v+1); Z/l^{v+1}) \to KU_0(P^{2m}(v+1); Z/l^v)$ is onto. On homotopy groups π is induced by the map, k, of Theorem 2.5(a). Therefore we have

$$\begin{aligned} \hat{H}_v \pi[f] &= D_* Q(fk)_*(A^{sl}_v)_*(z_{2+sld_v,0}) \\ &= D_* Q(f)_* Q(k)_*(A^{sl}_v)_*(z_{2+sld_v,0}) \\ &= D_* Q(f)_*(A^s_{v+1})_* k_*(z_{2+sld_v,0}) \\ &= D_* Q(f)_*(A^s_{v+1})_*(z_{2+sld_v,0}) \end{aligned}$$

since

$$k_*: KU_0(P^{2m}(v); Z/l^v) \to KU_0(P^{2m}(v+1); Z/l^v)$$

is an isomorphism. Therefore (3.13) commutes. A similar argument, using Theorem 2.5(b), shows that the following diagram commutes.

$$\begin{array}{ccc} \pi_2(X; Z/l^v) & \xrightarrow{\hat{H}_v} & KU_0(X; Z/l^v) \\ \downarrow l_* & & \downarrow l_* \\ \pi_2(X; Z/l^{v+1}) & \xrightarrow{\hat{H}_{v+1}} & KU_0(X; Z/l^{v+1}) \end{array} \tag{3.14}$$

From (3.13) and (3.14)

$$\varprojlim_v \hat{H}_v \quad \text{and} \quad \varinjlim_v \hat{H}_v$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

exist and therefore so do

$$\varprojlim_{\nu} \rho'_{\nu} \quad \text{and} \quad \varinjlim_{\nu} \rho'_{\nu}.$$

Finally we show that $\rho'_{\nu}=\rho_{\nu}$ by means of Section 3.11(b). Consider the external product in algebraic K-theory [**8**]

$$v: S^1 \wedge BGL\,A^+ \to BGLA[v, v^{-1}]^+.$$

The adjoint of v, together with the natural map, j, from $BGLA^+$, may be "added" to give a map

$$BGLA^+ \times \Omega BGLA^+ \xrightarrow{adj(v)+\Omega(j)} \Omega BGLA[v, v^{-1}]^+ \tag{3.15}$$

which is a homotopy equivalence when A is a regular ring, by [**8**] and the localisation sequence [**5**]. However, by [**19**], the localisation sequence exists with mod l^{ν} coefficients provided that l is invertible in A. Hence (3.15) is an equivalence (mod l^{ν}). Thus $BGLA[v, v^{-1}]^+$ behaves like a "delooping" of $BGLA^+$ in the sense that the homomorphism, σ_v, defined by the diagram,

$$\begin{array}{ccc} KU_{\alpha}(BGLA^+; Z/l^{\nu}) & \xrightarrow{\cong} & KU_{\alpha+1}(S^1 \wedge BGLA^+; Z/l^{\nu}) \\ & {\scriptstyle \sigma_v} \searrow \quad \swarrow {\scriptstyle v_*} & \\ & KU_{\alpha+1}(BGLA[v, v^{-1}]^+; Z/l^{\nu}) & \end{array}$$

satisfies the same properties with respect to Dyer–Lashof operations as does the usual suspension homomorphism

$$\sigma_*: KU_{\alpha}(\Omega X; Z/l^{\nu}) \to KU_{\alpha+1}(X; Z/l^{\nu}).$$

For example, $(\sigma_v)_*$ annihilates decomposables.

Therefore, by Section 3.11(b),

$$(\sigma_v)_*(H(y)) = (\sigma_v)_*(H_K(y)) + \sum_{j=1}^{N(\nu)} a_{\nu,j} l_*(\sigma_v)_* Q(X^{j-1}(H_K(y))).$$

From [**10**, Theorem 1(v)]

$$(\sigma_v)_* Q(z) = \begin{cases} Q(\sigma_v)_*(z) & \text{if} \quad \deg(z) \equiv 0(2), \\ \pi((\sigma_v)_*(z))^l + lQ(\sigma_v)_*(z), & \text{if} \quad \deg(z) \equiv 1(2). \end{cases}$$

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

Therefore, for large integers T

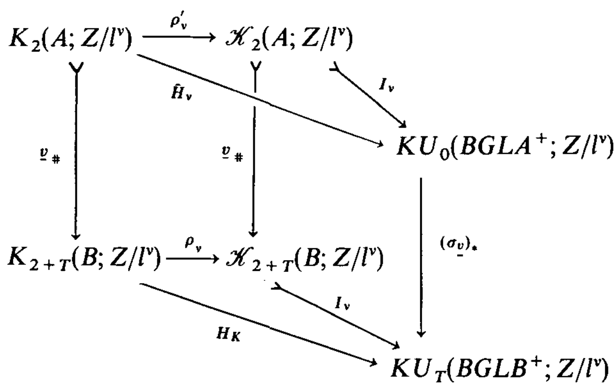
$$(\sigma_{v_T})_*(\sigma_{v_{T-1}})_*\cdots(\sigma_{v_1})_*[\hat{H}_\nu(y)-H_K(y)]=0 \tag{3.16}$$

in $KU_*(BGLB^+; Z/l^\nu)$, where

$$B=A[v_1, v_1^{-1}, \ldots, v_T, v_T^{-1}].$$

Set $\underline{v}=v_Tv_{T-1}\ldots v_1$ and $\sigma_{\underline{v}}=\sigma_{v_T}\ldots v_{v_1}$.

Consider the following commutative diagram, for T large.



In (3.17) the lower triangle commutes if $2+T\geqq q$ in (3.3), $(\sigma_{\underline{v}})_*\hat{H}_\nu=H_K\underline{v}_\#$ by (3.16), $(\sigma_v)_*I_\nu=I_\nu\underline{v}_\#$ by well-known properties of the Hurewicz map (which induces I_ν) and so $\underline{v}_\#\rho'_\nu=\rho_\nu\underline{v}_\#$, as I_ν is injective. On the other hand the natural map satisfies $\underline{v}_\#\rho_\nu=\rho_\nu v_\#$ so $\rho_\nu=\rho'_\nu$ because $\underline{v}_\#$ is one-one.

REFERENCES

1. J. F. Adams, On the groups $J(X)$, IV, *Topology* **5** (1966), 21–71.

2. J. F. Adams, *Stable Homotopy and Generalised Homology* (Chicago Lecture Notes in Mathematics, 1974).

3. W. Browder, *Algebraic K-theory with Coefficients Z/p* (Lecture Notes in Mathematics No. 654, Springer-Verlag, 1978), 40–84.

4. W. G. Dwyer, E. M. Friedlander, V. P. Snaith and R. W. Thomason, Algebraic K-theory eventually surjects onto topological K-theory, *Inventiones Math.* **66** (1982), 481–491.

5. D. Grayson (after D. G. Quillen), *Higher Algebraic K-theory* II (Lecture Notes in Mathematics, No. 551, 0000, Springer-Verlag), 217–240.

6. L. Hodgkin, *Dyer–Lashof Operations in K-theory* (London Math. Soc. Lecture Notes No. 11, 1974), 27–33.

7. L. Hodgkin and V. P. Snaith, The K-theory of some more well-known spaces, *Illinois J. Math* (2) **22** (1978), 270–278.

https://doi.org/10.1017/S0013091500003217 Published online by Cambridge University Press

8. J-L. Loday, *K*-théorie algébrique et représentations de groupes, *Ann. Sci. Ec. Norm. Sup* 4[e] *serie* **9** (1976), 309–377.

9. I. Madsen, Higher torsion in *SG* and *BSG*, *Math. Zeit.* **143** (1975), 55–80.

10. J. E. McClure, Dyer–Lashof operations in *K*-theory, *Bull. Amer. Math. Soc.* (1) **8** (1983), 67–72.

11. J. E. McClure and V. P. Snaith, On the *K*-theory of the extended power construction, *Math. Proc. Cambs. Philos. Soc.* (2) **92** (1982), 263–274.

12. J. P. May, *Geometry of Iterated Loopspaces* (Lecture Notes in Mathematics No. 271, 0000, Springer-Verlag).

13. J. P. May (with contributions by Frank Quinn, Nigel Ray and Jørgen Tornehave), E_∞-*ring Spaces and* E_∞-*ring Spectra* (Lecture Notes in Mathematics No. 577, 0000, Springer-Verlag).

14. V. P. Snaith, *Dyer–Lashof Operations in K-theory* (Lecture Notes in Mathematics No. 496, 0000, Springer-Verlag), 100–299.

15. V. P. Snaith, Unitary *K*-homology and the Lichtenbaum–Quillen conjecture on the algebraic *K*-theory of schemes, *Algebraic Top. Conf., Aarhus* 1982 (Lecture Notes in Mathematics No. 1051, 0000, Springer-Verlag), 128–155.

16. V. P. Snaith, *Algebraic Cobordism and K-theory* (Mem. A. M. Soc. No. 221, 1979).

17. R. W. Thomason, Algebraic *K*-theory and étale cohomology (*preprint* M.I.T., 1981).

18. R. W. Thomason, The Lichtenbaum–Quillen conjecture for $K/l^*[1/\beta]$, *Proc. Current Trends in Algebraic Topology* (Can. Math. Soc. Conf. Proc. Series No. 2, Part 1, 1982), 117–140.

19. C. A. Weibel, *Mayer–Vietoris Sequences* (Lecture Notes in Mathematics No. 966, 0000, Springer-Verlag), 390–407.

Department of Mathematics
University of Western Ontario
London, Ontario
N6A 5B7 Canada