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A NOTE ON BUNDLE GERBES AND INFINITE-DIMENSIONALITY

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Dedicated to Alan Carey, on the occasion of his 60th birthday

Abstract

Let (P, Y) be a bundle gerbe over a fibre bundle $Y \to M$. We show that if M is simply connected and the fibres of $Y \to M$ are connected and finite-dimensional, then the Dixmier–Douady class of (P, Y) is torsion. This corrects and extends an earlier result of the first author.

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1. Introduction

The idea of bundle gerbes [9] had its original motivation in attempts by the first author and Alan Carey to geometrise degree-three cohomology classes. This, in turn, arose from a shared interest in anomalies in quantum field theory resulting from nontrivial cohomology classes in the space of connections modulo gauge transformations. Even in the earliest of their joint papers on anomalies [6], which demonstrates that the Wess–Zumino–Witten term can be understood as holonomy for a line bundle on the loop group, there is a bundle gerbe, at that time unnoticed, lurking in the background. It was not until some time later that they realised that a better interpretation of the Wess–Zumino–Witten term for a map of a surface into a compact Lie group is as the surface holonomy of the pullback of the basic bundle gerbe over that group [5].

In this work we are concerned with the relationship between bundle gerbes and infinite-dimensionality. It is well-known [1, 3] that there is a distinct difference in twisted K-theory over a manifold M between the case where the twist $\alpha \in H^3(M, \mathbb{Z})$ is torsion and the case where it is of infinite order. The latter seems to necessitate infinite-dimensional constructions in a way that the former does not.

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A similar situation holds in the case of geometric realisations of the twist α as gerbes and bundle gerbes. In particular, in [9] it was claimed by the first author that the following was true.

THEOREM 1.1. *Let $Y \to M$ be a fibre bundle with finite-dimensional 1-connected fibres. Let M also be 1-connected. Then any bundle gerbe (P, Y) over M has exact three-curvature and hence torsion Dixmier–Douady class.*

Unfortunately the proof given in [9] is incorrect. We will explain why this is the case and give a correct proof below. Moreover we will extend this result to the case that the fibre is just connected. In addition, we will give examples of bundle gerbes with nontorsion Dixmier–Douady classes for various cases where we relax the hypotheses on the fibre and base.

2. Bundle gerbes

We quickly review here the basic results on bundle gerbes needed to understand the proof and later examples. The reader is referred to [9–11] for further details and additional references.

2.1. Basic definitions. Let $\pi : Y \to M$ be a surjective submersion and denote by $Y^{[p]}$ the p-fold fibre product

$$Y^{[p]} = \{(y_1, \ldots, y_p) \mid \pi(y_1) = \cdots = \pi(y_p)\} \subset Y^p.$$

For each $i = 1, \ldots, p + 1$, define the projection $\pi_i : Y^{[p+1]} \to Y^{[p]}$ to be the map that omits the ith element.

Here and elsewhere, if Q and R are two U(1) bundles, then we define their *product* $Q \otimes R$ to be the quotient of the fibre product of Q and R by the U(1) action $(q, r)z = (qz, rz^{-1})$, with the induced right action of U(1) on equivalence classes being given by

$$[q, r]w = [q, rw] = [qw, r].$$

In other words, observe that the fibre product is a $\mathrm{U}(1) \times \mathrm{U}(1)$ bundle and quotient by the subgroup $\{(z, z^{-1}) \mid z \in \mathrm{U}(1)\}$.

In addition, if P is a U(1) bundle, we denote by P^* the U(1) bundle with the same total space as P but with the action of U(1) changed to its inverse, thus if $u \in P^*$ and $z \in \mathrm{U}(1)$, then z acts on u by sending it to uz^{-1}. We will refer to P^* as the *dual* U(1) bundle to P.

If L and J are the hermitian line bundles associated to P and Q respectively, then there are canonical isomorphisms between $L \otimes J$ and the hermitian line bundle associated to $P \otimes Q$, as well as canonical isomorphisms between the dual line bundle L^* and the hermitian line bundle associated to P^*.

If $Q \to Y^{[p]}$ is a U(1) bundle, then we define a new U(1) bundle $\delta(Q) \to Y^{[p+1]}$ by

$$\delta(Q) = \pi_1^*(Q) \otimes \pi_2^*(Q)^* \otimes \pi_3^*(Q) \otimes \cdots .$$

It is straightforward to check that $\delta(\delta(Q))$ is canonically trivial as a U(1) bundle.

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We then have the following definition.

DEFINITION 2.1 (See [9]). A *bundle gerbe* over M is a pair (P, Y), where $Y \to M$ is a surjective submersion and $P \to Y^{[2]}$ is a U(1) bundle satisfying the following two conditions.

(1) There is a *bundle gerbe multiplication*, which is a smooth isomorphism

$$m : \pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$$

of U(1) bundles over $Y^{[3]}$.

(2) This multiplication is associative, that is, if $P_{(y_1,y_2)}$ denotes the fibre of P over (y_1, y_2), then the following diagram commutes for all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \longrightarrow & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \downarrow & & \downarrow \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} & \longrightarrow & P_{(y_1,y_4)} \end{array}$$

It is easy to check that for every $y \in Y$, there is a unique element $e \in P_{(y,y)}$ such that $ep = p \in Y_{(y,z)}$ for all $p \in Y_{(y,z)}$ and $qe = q \in Y_{(x,y)}$ for all $q \in Y_{(x,y)}$. Also, for any $p \in P_{(x,y)}$, there is a unique $p^{-1} \in P_{(y,x)}$ such that $pp^{-1} = e = p^{-1}p$.

2.2. Triviality and the Dixmier–Douady class. Bundle gerbes are higher dimensional analogues of line bundles. Accordingly they share many of the familiar properties of line bundles: just as we can pull back line bundles by smooth maps, form duals and take tensor products, we can do the same for bundle gerbes.

If (P, Y) is a bundle gerbe over M, then we can form the *dual* bundle gerbe (P^*, Y) by setting $P^* \to Y^{[2]}$ to be the dual of the U(1) bundle P in the sense described earlier. The process of forming duals commutes with taking pullbacks and forming tensor products and so we see that the bundle gerbe multiplication on P induces a bundle gerbe multiplication on P^* in a canonical way.

If (P, Y) and (Q, X) are bundle gerbes over M, then we can form a new bundle gerbe $(P \otimes Q, Y \times_M X)$ over M called the *tensor product* of P and Q. Here the surjective submersion is the fiber product $Y \times_M X \to M$ and $P \otimes Q$ is the U(1) bundle on $(Y \times_M X)^{[2]}$ whose fibre at $((y_1, x_1), (y_2, x_2))$ is given by

$$P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}.$$

The bundle gerbe multiplication on $P \otimes Q$ is defined in the obvious way, using the bundle gerbe multiplications on P and Q.

Note that if $Y = X$, then we can form the tensor product bundle gerbe in a slightly different way. We use the original surjective submersion $Y \to M$, and define $P \otimes Q$ to be the U(1) bundle with fiber $P_{(y_1,y_2)} \otimes Q_{(y_1,y_2)}$ at $(y_1, y_2) \in Y^{[2]}$. The bundle gerbe multiplication is again induced from the multiplications on P and Q. We will call the bundle gerbe $(P \otimes Q, Y)$ the *reduced* tensor product of P and Q.

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A bundle gerbe (P, Y) over M is said to be *trivial* if there is a U(1) bundle Q on Y such that $P = \delta(Q)$ and the bundle gerbe multiplication on P is given by the isomorphism

$$Q^*_{y_1} \otimes Q_{y_2} \otimes Q^*_{y_2} \otimes Q_{y_3} \cong Q^*_{y_1} \otimes Q_{y_3}$$

resulting from the canonical pairing between Q_{y_2} and $Q^*_{y_2}$.

Just as every line bundle L on M has a characteristic class in $H^2(M, \mathbb{Z})$, the Chern class $c_1(L)$ of L, every bundle gerbe (P, Y) over M has a characteristic class in $H^3(M, \mathbb{Z})$. This characteristic class is called the *Dixmier–Douady* class and is denoted $DD(P, Y)$. We construct it in terms of Čech cohomology as follows. Choose a good cover $\mathcal{U} = \{U_\alpha\}$ of M [2] with sections $s_\alpha : U_\alpha \to Y$ of $\pi : Y \to M$. Then

$$(s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$$

is a section. Choose a section $\sigma_{\alpha\beta}$ of $P_{\alpha\beta} = (s_\alpha, s_\beta)^*(P)$. That is, $\sigma_{\alpha\beta}$ is a map such that

$$\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to P,$$

with $\sigma_{\alpha\beta}(x) \in P_{(s_\alpha(x), s_\beta(x))}$. Over triple overlaps,

$$m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x)) = g_{\alpha\beta\gamma}(x)\sigma_{\alpha\gamma}(x) \in P_{(s_\alpha(x), s_\gamma(x))}$$

for $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$. This defines a cocycle that represents the Dixmier–Douady class

$$DD(P, Y) = [g_{\alpha\beta\gamma}] \in H^2(M, \mathrm{U}(1)) = H^3(M, \mathbb{Z}).$$

The Dixmier–Douady class of P is the obstruction to (P, Y) being trivial, in the sense that $DD(P, Y)$ vanishes if and only if (P, Y) is isomorphic to a trivial bundle gerbe. Note also that the Dixmier–Douady class is compatible with forming tensor products in the sense that

$$DD(P \otimes Q, Y \times_M X) = DD(P, Y) + DD(Q, X).$$

Likewise, for the reduced tensor product,

$$DD(P \otimes Q, Y) = DD(P, Y) + DD(Q, Y).$$

We also need to understand the image of the Dixmier–Douady class in real cohomology. This can be defined in terms of de Rham cohomology as in the following section.

2.3. Connections, curving and the real Dixmier–Douady class. Let $\Omega^q(Y^{[p]})$ be the space of differential q-forms on $Y^{[p]}$. Define a homomorphism

$$\delta : \Omega^q(Y^{[p]}) \to \Omega^q(Y^{[p+1]}) \quad \text{by } \delta = \sum_{i=1}^{p+1} (-1)^{i-1} \pi_i^*. \tag{2.1}$$

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These maps form the *fundamental complex*

$$0 \to \Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \cdots ,$$

which is exact [9]. If (P, Y) is a bundle gerbe on M, then a *bundle gerbe connection* is a connection ∇ on P that commutes with the bundle gerbe multiplication. If F_∇ is the curvature of a bundle gerbe connection ∇, then $\delta(F_\nabla) = 0$ so, from the exactness of the fundamental complex, $F_\nabla = \delta(f)$ for some two-form $f \in \Omega^2(Y)$. A choice of such an f is called a *curving* for ∇. From the exactness of the fundamental complex we see that the curving is only unique up to addition of two-forms pulled back to Y from M. Given a choice of curving f, we have $\delta(df) = d\delta(f) = dF_\nabla = 0$, so that $df = \pi^*(\omega)$ for a closed three-form ω on M called the *three-curvature* of ∇ and f. The de Rham class

$$\left[\frac{1}{2\pi i}\omega\right] \in H^3(M, \mathbb{R})$$

is an integral class, which is the image in real cohomology of the Dixmier–Douady class of (P, Y). For convenience we call this the real Dixmier–Douady class of (P, Y).

2.4. The lifting bundle gerbe. For the sake of completeness, and because we use it in the examples in the last section, let us review the construction of the lifting bundle gerbe [9]. Let $P \to M$ be a principal G bundle and note that there is a natural function $\tau : P^{[2]} \to G$ defined by $p_1\tau(p_1, p_2) = p_2$. Assume moreover that G has a central extension

$$\mathrm{U}(1) \to \widehat{G} \to G.$$

Regarding this as a U(1) bundle $\widehat{G} \to G$ and pulling it back with τ defines a U(1)-bundle $Q \to P^{[2]}$. It is easy to check that the multiplication in $\widehat{G}$ induces a bundle gerbe product. The Dixmier–Douady class of this bundle gerbe has a well-known geometric interpretation as the obstruction to lifting the G bundle P to a $\widehat{G}$ bundle.

3. The theorem

THEOREM 3.1. *Let $Y \to M$ be a fibre bundle with finite-dimensional* 1*-connected fibres. Let M also be* 1*-connected. Then any bundle gerbe (P, Y) over M has exact three-curvature and hence torsion Dixmier–Douady class.*

As stated earlier the proof in [9] is incorrect but it is possible to fix it as follows. Consider first the exact statement of the results in [7] in the case of two-forms.

THEOREM 3.2 (See [7, Theorem 1]). *Let $F \to E \to B$ be a differentiable fibre bundle carrying a field ω of two-forms on the vertical bundle $\mathcal{V}$, defining a closed form on each fibre. Then there is a closed form on E extending ω if and only if there is a de Rham cohomology class c on E whose restriction to each fibre is the class determined by ω.*

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Although it is not spelt out in the statement, the construction assumes that the class of the extension of ω is c. The same authors also prove the following.

THEOREM 3.3 (See [7, Theorem 2]). *Let $F \to E \to B$ be a fibre space with F and B 1-connected. If the restriction map $H^2(E, \mathbb{R}) \to H^2(F, \mathbb{R})$ is not surjective, then $H^{2k}(F, \mathbb{R}) \neq 0$ for all $k \geq 0$.*

We use these results to prove Theorem 3.1. Let (P, Y) be the bundle gerbe. If $m \in M$, then denote by Y_m the fibre of $Y \to M$ over m. Notice that because M is 1-connected, if $m, m' \in M$ there is a unique homotopy equivalence between Y_m and $Y_{m'}$ and hence a unique identification of $H^2(Y_m)$ and $H^2(Y_{m'})$ for any choice of coefficients.

If $m \in M$, then there is a restriction map $H^2(Y, \mathbb{Q}) \to H^2(Y_m, \mathbb{Q})$ that induces an onto map $H^2(Y, \mathbb{R}) \to H^2(Y_m, \mathbb{R})$ by Theorem 3.3. It is easy to see that this implies that $H^2(Y, \mathbb{Q}) \to H^2(Y_m, \mathbb{Q})$ is also onto. Indeed, choose a basis for $H^2(Y, \mathbb{Q})$ and for $H^2(Y_m, \mathbb{Q})$. Then the restriction map is given by a matrix with rational entries. So its row reduced echelon form has rational entries and one can find a rational vector mapping to any rational vector.

Let ∇ be a bundle gerbe connection with curvature F and curving f. Fix $y_0 \in Y_m$ and define $\iota : Y_m \to Y^{[2]}$ by $\iota(y) = (y_0, y)$. Then $\pi_1 \circ \iota(y) = y$ and $\pi_2 \circ \iota(y) = y_0$, so that

$$\iota^*(F) = \iota^*\delta f = \iota^*\pi_1^*(f) - \iota^*\pi_2^*(f) = f.$$

Hence f restricted to Y_m is integral and certainly rational. We deduce from Theorem 3.3 that there is a rational class in $H^2(Y, \mathbb{Q})$ extending the class defined by f on any fibre and, moreover, it can be represented by a closed two-form ρ from Theorem 3.2.

Rationality implies that there is some integer n such that $n\rho$ is an integral two-form on Y. We form the nth reduced tensor power P^n of P; it has curving and curvature that are n times the curving and curvature of P, and $DD(P^n, Y) = nDD(P, Y)$. As we are trying to show that $DD(P, Y)$ is a torsion class it suffices to show that $DD(P^n, Y)$ is a torsion class and so we may as well assume that $n = 1$ or, in other words, that ρ is integral.

As Y_m and M are 1-connected so also is Y, and hence ρ defines a U(1) bundle $Q \to Y$ whose curvature is ρ. Consider the bundle gerbe $P \otimes \delta(Q^*) \to Y^{[2]}$. This has curvature $F - \delta(\rho)$ with curving $f - \rho$ that is zero restricted to the fibres of $Y \to M$. It follows that F is zero restricted to the fibres of $Y^{[2]} \to M$ as $F = \delta(f - \rho)$. Since the fibres of $Y \to M$ are 1-connected, the fibres of $Y^{[2]} \to M$ are 1-connected and so we can descend $P \otimes \delta(Q^*)$ to a bundle $R \to M$ by taking covariantly constant sections over the fibres of $Y^{[2]} \to M$.

This descended bundle will have connection a and curvature F_a whose pullback to $Y^{[2]}$ is the connection ∇ and curvature $F - \delta(\rho)$ of $P \otimes \delta(Q^*)$. However, now we have a two-form F_a on M whose pullback to $Y^{[2]}$ is zero under δ. If we denote

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the projection from $Y^{[p]}$ to M by $\pi^{[p]}$ and let $\pi_i : Y^{[p]} \to Y^{[p-1]}$ be one of the usual projections, then $\pi^{[p-1]} \circ \pi_i = \pi^{[p]}$ and, in particular, $\pi^{[2]} \circ \pi_i = \pi^{[3]}$. It follows that

$$\begin{aligned} 0 &= \delta(\pi^{[2]*}(F_a)) \\ &= (\pi^{[2]} \circ \pi_1)^*(F_a) - (\pi^{[2]} \circ \pi_2)^*(F_a) + (\pi^{[2]} \circ \pi_3)^*(F_a) \\ &= (\pi^{[3]})^*(F_a) \end{aligned}$$

and $(\pi^{[3]})^*$ is injective so $F_a = 0$. Hence $F - \delta(\rho) = 0$ and the bundle gerbe $P \otimes \delta(Q^*)$ has zero three-curving and thus torsion Dixmier–Douady class. However, $DD(P) = DD(P \otimes \delta(Q^*))$ and thus is also torsion. This proves Theorem 3.1.

REMARK 3.4. For the interested reader, we note that the mistake in the original proof in [9] was to claim that because the forms $f - \rho$ and $d(f - \rho)$ were vertical, in the sense of restricting to zero on fibres the form $f - \rho$ descended to M. This is, of course, not true. What is true is that if a form μ on the total space of a fibre bundle and its exterior derivative $d\mu$ are vertical in the stronger sense of vanishing when contracted with any vertical vector, then μ descends to the base.

We call a bundle gerbe (P, Y) over M a *finite bundle gerbe* if Y is a fibre bundle over M with finite-dimensional fibres. We can restate Theorem 3.1 as follows.

THEOREM 3.5. *Let (P, Y) be a finite bundle gerbe over M. If M and the fibres of $Y \to M$ are 1-connected then (P, Y) has torsion Dixmier–Douady class.*

We now show how to extend this result to the case of fibres that are only connected. First we have a proposition.

PROPOSITION 3.6. *Let (P, Y) be a finite bundle gerbe over S^3 with connected fibre F. Then (P, Y) has torsion and hence zero Dixmier–Douady class.*

PROOF. We form the universal cover $\tilde{Y} \to Y$. Then we have the diagram.

$$\begin{tikzcd} \tilde{Y} \arrow[rr, "p"'] \arrow[dr, "\tilde{\pi}"'] & & Y \arrow[dl, "\pi"] \\ & S^3 & \end{tikzcd}$$

Since $Y \to S^3$ is locally trivial and $\tilde{Y} \to Y$ is a covering space, $\tilde{Y} \to S^3$ is locally trivial with fiber $\tilde{F}$, where $\tilde{F}$ denotes the pullback of $\tilde{Y} \to Y$ under the inclusion of the fiber $F \subset Y$. Consider the long exact homotopy sequences of the fibrations $\tilde{Y} \to S^3$ and $Y \to S^3$. By naturality, we have the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2(S^3) \arrow[r] \arrow[d] & \pi_1(\tilde{F}) \arrow[r] \arrow[d] & \pi_1(\tilde{Y}) \arrow[r] \arrow[d] & \pi_1(S^3) \arrow[d] \\ \cdots \arrow[r] & \pi_2(S^3) \arrow[r] & \pi_1(F) \arrow[r] & \pi_1(Y) \arrow[r] & \pi_1(S^3) \end{tikzcd}$$

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from which we conclude that $\pi_1(\tilde{F}) = 1$. Note that since $\tilde{Y} \to Y$ is a covering space and covering spaces pull back to covering spaces, $\tilde{F} \to F$ is also a covering space. In fact, $\tilde{F}$ is the universal covering of F.

In such a case as this the map $p : \tilde{Y} \to Y$ induces a map $p^{[2]} : \tilde{Y}^{[2]} \to Y^{[2]}$ and we can pull back $P \to Y^{[2]}$ to form a bundle gerbe $((p^{[2]})^*(P), \tilde{Y})$. It is straightforward from the explicit construction of the Dixmier–Douady class in Section 2.2 to show that $DD((p^{[2]})^*(P), \tilde{Y}) = DD(P, Y)$. Since $\tilde{F}$ is 1-connected Theorem 3.1 gives us that $DD(P, Y)$ is torsion. But $H^3(S^3, \mathbb{Z}) = \mathbb{Z}$ and so $DD(P, Y) = 0$. □

We can now prove the required result.

THEOREM 3.7. *Let (P, Y) be a finite bundle gerbe over a simply connected manifold M with connected fibre F. Then (P, Y) has torsion Dixmier–Douady class.*

PROOF. As M is a simply connected manifold, Hurewicz's theorem implies that the Hurewicz homomorphism

$$h : \pi_3(M) \to H_3(M, \mathbb{Z})$$

is onto. Recall that h is defined by choosing a generator $e \in H_3(S^3, \mathbb{Z})$ and setting $h([\alpha]) = \alpha_*(e)$. Recall also that there is a homomorphism

$$\rho : H^3(M, \mathbb{Z}) \to \mathrm{Hom}(H_3(M, \mathbb{Z}), \mathbb{Z}),$$

defined by pairing the cohomology and homology classes, whose kernel is the torsion subgroup of $H^3(M, \mathbb{Z})$.

Let (P, Y) be a finite bundle gerbe with Dixmier–Douady class $DD(P, Y)$. Then $\rho(DD(P, Y))$ can be determined by evaluating it on classes of the form $h([\alpha])$ to get

$$\begin{aligned} \rho(DD(P, Y)) = \rho(DD(P, Y))(h([\alpha])) &= \alpha^*(DD(P, Y)) \\ &= DD(\alpha^*(P), \alpha^*(Y)) = 0. \end{aligned}$$

Hence $DD(P, Y)$ is torsion. □

4. Examples

We consider some examples to see what can be said about the necessity of the conditions in Theorem 3.7. Note first that a bundle gerbe over a manifold M restricts to a bundle gerbe over any connected component of M. Thus there is nothing of interest to be lost by assuming, as we shall henceforth, that M is connected.

Before considering the constructions, we need to make two general remarks. First, we introduce some notation: given a map $a : Y^{[p]} \to A$ for some abelian group A, we define $\delta(a) : Y^{[p+1]} \to A$ by

$$\delta(a) = \pi_1^* a - \pi_2^* a + \pi_3^* a - + \cdots .$$

Second, if $Y \to M$ is a surjective submersion, then one way to define a bundle gerbe is to consider a function $c : Y^{[3]} \to \mathrm{U}(1)$, take $P \to Y^{[2]}$ to be the trivial $\mathrm{U}(1)$ bundle,

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and define a bundle gerbe product by

$$((y_1, y_2), \alpha)((y_2, y_3), \beta) = ((y_1, y_3), c(y_1, y_2, y_3)\alpha\beta).$$

This product is associative if and only if $\delta(c) = 1$. A bundle gerbe connection for Q is a one-form A on $Y^{[2]}$ satisfying $\delta(A) = h^{-1}\, dh$, where $\delta(A)$ is defined in equation (2.1). The curvature of A is dA. We can define curving and three-curvature in the usual way.

4.1. Bundle gerbes from open covers. It is perhaps worth remarking that, if $[g_{\alpha\beta\gamma}]$ is a representative cocycle for a class in $H^3(M, \mathbb{Z})$ with respect to an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M, we can define $Y_{\mathcal{U}}$ to be the disjoint union of the open sets in the cover $\mathcal{U}$ with the obvious projection $Y_{\mathcal{U}} \to M$. Then $g_{\alpha\beta\gamma}$ defines a function $c : Y_{\mathcal{U}}^{[3]} \to \mathrm{U}(1)$ as above and it is easy to check that this defines a finite bundle gerbe with Dixmier–Douady class $[g_{\alpha\beta\gamma}]$. In this example $Y_{\mathcal{U}} \to M$ is a surjective submersion but is, of course, unlikely to be a fibre bundle.

4.2. Cup-product bundle gerbes. A nice way to construct examples of bundle gerbes is via the cup-product construction (see, for example, [4, 8]). Suppose that we are given geometric representatives of classes α in $H^2(M, \mathbb{Z})$ and β in $H^1(M, \mathbb{Z})$ corresponding to a principal U(1) bundle Q on M and a smooth map $f : M \to S^1$ respectively. Then there is a bundle gerbe over M with Dixmier–Douady class equal to the cup product $\alpha \cup \beta$.

There are two ways in which this can be described that are of interest to us. In the first case, the bundle gerbe is of the form (P, Y), where Y is the $\mathbb{Z}$-bundle $f^*\mathbb{R}$ and $\mathbb{R} \to S^1$ is the universal $\mathbb{Z}$ bundle. In the second case, it is of the form (P, Y), where Y is the $\mathbb{Z} \times \mathrm{U}(1)$ fibre bundle that is the fibre product $f^*\mathbb{R} \times_M Q$ of $f^*(\mathbb{R})$ and Q. Notice that, in both cases, Y is disconnected.

Let us consider the first case in more detail. Take $Y = f^*\mathbb{R}$. Then there is a map $\tau : Y^{[2]} \to \mathbb{Z}$, defined by $y_2 = y_1\tau(y_1, y_2)$ for $(y_1, y_2) \in Y^{[2]}$, and we may define $P \to Y^{[2]}$ to be the U(1) bundle whose fibre at (y_1, y_2) is given by $Q_m^{\otimes\tau(y_1, y_2)}$, where $\pi(y_1) = \pi(y_2) = m$.

Likewise, in the second case Y is the fibre product $Q \times_M f^*\mathbb{R}$, so that Y is a principal $\mathrm{U}(1) \times \mathbb{Z}$ bundle over M. The group $\mathrm{U}(1) \times \mathbb{Z}$ fits into a central extension of Lie groups

$$\mathrm{U}(1) \to \mathrm{U}(1) \times \mathbb{Z} \times \mathrm{U}(1) \to \mathrm{U}(1) \times \mathbb{Z}$$

where the product on $\mathrm{U}(1) \times \mathbb{Z} \times \mathrm{U}(1)$ is defined by

$$(z_1, n_1, w_1) \cdot (z_2, n_2, w_2) = (z_1 z_2, n_1 + n_2, w_1 w_2 z_1^{n_1}).$$

We refer the reader to [4] for more details. The bundle gerbe (P, Y) is then given by the lifting bundle gerbe construction. One can check (see [4, Corollary 4.1.15]) that the Dixmier–Douady class $DD(P, Y)$ is given by the cup product $\alpha \cup \beta$.

As an example, let us take $M = S^2 \times S^1$. We let α denote the class in $H^2(M, \mathbb{Z})$ defined by pulling back the Hopf bundle $S^3 \to S^2$ via the projection to S^2 and we let β

denote the class in $H^1(M, \mathbb{Z})$ defined by the projection to S^1. In the first construction we take $Y = S^2 \times \mathbb{R} \to S^2 \times S^1$ and in the second $Y = S^3 \times \mathbb{R} \to S^2 \times S^1$.

This example can be greatly generalised. Suppose that G is a compact, simple, 1-connected Lie group with maximal torus T. Let $\mathfrak{t}$ denote the Lie algebra of T. Then there is a natural principal bundle $G \times \mathfrak{t} \to G/T \times T$ with structure group $T \times \pi_1(T)$. Given a bilinear form on the Lie algebra $\mathfrak{t}$ of T, one can define a central extension of groups

$$\mathrm{U}(1) \to T \times \pi_1(T) \times \mathrm{U}(1) \to T \times \pi_1(T)$$

(see, for example, [13]) and so we can form the corresponding lifting bundle gerbe.

4.3. Bundle gerbes on unitary groups. Theorem 3.7 implies that a finite bundle gerbe with connected fibres over a simply connected, simple compact Lie group G must be torsion. In particular, the basic bundle gerbe corresponding to the standard generator of $H^3(G, \mathbb{Z})$ cannot be a finite bundle gerbe with connected fibres. We have shown in [12] that when $G = \mathrm{SU}(n)$ it is possible to realise the bundle gerbe with Dixmier–Douady class the standard generator of $H^3(\mathrm{SU}(n), \mathbb{Z})$ as a finite bundle gerbe with disconnected fibres as follows. We define

$$Y = \{(X, \lambda) \mid \det(X - \lambda I) \neq 0\} \subset \mathrm{SU}(n) \times Z,$$

where Z denotes the *set* $\mathrm{U}(1)$ with the identity element removed. A point in $Y^{[2]}$ can be thought of as a triple (X, α, β), where neither of α or β is an eigenvalue of X. We define a hermitian line bundle over $Y^{[2]}$ by taking the fibre at (X, α, β) to be the determinant of the sum of the eigenspaces of X lying between α and β on Z, with respect to a certain ordering on Z. The corresponding $\mathrm{U}(1)$ bundle is the required bundle gerbe. Of course in this case the fibres of $Y \to \mathrm{SU}(n)$ are disconnected and it is not, in fact, a fibre bundle.

Other constructions of the basic bundle gerbe on a compact Lie group with the fibres of Y either disconnected or infinite-dimensional have been considered by other authors and are reviewed in the introduction to [12].

4.4. A bundle gerbe on the three-torus. Consider $T^3 = S^1 \times S^1 \times S^1$ and the projection $Y = \mathbb{R}^3 \xrightarrow{\pi} T^3$ that is induced by the standard projection $\mathbb{R} \to S^1$ given by $t \mapsto \exp(2\pi i t)$. Notice that the fibres of $\pi : Y \to T^3$ are disconnected and the base is, of course, not simply connected. Using constructions from [8], we show how to construct the bundle gerbe whose Dixmier–Douady class is the natural generator of $H^3(T^3, \mathbb{Z})$.

We will write $x = (x^1, x^2, x^3)$ for a vector x in Y. Note that $(x, y) \in Y^{[2]}$ if and only if $x - y \in \mathbb{Z}^3$. If $(x, y, z) \in Y^{[3]} \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, then define

$$\gamma(x, y, z) = (y^1 - z^1)(x^2 - y^2)x^3$$

and

$$c(x, y, z) = \exp(2\pi i \gamma(x, y, z)).$$

If $(x, y, z, w) \in Y^{[4]}$, then

$$\delta(\gamma)(x, y, z, w) = \gamma(y - x, z - y, w - z) \in \mathbb{Z},$$

so that $\delta(c) = 1$ and this defines a bundle gerbe. Writing points of $Y^{[3]}$ as (x, y, z), we can denote the projections as x, y, z, and we have $\mathbb{R}^3$ valued differential forms dx, dy and dz. As x, y and z differ by integers these forms are all equal, and we will denote the resulting form by $\theta = (\theta^1, \theta^2, \theta^3)$.

We can similarly define an $\mathbb{R}^3$-valued one-form on Y. Pulling it back by either of the projections $Y^{[2]} \to Y$ gives the form θ, so we will denote the form on Y by θ as well. Finally notice that θ^i, on Y, is the pullback from the ith copy of U(1) of the one-form $d\theta/2\pi$ that has total integral one on U(1). It is now easy to check that

$$A = -2\pi i(x^1 - y^1)(x^2)\theta^3 \quad \text{and} \quad f = 2\pi i x^1 \theta^2 \wedge \theta^3$$

give a connection and curving for this bundle gerbe. Notice that the curvature is $2\pi i \theta^1 \wedge \theta^2 \wedge \theta^3$, so the real Dixmier–Douady class is $(1/8\pi^3)\, d\theta^1 \wedge d\theta^2 \wedge d\theta^3$ on T^3 as we require.

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