Can. J. Math., Vol. XXIX, No. 6, 1977, pp. 1277–1286

EXTENSIONS OF CLOSURE SPACES

K. C. CHATTOPADHYAY AND W. J. THRON

1. Introduction. Extension theory has been intensively studied for completely regular spaces and is fairly well developed for T_0-topological spaces. (See, for example, [**1**] and [**5**]). However, except for definitions of some of the basic concepts in [**4**] and results on embedding of closure spaces in cubes in [**2**] and [**7**], ours is the first study of the general theory of extensions of G_0-closure spaces. (Definitions will be given following these introductory paragraphs).

The main results of this article are the following. In Section 2 we give essentially (except for one condition) all G_0-extensions of a given G_0-space with a prescribed dual trace system X^*. In Section 3 we study some special closure operators and consider the question when an extension is topological. In Section 4 we obtain results on separation properties of extensions, and in Section 5 we are concerned with compact extensions.

Though a good many of the results parallel corresponding theorems for extensions of topological spaces there are also some substantial differences. In particular the concept of a "base for the closed sets of a space" has no meaningful analogue for closure spaces and, partly because of this, there appears to be no extension, which plays the role of the principal extension of a topological space, in the general case. It is also of interest to see the topological extensions of a topological space in relation to all possible extensions of that space. Finally, we develop the theory in terms of closure operators and adherence grills rather than in terms of neighborhoods and neighborhood filters.

A function $c\colon \mathfrak{P}(X) \to \mathfrak{P}(X)$ is called a *closure operator on* X if it satisfies the following three conditions:

C_1: $c(\emptyset) = \emptyset$,
C_2: $c(A) \supset A$,
C_3: $c(A \cup B) = c(A) \cup c(B)$.

A pair (X, c) where c is a closure operator on the set X, is called a *closure space.* These concepts are generalizations of the more familiar Kuratowski closure operator and topological space, respectively. A *Kuratowski closure operator* is assumed to satisfy C_1 to C_3 as well as:

C_4: $c(c(A)) \subset c(A)$.

A pair (X, c), where c is a Kuratowski closure operator on X, is called a *topological space.*

Received December 30, 1976 and in revised form May 31, 1977.

https://doi.org/10.4153/CJM-1977-127-6 Published online by Cambridge University Press

Closure spaces were introduced by Čech [**2**]. They form the basic spatial structures in his treatise. One advantage of this approach is that it provides a convenient basis for the study of general proximity and uniform structures.

In our development, which is strongly influenced by the articles of Gagrat and Naimpally [**3**] and Thron [**6**], the concept of a grill plays an important role. A *grill on* X is a collection $\mathfrak{G}$ of subsets of X satisfying

$$G_1\colon B \supset A \in \mathfrak{G} \Rightarrow B \in \mathfrak{G},$$
$$G_2\colon A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G} \quad \text{or} \quad B \in \mathfrak{G},$$
$$G_3\colon \emptyset \notin \mathfrak{G}.$$

Let (X, c) be a closure space. Then a grill $\mathfrak{G}$ on X is called a *c-grill* with respect to (X, c) if

$$A \in \mathfrak{G} \quad \Leftrightarrow \quad c(A) \in \mathfrak{G}.$$

Dual to the concept of neighborhood filter of a point is that of *adherence grill of a point* x. By it we mean the grill

$$\mathfrak{G}_c(x) = [A: x \in c(A)].$$

Since

$$c(A) = [x: A \in \mathfrak{G}_c(x)]$$

it is clear that knowledge of all $\mathfrak{G}_c(x)$, $x \in X$, determines c completely just as knowledge of c determines all $\mathfrak{G}_c(x)$. The following lemma is an easy consequence of the appropriate definitions.

LEMMA 1. *If* (X, c) *is a closure space then* $\mathfrak{G}_c(x)$ *is a grill on* X, *for all* $x \in X$. *If for each* $x \in X$ *the family* $\mathfrak{G}_x$ *is a grill on* X *containing* $[x]$, *then the operator* $g\colon \mathfrak{P}(X) \to \mathfrak{P}(X)$ *defined by*

$$g(A) = [x: A \in \mathfrak{G}_x]$$

is a closure operator on X. *If* (X, c) *is a topological space then all* $\mathfrak{G}_c(x)$ *are c-grills. If all* $\mathfrak{G}_x$ *are g-grills then* (X, g) *is a topological space.*

A closure space (X, c) shall be called a G_0-*space* if

$$\mathfrak{G}_c(x_1) = \mathfrak{G}_c(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

We note that the condition

$$x_1 \in c([x_2]) \quad \text{and} \quad x_2 \in c([x_1]) \Rightarrow x_1 = x_2,$$

which is the T_0-separation axiom, implies but is not equivalent to, the G_0-axiom. We will have more to say about separation axioms for closure spaces in Section 4. In what follows there is always an underlying nonempty set X and frequently also a set $Y \supset X$. It will be convenient to denote elements of X or Y by $x, y, \ldots$, subsets by $A, B, \ldots$. Families of subsets will be denoted by $\mathfrak{A}$, $\mathfrak{B}, \ldots$. In particular, $\mathfrak{A}$, $\mathfrak{B}$ will be used for ultrafilters and $\mathfrak{G}$ for grills. Letters

$\alpha, \beta, \gamma, \ldots$ shall be used for collections of families of sets (i.e., $\alpha \subset \mathfrak{P}(\mathfrak{P}(X))$). There will be some exceptions to these conventions.

2. Extensions of closure spaces. Let $\psi : X \to Y$ be an injection; let c be a closure operator on X and k be a closure operator on Y. Then $E = (\psi, (Y, k))$ is called an *extension of* (X, c) if

(1) $\psi(c(A)) = k(\psi(A)) \cap \psi(X), \quad \text{for all } A \subset X,$

and

(2) $k(\psi(X)) = Y.$

Since ψ is an injection, (1) insures that ψ is a homeomorphism from (X, c) onto $(\psi(X), k^1)$, where $k^1(B) = k(B) \cap \psi(X)$, for all $B \subset \psi(X)$, is the closure operator induced on $\psi(X)$ by the closure operator k on Y. Condition (2) insures that $\psi(X)$ is dense in (Y, k).

Two extensions $E_1 = (\psi_1, (Y_1, k_1))$ and $E_2 = (\psi_2, (Y_2, k_2))$ of the same space (X, c) are called *equivalent* if there exists a homeomorphism χ from (Y_1, k_1) onto (Y_2, k_2) such that on X, $\chi \circ \psi_1 = \psi_2$. The extension E_1 is said to be *greater than* the extension E_2 if there exists a continuous function θ from (Y_1, k_1) onto (Y_2, k_2) such that on X, $\theta \circ \psi_1 = \psi_2$.

Associated with each extension is its *dual trace system*

$$X^* = X^*(E) = [\tau(y, E) \colon y \in Y],$$

where

$$\tau(y) = \tau(y, E) = [A \colon y \in k(\psi(A))].$$

We speak of dual trace systems X^* and *dual traces* $\tau(y)$ since the terms "trace system" and "trace" are usually reserved for the families of filters and individual filters, respectively, which are the traces on X of the neighborhood filters of y on Y.

The adherence grills on Y are related to the dual traces $\tau(y)$ as follows:

$$\begin{aligned} \mathfrak{E}_k(y) &= [B \colon y \in k(B)] \\ &= [B \colon y \in k((B \cap \psi(X)) \cup (B \sim \psi(X)))]. \end{aligned}$$

If we set $A = \psi^{-1}(B)$ we obtain

(3) $$\begin{aligned} \mathfrak{E}_k(y) &= [B \colon y \in k(\psi(A) \cup (B \sim \psi(X)))] \\ &= [B \colon y \in k(\psi(A))] \cup [B \colon y \in k(B \sim \psi(X))] \\ &= [B \supset \psi(A) \colon A \in \tau(y)] \cup [B \colon y \in k(B \sim \psi(X))]. \end{aligned}$$

The following lemma is easily established.

LEMMA 2. (a) *For all extensions E and all $y \in Y$ the trace $\tau(y, E)$ is a grill on X.*

(b) $\tau(\psi(x), E) = \mathfrak{G}_c(x)$.

(c) *Equivalent extensions have identical dual trace systems.*

(d) *If (Y, k) is a topological space then all dual traces $\tau(y)$ are c-grills.*

We are now able to state and prove our main result.

THEOREM 1. *Let (X, c) be a given G_0-closure space. Let X^* be a collection of grills on X satisfying*

$$[\mathfrak{G}_c(x): x \in X] \subset X^*.$$

Define

$$A^* = [\mathfrak{G}: \mathfrak{G} \in X^*, A \in \mathfrak{G}],$$
$$\varphi: X \to X^* \quad by \quad \varphi(x) = \mathfrak{G}_c(x)$$
$$h_r(\alpha) = (\varphi^{-1}(\alpha))^* \cup r(\alpha \sim \varphi(X)), \quad \alpha \subset X^*$$

where $r: \mathfrak{P}(X^* \sim \varphi(X)) \to \mathfrak{P}(X^*)$ satisfies

$$(4) \quad r(\emptyset) = \emptyset, \quad r(\beta) \supset \beta, \quad r(\beta_1 \cup \beta_2) = r(\beta_1) \cup r(\beta_2).$$

Then $(\varphi, (X^*, h_r))$ is a G_0-extension of (X, c) with dual traces $\tau(\mathfrak{G}) = \mathfrak{G}$, for all $\mathfrak{G} \in X^*$. Moreover all extensions of (X, c) on X^* with dual traces $\tau(\mathfrak{G}) = \mathfrak{G}$, for all $\mathfrak{G} \in X^*$, can be obtained by suitable choices of r.

Proof. Clearly φ is an injection into X^*. Moreover

$$h_r(\varphi(X)) = X^* \cup r(\phi) = X^*$$

so that $\varphi(X)$ is dense in X^*. Since $\tau(\mathfrak{G}) = [B: \mathfrak{G} \in h_r(\varphi(B))]$ the condition $\tau(\mathfrak{G}) = \mathfrak{G} = [B: B \in \mathfrak{G}]$ is equivalent to

$$B \in \mathfrak{G} \Leftrightarrow \mathfrak{G} \in h_r(\varphi(B)).$$

This is the case if and only if $h_r(\varphi(B)) = B^*$.

If h is any closure operator on X^* then it is additive and hence

$$h(\alpha) = h(\alpha \cap \varphi(X)) \cup h(\alpha \sim \varphi(X)).$$

Now $\alpha \cap \varphi(X) = \varphi(\varphi^{-1}(\alpha))$ and hence the requirement $\tau(\mathfrak{G}) = \mathfrak{G}$ for all $\mathfrak{G} \in X^*$ is equivalent to

$$h(\alpha \cap \varphi(X)) = (\varphi^{-1}(\alpha))^* \quad \text{and} \quad h(\varphi(A) \sim \varphi(X)) = h(\emptyset) = \emptyset.$$

Thus we can write

$$h_r(\alpha) = (\varphi^{-1}(\alpha))^* \cup r(\alpha \sim \varphi(X)),$$

where r is defined on $\mathfrak{P}(X^* \sim \varphi(X))$ with values in $\mathfrak{P}(X^*)$ and is arbitrary except for satisfying the condition (4). We now note that

$$\begin{aligned} h_r(\varphi(A)) \cap \varphi(X) &= A^* \cap \varphi(X) \\ &= [\mathfrak{G}: \mathfrak{G} = \mathfrak{G}_c(x), A \in \mathfrak{G}] \\ &= [\mathfrak{G}_c(x): x \in c(A)] = \varphi(c(A)). \end{aligned}$$

Hence φ is a homeomorphism onto $(\varphi(X), h_r^1)$ where $h_r^1(\beta) = h_r(\beta) \cap \varphi(X)$, and $(\varphi, (X^*, h_r))$ is an extension of (X, c).

Note that it follows from (3) that for every $\mathfrak{E} \in X^*$

$$\mathfrak{E}_{h_r}(\mathfrak{E}) \cap \mathfrak{P}(\varphi(X)) = [\varphi(A) : A \in \mathfrak{E}].$$

Thus $\mathfrak{E} \neq \mathfrak{E}'$ implies $\mathfrak{E}_{h_r}(\mathfrak{E}) \neq \mathfrak{E}_{h_r}(\mathfrak{E}')$ and hence (X^*, h_r) is a G_0-space.

Finally, if $(\psi, (Y, k)$ is an extension of (X, c) with dual trace system X^* then the function $\tau : Y$ onto X^* may not be a bijection. Such extensions are not included among the ones discussed above. However, if τ is bijection onto X^*, then it is equivalent to an extension on X^*, and hence is subsumed under the extensions described above.

3. Special choices for r and topological extensions. From now on we shall assume that all spaces (X, c) are G_0-spaces. Choosing $r(\beta) = \beta$, for all $\beta \subset X^*$, is a permissible choice for r in h_r. Since the identity mapping from (X, c) to (X, c') is continuous if $c'(A) \supset c(A)$, for all $A \subset X$, we conclude that the extension $(\varphi, (X^*, h_1))$, where

$$h_1(\alpha) = (\varphi^{-1}(\alpha))^* \cup (\alpha \sim \varphi(X)) = (\varphi^{-1}(\alpha))^* \cup \alpha,$$

is the largest among all extensions $(\varphi, (X^*, h_1))$ of (X, c). It is called the *simple extension of* (X, c) *with dual trace system* X^*. Similarly $(\varphi, (X^*, h_0))$ where

$$h_0(\alpha) = (\varphi^{-1}(\alpha))^* \cup r_0(\beta), \quad \text{and} \quad r_0(\emptyset) = \emptyset, r_0(\beta) = X^* \text{ if } \beta \neq \emptyset,$$

is the smallest among all extensions $(\varphi, (X^*, h_r))$.

Let $(\psi, (Y, k))$ be an extension of (X, c) with trace system X^*. Then τ from Y to X^* is continuous from (Y, k) to (X^*, h), provided $h(\alpha) = \tau(k(\tau^{-1}(\alpha)))$, for all $\alpha \subset X^*$. Since τ is $1 - 1$ on $\psi(X)$ the pair $(\tau \circ \psi, (X^*, h))$ is an extension of (X, c). We note that $\tau \circ \psi = \varphi$ and that the dual traces of the points $\mathfrak{E}$ in $(\tau \circ \psi, (X^*, h))$ are exactly the grills $\mathfrak{E}$ themselves. It follows that every extension of a G_0-space (X, c) with trace system X^* is larger than the extension $(\varphi, (X^*, h_0))$. We thus have established the following result.

THEOREM 2. *Let (X, c) be a G_0-space. Then the simple extension $(\varphi, (X^*, h_1))$ is the largest among all extensions $[\varphi, (X^*, h_r))$. The extension $(\varphi, (X^*, h_0))$ is the smallest among all extensions of (X, c) with dual trace system X^*.*

While in many respects the extension theory for closure spaces parallels that of topological spaces, there are some important differences. The concept of a base for the closed sets has no analogue for closure spaces, and largely because of that there is no significant general closure operator that corresponds to the Kuratowski closure operator

$$d(\alpha) = \bigcap [A^* : \alpha \subset A^*].$$

Now consider

$$\begin{aligned} h_d(\alpha) &= (\varphi^{-1}(\alpha))^* \cup d(\alpha) \\ &= (\varphi^{-1}(\alpha))^* \cup d(\alpha \sim \varphi(x)). \end{aligned}$$

The equality follows from the fact that $d(\alpha)$ is always a closure operator and hence $d(\alpha) = d(\alpha \sim \varphi(X)) \cup d(\varphi(\varphi^{-1}(\alpha)))$, together with the observation that

$$d(\varphi(A)) \subset A^*$$

always holds. In general $d(\varphi(A)) \neq A^*$. However if (X, c) is a topological space, which insures that all $\mathfrak{G}_c(x)$ are c-grills, and if in addition all other $\mathfrak{G} \in X^*$ are also c-grills then $A^* = (c(A))^*$ for all $A \subset X$. Now $\varphi(A) \subset B^*$ $\Leftrightarrow A \subset c(B) \Leftrightarrow c(A) \subset c(B)$ if and only if

$$A^* = (c(A))^* \subset (c(B))^* = B^*.$$

Thus finally, $d(\varphi(A)) = A^*$ if all $\mathfrak{G} \in X^*$ are c-grills. Hence in this case $h_d = d$ and thus all $\tau(\mathfrak{G}) = \mathfrak{G}$ so that $(\varphi, (X^*, d))$ is a topological extension of (X, c) with dual trace system X^*.

If h' is any other Kuratowski closure operator on X^* with dual trace system X^* (we are continuing to assume that all $\mathfrak{G} \in X^*$ are c-grills) then we must have $h'(\varphi(A)) = A^*$ and hence $A^* = h'(A^*)$, for all $A \subset X$. Thus all A^* are closed sets in (X^*, h) and hence $h'(\alpha) \subset d(\alpha)$, for all $\alpha \subset X^*$.

It follows that $(\varphi, (X^*, d))$ is the smallest topological extension of (X, c) with trace system X^*, all of whose elements are c-grills. The extension $(\varphi, (X^*, d))$ is known as the *principal* (*strict*) *topological extension* of the T_0-topological space (X, c).

If not all elements of X^* are c-grills then the extension $(\varphi, (X^*, h_d))$ has no known extremal properties, but it does yield a compactification under a certain additional condition (see Section 5).

We conclude this section by observing that, except in trivial special cases, the extension $(\varphi, (X^*, h_0))$ is not topological even if all $\mathfrak{G} \in X^*$ are c-grills.

4. Separation axioms of extensions. Separation axioms on closure spaces have different implications than the identically stated axioms have on topological spaces. In order to avoid confusion it thus appears to be desirable to assign other letters to them. A closure space (X, c) is called a D_0-*space* if

$$x \in c([y]) \quad \text{and} \quad y \in c([x]) \Rightarrow x = y.$$

A closure space is a D_1-*space* if

$$c([x]) = [x], \quad \text{for all } x \in X.$$

Before defining the D_2-axiom it will be helpful to introduce further results and notation about grills (for more details see [**6**]). Grills are always unions of ultrafilters and every union of ultrafilters is a grill. By $\Omega(X)$ we shall mean the

https://doi.org/10.4153/CJM-1977-127-6 Published online by Cambridge University Press

set of all ultrafilters on X. If $\mathfrak{G}$ is a grill on X then

$$\mathfrak{G}^+ = [\mathfrak{A} : \mathfrak{A} \in \Omega(X), \mathfrak{A} \subset \mathfrak{G}].$$

Thus $\mathfrak{G}^+ \subset \Omega(X)$. For convenience we shall write $\mathfrak{G}_c^+(x)$ instead of $(\mathfrak{G}_c(x))^+$.
If f: $X \to Y$ and $\mathfrak{A} \in \Omega(X)$ then

$$f(\mathfrak{A}) = [B: B \subset Y, B \supset f(U) \text{ for some } U \in \mathfrak{A}]$$

is an ultrafilter on Y. Moreover, if f is an injection, then

$$\mathfrak{A} \neq \mathfrak{B} \Rightarrow f(\mathfrak{A}) \neq f(\mathfrak{B}).$$

Let $A \subset X$ then $\Omega(X)$ can be decomposed into two disjoint sets

$$\Omega(X) = \Omega_1^A(X) \cup \Omega_2^A(X),$$

defined as follows

$$\Omega_1^A(X) = [\mathfrak{A}: A \in \mathfrak{A} \in \Omega(X)],$$
$$\Omega_2^A(X) = [\mathfrak{A}: X \sim A \in \mathfrak{A} \in \Omega(X)].$$

The Hausdorff separation axiom for topological spaces can be stated as follows:

$$x_1, x_2 \in X, x_1 \neq x_2 \quad \Rightarrow \quad \nexists\, \mathfrak{A} \in \Omega(X) \text{ such that } \mathfrak{N}_{x_1} \cup \mathfrak{N}_{x_2} \subset \mathfrak{A}.$$

The dual of this statement is

$$x_1, x_2 \in X, x_1 \neq x_2 \quad \Rightarrow \quad \mathfrak{G}_c^+(x_1) \cap \mathfrak{G}_c^+(x_2) = \emptyset,$$

and we shall use it as our definition for a D_2-*space* for general closure spaces. That there are D_2-spaces which are not topological is shown in [**2**, 27B9(c)].
An inspection of the D_0-axiom yields the following theorem.

THEOREM 3. *An extension* $(\varphi, (X^*, h_r))$ *of a* G_0-*space* (X, c) *is a* D_0-*extension if and only if all of the following conditions are satisfied:*
(a$_0$) (X, c) *is a* D_0-*space*,
(b$_0$) $[x] \in \mathfrak{G} \in X^* \sim \varphi(X) \Rightarrow \mathfrak{G}_c(x) \notin r([\mathfrak{G}])$,
(c$_0$) $\mathfrak{G}_c(y) \in r([\mathfrak{G}]), \mathfrak{G} \in X^* \sim \varphi(X) \Rightarrow [y] \notin \mathfrak{G}$,
(d$_0$) $\mathfrak{G} \neq \mathfrak{G}', \mathfrak{G}, \mathfrak{G}' \in X^* \sim \varphi(X)$ *and* $\mathfrak{G}' \in r([\mathfrak{G}]) \Rightarrow \mathfrak{G} \notin r([\mathfrak{G}'])$.

For r_0 we have for all $\mathfrak{G} \in X^* \sim \varphi(X)$

$$r_0([\mathfrak{G}]) = X^*,$$

and hence, if $|X^* \sim \varphi(X)| \geqq 2$, $(\varphi, (X^*, h_0))$ is not a D_0-extension. It is, however, as we saw in Theorem 1, a G_0-extension.
To study D_1-extensions we note that

$$h_r([\mathfrak{G}]) = \begin{cases} [x]^* & \text{if } \mathfrak{G} = \mathfrak{G}_c(x), \\ r([\mathfrak{G}]) & \text{if } \mathfrak{G} \in X^* \sim \varphi(X). \end{cases}$$

Thus (X^*, h_r) is a D_1-space if and only if

$$[x]^* = [\mathfrak{C}_c(x)] \quad \text{for all } x \in X$$

and

$$r([\mathfrak{C}]) = [\mathfrak{C}] \quad \text{for all } \mathfrak{C} \in X^* \sim \varphi(X).$$

The first condition is equivalent to: for all $\mathfrak{C} \in X^*$, $[x] \in \mathfrak{C}$ if and only if $\mathfrak{C} = \mathfrak{C}_c(x)$. This condition insures that (X, c) is a D_1-space. Theorem 4 now follows easily.

THEOREM 4. *An extension $(\varphi, (X^*, h_r))$ of a G_0-space (X, c) is a D_1-extension if and only if the two conditions below are satisfied.*

(a$_1$) *for all $\mathfrak{C} \in X^*$, $[x] \in \mathfrak{C}$ if and only if $\mathfrak{C} = \mathfrak{C}_c(x)$,*

(a$_2$) *$r([\mathfrak{C}]) = [\mathfrak{C}]$, for all $\mathfrak{C} \in X^* \sim \varphi(X)$. $(\varphi, (X^*, h_r))$ will be a D_1-extension only if (X, c) is a D_1-space.*

From the expression obtained for $\mathfrak{C}_k(y)$ in (3) it follows immediately that a necessary condition for $(\psi, (Y, k))$ to be a D_2-extension is that $y_1 \neq y_2 \Rightarrow (\tau(y_1))^+ \cap (\tau(y_2))^+ = \emptyset$. Hence $\tau: Y \to X^*$ must be a bijection and all D_2-extensions are equivalent to extensions of the type $(\varphi, (X^*, h_r))$.

Formula (3) yields

$$\mathfrak{C}_{h_r}{}^+(\mathfrak{C}) = [\varphi[\mathfrak{A}]: \mathfrak{A} \in \mathfrak{C}^+]$$
$$\cup\, [\mathfrak{B}: \mathfrak{B} \in \Omega(X^*), \varphi(X) \notin \mathfrak{B}, \forall \beta \in \mathfrak{B}, \mathfrak{C} \in r(\beta \sim \varphi(X))].$$

Introduce

$$\Gamma_1(\mathfrak{C}) = [\varphi(\mathfrak{A}): \mathfrak{A} \in \mathfrak{C}^+]$$

and

$$\Gamma_2(\mathfrak{C}) = [\mathfrak{B}: \mathfrak{B} \in \Omega(X^*), \varphi(X) \notin \mathfrak{B}, \forall \beta \in \mathfrak{B}, \mathfrak{C} \in r(\beta \sim \varphi(X))].$$

Then $\Gamma_1(\mathfrak{C}) \subset \Omega_1{}^{\varphi(X)}(X^*)$ and $\Gamma_2(\mathfrak{C}) \subset \Omega_2{}^{\varphi(X)}(X^*)$ and hence $\Gamma_1(\mathfrak{C}_1) \cap \Gamma_2(\mathfrak{C}_2) = \emptyset$ for all $\mathfrak{C}_1, \mathfrak{C}_2 \in X^*$. It follows that

$$\mathfrak{C}_{h_r}{}^+(\mathfrak{C}_1) \cap \mathfrak{C}_{h_r}{}^+(\mathfrak{C}_2) = (\Gamma_1(\mathfrak{C}_1) \cap \Gamma_1(\mathfrak{C}_2)) \cup (\Gamma_2(\mathfrak{C}_1) \cap \Gamma_2(\mathfrak{C}_2)).$$

Since $\Gamma_1(\mathfrak{C}_1) \cap \Gamma_1(\mathfrak{C}_2) = \emptyset$ if and only if $\mathfrak{C}_1{}^+ \cap \mathfrak{C}_2{}^+ = \emptyset$, we have now proved the following theorem.

THEOREM 5. *In order that an extension of a G_0-space (X, c) with trace system X^*, be a D_2-extension it is necessary that (X, c) be a D_2-space and that the extension be equivalent to one of the form $(\varphi, X^*, h_r))$. In order that $(\varphi, X^*, h_r))$ be a D_2-extension it is necessary and sufficient that, for all $\mathfrak{C}_1, \mathfrak{C}_2 \in X^*$, $\mathfrak{C}_1 \neq \mathfrak{C}_2$,*

$$\mathfrak{C}_1{}^+ \cap \mathfrak{C}_2{}^+ = \emptyset \quad \textit{and} \quad \Gamma_2(\mathfrak{C}_1) \cap \Gamma_2(\mathfrak{C}_2) = \emptyset.$$

The condition $\mathfrak{C}_1{}^+ \cap \mathfrak{C}_2{}^+ = \emptyset$ is a condition on X^* alone. The second requirement depends on both r and X^*.

In particular we have for h_1, $\Gamma_2(\mathfrak{C}) = [\mathfrak{B}(\mathfrak{C})]$, where $\mathfrak{B}(\mathfrak{C}) = [\beta: \mathfrak{C} \in \beta \in X^*]$

is the principal ultrafilter of $\mathfrak{E}$ in X^*. Hence $(\varphi, (X^*, h_1))$ is a D_2-extension if and only if $\mathfrak{E}_1^+ \cap \mathfrak{E}_2^+ = \emptyset$ for all $\mathfrak{E}_1, \mathfrak{E}_2 \in X^*$, $\mathfrak{E}_1 \neq \mathfrak{E}_2$.

5. Compactifications. An extension $(\psi, (Y, k))$ is a *compactification* if (Y, k) is a compact closure space. Čech [**2**, 41A3] defines a closure space to be compact if and only if every filter has a cluster point. Using his definition of a cluster point we are thus led to the statement:

A closure space (X, c) is *compact* if, for every filter $\mathfrak{F}$ on X,

$$\cap [c(F) : F \in \mathfrak{F}] \neq \emptyset.$$

For our purposes the following characterization of compactness is even more useful.

LEMMA 3. *A closure space (X, c) is compact if and only if $[\mathfrak{E}_c^+(x) : x \in X]$ is a cover of $\Omega(X)$.*

Proof. $\cap [c(F): F \in \mathfrak{F}] \neq \emptyset$ if and only if there exists an x such that $x \in \cap [c(F): F \in \mathfrak{F}]$. This is equivalent to $\mathfrak{F} \subset \mathfrak{E}_c(x)$. Since every filter is contained in some ultra filter, $\mathfrak{F} \subset \mathfrak{E}_c(x)$ is equivalent to $\mathfrak{A} \subset \mathfrak{E}_c(x)$, and hence every $\mathfrak{A} \in \Omega(X)$ must be an element of some $\mathfrak{E}_c^+(x)$.

In terms of it we have:

THEOREM 6. *An extension $(\psi, (Y, k))$ of a G_0-space (X, c) is a compactification of (X, c) if and only if the two conditions below are satisfied:*
(a) *the family $[\tau^+(y): y \in Y]$ covers $\Omega(X)$,*
(b) *the family $[\sigma^+(y): y \in Y]$ covers $\Omega_2^{\psi(X)}(Y)$.*
Here $\sigma(y) = [B: y \in k(B \sim \psi(X))]$.

Proof. We note that $\sigma(y)$ is a grill on Y and that every element of it has a non-null intersection with $Y \sim \psi(X)$. Hence $\sigma^+(y) \subset \Omega_2^{\psi(X)}(Y)$. Similarly,

$$[B \subset \psi(A): A \in \tau(y)]^+ \subset \Omega_1^{\psi(X)}(Y).$$

In order that $[\mathfrak{E}_k^+(y): y \in Y]$ cover $\Omega(Y)$ it is therefore necessary and sufficient that the sets $\sigma^+(y)$ cover $\Omega_2^{\psi(X)}(Y)$ and the sets $[B \supset \psi(A): A \in \tau(y)]^+$ cover $\Omega_1^{\psi(X)}(Y)$, as y ranges over Y. The second requirement is equivalent to condition (a) of the theorem.

The extension $(\varphi, (X^*, h_0))$ is a compactification provided $[\mathfrak{E}^+: \mathfrak{E} \in X^*]$ covers $\Omega(X)$, for in this case $\sigma^+(y) = \Gamma_2(\mathfrak{E}) = \Omega_2^{\varphi(X)}(X^*)$ and hence condition (b) is also satisfied.

$(\varphi, (X^*, h_1))$ cannot be a compactification if $X^* \sim \varphi(X)$ is infinite. This is true since $\sigma^+(\mathfrak{E}) = \Gamma_2(\mathfrak{E}) = [\mathfrak{B}(\mathfrak{E})]$ in this case, but $\Omega^{\varphi_2(X)}(X^*)$ does contain non principal ultrafilters.

A collection X^* of grills is said to satisfy *property P* if

$$\mathfrak{A}_h \subset \mathfrak{A}, |\mathfrak{A}_h| < \aleph_0, \exists \mathfrak{G} \in X^* \text{ such that } \mathfrak{A}_h \subset \mathfrak{G}$$
$$\Rightarrow \exists \mathfrak{G}_0 \in X^* \text{ such that } \mathfrak{A} \subset \mathfrak{G}_0.$$

It is known that property P is satisfied by the family of maximal Π-clans with respect to a given proximity Π (see, for example [**3**]).

For a collection X^* satisfying property P and the additional condition that for every $x \in X$ there exists a $\mathfrak{G}_x \in X^*$ with $[x] \in \mathfrak{G}_x$ (this is the case when $[\mathfrak{G}_c(x): x \in X] \subset X^*$), let $\mathfrak{A}$ be an arbitrary ultrafilter. If $\mathfrak{A}_k$ is any finite subcollection of $\mathfrak{A}$ then $\bigcap [\mathfrak{A}: U \in \mathfrak{A}_k] \supset [x_k]$ and hence $\mathfrak{A}_k \subset \mathfrak{G}_{x_k}$. Thus $\exists \mathfrak{G} \in X^*$ such that $\mathfrak{A} \subset \mathfrak{G}$ and $\Omega(X)$ is covered by the family $[\mathfrak{G}^+: \mathfrak{G} \in X^*]$.

We conclude this section with the following theorem.

THEOREM 7. *Let* (X, c) *be a* G_0-*space and let* X^* *be a trace system on* (X, c) *satisfying property P. Then* (X^*, d) *is a compact space and* $(\varphi, X^*, h_\tau))$ *is a compactification of* (X, c) *provided* $h_\tau(\alpha) \supset d(\alpha)$, *for all* $\alpha \subset X^*$. *In particular* $(\varphi, (X^*, h_d))$ *is a compactification of* (X, c).

Proof. (X^*, d) is a topological space and the sets A^* form a base for its closed sets. To prove its compactness it therefore suffices to consider filters with base elements A^*. If $[A_i^*: i \in I]$ forms a base of a filter on X^*, then there exists $\mathfrak{G} \in \bigcap [A_{ik}^*: k = 1, \ldots n]$. Hence $[A_{ik}: k = 1, \ldots, n] \subset \mathfrak{G}$. Property P then insures the existence of a $\mathfrak{G}_0 \in X^*$ such that $[A_i: i \in I] \subset \mathfrak{G}_0$. This, however, is equivalent to $\mathfrak{G}_0 \in \bigcap [A_i^*: i \in I]$ and hence (X^*, d) is compact. The second assertion of the theorem follows from the fact that continuous images of compact spaces are compact. For closure spaces this is proved in [**2**, 41A14].

REFERENCES

1. B. Banaschewski, *Extensions of topological spaces*, Can. Math. Bull. *7* (1964), 1–22.

2. E. Čech, *Topological spaces*, rev. ed. (Publ. House Czech. Acad. Sc. Prague, English transl. Wiley, New York, 1966).

3. M. S. Gagrat and S. A. Naimpally, *Proximity approach to extension problems*, Fund. Math. *71* (1971), 63–76.

4. M. S. Gagrat and W. J. Thron, *Nearness structures and proximity extensions*, Trans. Amer. Math. Soc. *208* (1975), 103–125.

5. W. J. Thron, *Topological structures* (Holt, Rinehart and Winston, New York, 1966).

6. ——— *Proximity structures and grills*, Math. Ann. *206* (1973), 35–62.

7. W. J. Thron and R. H. Warren, *On the lattice of proximities of Čech compatible with a given closure space*, Pac. J. Math. *49* (1973), 519–535.

Panjab University,
Chandigarh, India;
University of Colorado,
Boulder, Colorado