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INJECTIVE SHEAVES OF ABELIAN GROUPS: A COUNTEREXAMPLE

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It has been claimed that a sheaf of abelian groups on a Hausdorff space in which the compact open sets form a basis is injective in the category of all such sheaves whenever its group of global elements is divisible (Dobbs [**1**]). The purpose of this note is to present an optimal counterexample to this by showing, more generally, that on any non-discrete T_0-space there exists a sheaf of the type in question which is not injective.

Recall that a sheaf A of abelian groups on a space X assigns to each open set U in X an abelian group AU and to each pair U, V of open sets in X such that $V \subseteq U$ a group homomorphism, denoted $s \rightsquigarrow s|V$, satisfying the familiar sheaf conditions ([**3**, p. 246]) which make A a special type of contravariant functor from the category given by the inclusion relation between the open sets of X into the category **Ab** of abelian groups, and that a map between sheaves A and B of abelian groups is a natural transformation $h:A \to B$, with component homomorphisms $h_U:AU \to BU$. In the following, **Ab**ShX will be the category with these A as objects and these $h:A \to B$ as maps (= morphisms).

Our example is based on the topological fact that any non-discrete T_0-space has dense open proper subsets. To see this, let X be any T_0-space without such subset and U an arbitrary open set in X. Then $X = U \cup U^*$ where U^* is the largest open set in X disjoint from U: for any open set V, if $V \cap (U \cup U^*) = \emptyset$ then $V \cap U = \emptyset$, hence $V \subseteq U^*$, and thus $V = \emptyset$; this shows $U \cup U^*$ is dense and therefore the whole space, by hypothesis. As a result, U^* is actually the complement of U, and consequently every open set in X is also closed; since X is T_0, this makes it Hausdorff, thus T_1, and therefore discrete.

Now, for any non-discrete T_0-space X, let W be a dense open proper subset of X, and take some point $p \in X$ outside W. Then, define sheaves A and B of abelian groups on X as follows: For each open set U, AU is the group of all locally constant functions on U with values in the additive group $\mathbf{Q}$ of rational numbers (i.e., the continuous maps from U into $\mathbf{Q}$ as discrete group) which are zero at p if $p \in U$, and $AU \to AV$ is the usual restriction map for $V \subseteq U$; further, let $BU = A(U \cap W)$ and take the restriction maps $BU \to BV$, $V \subseteq U$, as given by A. Then, the

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restriction maps $AU \to A(U \cap W) = BU$ provide a map $i:A \to B$ in $\mathbf{Ab}ShX$ which is a monomorphism since $U \cap W$ is always dense in U and any two continuous maps from U into any Hausdorff space which coincide on a dense subset of U are equal. Moreover, this monomorphism is not an isomorphism: $AX \to BX = AW$ fails to be onto since no non-zero constant $s \in AW$ can be the restriction of a $t \in AX$ because t vanishes on some neighbourhood V of p and therefore on the non-empty set $V \cap W$.

Note that for this $A \in \mathbf{Ab}ShX$, not only AX but in fact all AU are divisible, being actually vector spaces over the rational number field. Now, suppose A is injective in $\mathbf{Ab}ShX$. Then, the above monomorphism $i:A \to B$ has a left inverse $h:B \to A$, and for each open U one has the following diagram:

$$\begin{array}{ccccc} AU & \xrightarrow{i_U} & BU & \xrightarrow{h_U} & AU \\ \downarrow & & \downarrow & & \downarrow \\ A(U \cap W) & \xrightarrow[i_{U \cap W}]{} & B(U \cap W) & \xrightarrow[h_{U \cap W}]{} & A(U \cap W) \end{array} \qquad (\downarrow \text{ restrictions})$$

where $A(U \cap W) = BU = B(U \cap W)$, and $i_{U \cap W}$ and $BU \to B(U \cap W)$ are the identity map. Since $hi = \mathrm{id}_A$, this implies that $h_{U \cap W}$ is also the identity map, and hence h_U has a left inverse. However, the latter makes h a monomorphism and therefore i an isomorphism, a contradiction.

This shows:

THEOREM. *On any non-discrete T_0-space X, there exists a sheaf A of abelian groups which is not injective but for which all component groups AU, U open in X, are divisible.*

Note that this result is best possible for T_0-spaces: If X is discrete then it is well-known that $\mathbf{Ab}ShX$ is equivalent to the product category $\mathbf{Ab}^X$ via the functor $A \rightsquigarrow (A\{x\})_{x \in X}$; hence $A \in \mathbf{Ab}ShX$ is indeed injective whenever $AX \cong \prod A\{x\}$ $(x \in X)$ is divisible, as was also observed in [**1**, Remark 4].

It should be added that the restriction to T_0-spaces is essentially immaterial, in the following sense: For any space X, let X_0 be its T_0-reflection, i.e., the T_0-space obtained from X as quotient space modulo the equivalence relation identifying all points with equal neighbourhood filters (see also [**2**, p. 80]), and $q:X \to X_0$ the quotient map. Then $q^{-1}q(U) = U$ for each open set U of X since $x \in U$ and $q(y) = q(x)$, i.e., y has the same neighbourhoods as x, evidently implies $y \in U$. This shows X and X_0 have isomorphic lattices of open sets, and since sheaves only depend on the latter one has $\mathbf{Ab}ShX \cong \mathbf{Ab}ShX_0$. Hence we have the following general result:

THEOREM. *For any space X, the $A \in \mathbf{Ab}ShX$ with divisible AX are injective in $\mathbf{Ab}ShX$ if and only if X has discrete T_0-reflection.*

Remark 1. For the Hausdorff spaces whose compact open sets form a basis (i.e., the locally compact zero-dimensional Hausdorff spaces) which are not discrete our example contradicts Theorem 6 of [**1**]. The error occurs in lines 3–5 of the final paragraph of the proof (top of p. 1036). It is claimed there that, for any $A \in \mathbf{Ab}ShX$ and any direct summand M of AX, the presheaf P of abelian groups given as

$$PX = M, \quad PU = 0 \quad (U \neq X)$$

is a subpresheaf of A, but this obviously only holds if all restriction maps $AX \to AU(U \neq X)$ anull M, which in turn makes M zero since X can be covered by proper open subsets.

Remark 2. There is an alternative argument for the type of example given here which requires more sheaf theory but reduces the ultimate checking that a certain sheaf of abelian groups is not injective to a single space, the Sierpinski space $\mathbf{S}$ of two points 0 and 1 with the open sets $\emptyset$, $\{1\}$, and $\{0, 1\}$. This uses the functor $\mathbf{Ab}ShX \to \mathbf{Ab}Sh\mathbf{S}$ induced by the characteristic function $X \to \mathbf{S}$ of a dense open proper subset W of X (sending A to the sheaf assigning AX to $\{0, 1\}$ and AW to $\{1\}$) which is known to preserve injectives. For the A given above, $AX \to AW$ is one-one but not onto, as previously shown, and one then concludes the argument by the general observation that a map

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C} \\ C & \xrightarrow[\mathrm{id}_C]{} & C \end{array} \qquad (h \text{ monic})$$

in the arrow category of $\mathbf{Ab}$ (which is equivalent to $\mathbf{Ab}Sh\mathbf{S}$) has an inverse only if h is an isomorphism, so that no proper monomorphism can be injective. Clearly, this argument is less elementary than the previous one (and therefore we omit the details), but it shows that the fact we are dealing with here for arbitrary non-discrete T_0-spaces may be reduced to a suitable instance for $\mathbf{S}$. Note, however, that not every instance is suitable; thus, Example 3 of [**1**], which also gives the desired result for $\mathbf{S}$, does not seem to work for the present purpose.

We close with a few further observations.

(1) The same arguments as above provide sheaves of vector spaces, with an arbitrary field in place of $\mathbf{Q}$, which are not injective, in contrast with the fact that all vector spaces over a given field are injective.

(2) If R is any (left) noetherian ring, M any non-trivial injective (left) R-module, and X a non-discrete T_0-space then our construction with M

in place of **Q** shows there exists a sheaf A of R-modules on X which is not injective in the category of all such sheaves but for which each AU, U open in X, is injective. Each AU is the up-directed union of submodules isomorphic to some power of M, given by all those functions constant on the members of the same open partition of U; now, the latter are injective, and since R is noetherian the same then follows for AU.

(3) For any Boolean space X (i.e., X is zero-dimensional compact Hausdorff), the category **Ab***ShX* is equivalent to the category of modules over the ring $\mathbf{Z}(X)$ of continuous integer-valued functions on X, via the functor $A \rightsquigarrow AX$, $A \in$ **Ab***ShX*. This is an immediate consequence of [**4**, Theorem 7.5, p. 29] once one observes the obvious fact that **Ab***ShX* is also the category of modules for the ringed space given by the continuous integer-valued functions on the open sets of X. Thus, our construction shows that for any infinite Boolean space, there exist $\mathbf{Z}(X)$-modules with divisible additive group which are not injective.

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