J. Austral. Math. Soc. **21** (Series A) (1976), 257–266.

SIX CLASSES OF THEORIES*

H. JEROME KEISLER

(Received 14 April 1975)

A theory T is said to κ-stable if, given a pair of models $\mathfrak{A} \subset \mathfrak{B}$ of T with $\mathfrak{A}$ of power κ, there are only κ types of elements of $\mathfrak{B}$ over $\mathfrak{A}$ (types are defined below). This notion was introduced by Morley (1965) who gave a powerful analysis of ω-stable theories. Shelah (1971) showed that there are only four possibilities for the set of κ in which a countable theory is stable. This partition of all theories into four classes (ω-stable, superstable, stable, and unstable theories) has proved to be of great value. However, most familiar examples of theories are unstable.

In this paper we use the cardinal numbers of the sets of types in $\mathfrak{B}$ over $\mathfrak{A}$ to partition the unstable theories into three more classes, so altogether there are six classes of countable theories.

This is an expository paper which has statements of the theorems and examples, but no proofs. It is based on a lecture to the Australian Logic Summer School at Monash University in January 1974. The work was supported in part by an NSF Grant.

1. Stability Function

We assume throughout that L is a countable first order language, and T is a theory in L which has infinite models. By a *basic formula* we mean a finite conjunction of atomic and negated atomic formulas. Given a model $\mathfrak{A}$ for L with universe A, L_A is the expansion of L formed by adding a constant symbol for each $a \in A$. κ and λ denote infinite cardinals. Otherwise we follow the notation of Chang and Keisler (1973).

DEFINITION. Let $\mathfrak{A}$ be a model of T. If $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{B} \models T$, and $b \in B$, the *type* of b over $\mathfrak{A}$ is the set of all basic formulas $\varphi(x)$ in L_A such that $\mathfrak{B} \models \varphi[b]$, see Morley

* This is the first of a series of survey papers which will appear from time to time in the Journal.

https://doi.org/10.1017/S1446788700018541 Published online by Cambridge University Press

(1965). The *Stone space* of $\mathfrak{A}$, $S(\mathfrak{A}, T)$, is the set of all types over $\mathfrak{A}$. $S(\mathfrak{A}, T)$ has the topology such that for each basic formula $\varphi(x)$ of L_A, the set of all types containing $\varphi(x)$ is a basic open set.

It follows from the compactness theorem that $S(\mathfrak{A}, T)$ is a compact totally disconnected Hausdorff space.

Definition. Given two models $\mathfrak{A}, \mathfrak{B}$ of T with $\mathfrak{A} \subset \mathfrak{B}$, the set of all types of elements of $\mathfrak{B}$ over $\mathfrak{A}$ is denoted by $S(\mathfrak{A}, \mathfrak{B})$.

In general, $S(\mathfrak{A}, \mathfrak{B})$ is a subset of $S(\mathfrak{A}, T)$. This note deals with the cardinalities of the sets $S(\mathfrak{A}, \mathfrak{B})$ of types, not with the topology of the Stone space $S(\mathfrak{A}, T)$. There is a trivial lower and upper bound.

Proposition 1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are infinite models of T with $\mathfrak{A} \subset \mathfrak{B}$,*

$$|A| \leqq |S(\mathfrak{A}, \mathfrak{B})| \leqq |S(\mathfrak{A}, T)| \leqq 2^{|A|}.$$

This is because the elements of A all have different types over $\mathfrak{A}$, and each type is a set of formulas of L_A.

Remark. It may happen that two types over $\mathfrak{A}$ cannot both be realized in a single extension $\mathfrak{B} \supset \mathfrak{A}$, so there may be no extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $S(\mathfrak{A}, \mathfrak{B}) = S(\mathfrak{A}, T)$. However, we always have

$$S(\mathfrak{A}, T) = \cup \{S(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \subset \mathfrak{B}\}.$$

Definition. The *stability function* of T is the function f_T on infinite cardinals κ defined by

$$f_T(\kappa) = \sup\{|S(\mathfrak{A}, \mathfrak{B})| : \mathfrak{A}, \mathfrak{B} \text{ are models of } T, \mathfrak{A} \subset \mathfrak{B}, |A| = \kappa\}.$$

T is κ-stable if $f_T(\kappa) = \kappa$.

By Proposition 1, $\kappa \leqq f_T(\kappa) \leqq 2^\kappa$.

Shelah raised the question: *which cardinal functions are stability functions of theories?*

In this paper we announce a solution to this problem.

2. The Classification Theorem

Definition. The *Dedekind function* is defined by

$$\text{ded}(\kappa) = \sup\{\lambda : \text{There is a linear ordering of power } \kappa \text{ which has } \lambda \text{ dedekind cuts}\}.$$

It is easily seen that the theory LO of linear ordering has the stability function

$$f_{LO}(\kappa) = \text{ded}(\kappa).$$

In fact, for every extension T of LO with an infinite model,

$$f_T(\kappa) = \operatorname{ded}(\kappa)$$

because every linear ordering can be imbedded in a model of T.

PROPOSITION 2. (*Hausdorff*). $\kappa < \operatorname{ded} \kappa$, $\kappa^{\omega} \leqq \operatorname{ded} \kappa \leqq 2^{\kappa}$.

THEOREM A. *For every countable theory T with an infinite model, the stability function of T is one of the following six functions.*

$$\kappa,\ \kappa + 2^{\omega},\ \kappa^{\omega},\ \operatorname{ded} \kappa,\ (\operatorname{ded} \kappa)^{\omega},\ 2^{\kappa}.$$

Notice that these functions are increasing in the order listed.

It is convenient to have names for each of these six types of theories. Some related terminology of Shelah (1971) suggests the following.

$f_T(\kappa)$	Name
κ	T is ω-stable
$\kappa + 2^{\omega}$	T is strictly superstable
κ^{ω}	T is strictly stable
$\operatorname{ded} \kappa$	T is ordered
$(\operatorname{ded} \kappa)^{\omega}$	T is multiply ordered
2^{κ}	T is independent

Theorem A breaks up into five parts, one proved by Morley, three by Shelah, and one by the author. These five theorems give syntactical characterizations for the stability function of T. They all have a similar pattern and together imply Theorem A.

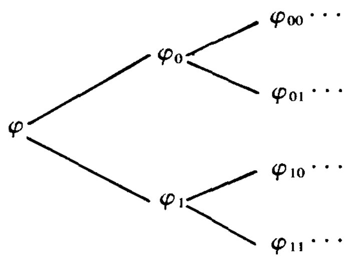
The symbol X^{∞} denotes the set of all finite sequences of elements of X. $\vec{x}$ and $\vec{a}$ denote finite sequences of variables from L and constants from a set A, respectively. By a basic formula we shall mean a basic formula in some expansion L_A of L.

THEOREM 1. (Morley (1965)). (i) $\rightarrow$ (ii) $\rightarrow$ (iii).

(i) $\kappa < f_T(\kappa)$ *for some* κ.

(ii) *There are basic formulas* $\varphi_s(x)$, $s \in 2^{\infty}$, *such that* $\varphi_{s1}(x) = \neg\varphi_{s0}(x)$ *for all s, and*

$$T \cup \{\varphi_{t \restriction n}(c_t) \colon t \in 2^{\omega}, n < \omega\} \quad \textit{is consistent.}$$



https://doi.org/10.1017/S1446788700018541 Published online by Cambridge University Press

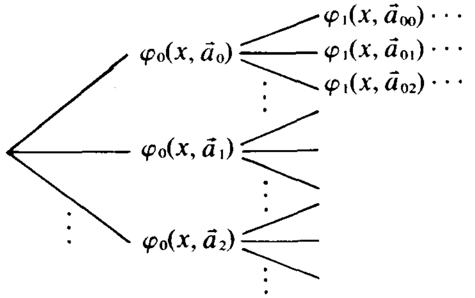
(iii) *$\kappa + 2^\omega \leqq f_T(\kappa)$ for all κ.*

THEOREM 2. (Shelah (1971)). (i) $\to$ (ii) $\to$ (iii).

(i) *For some κ, $\kappa + 2^\omega < f_T(\kappa)$.*

(ii) *There exist basic formulas $\varphi_n(x, \vec{y})$, $n < \omega$, such that T is consistent with*

$$\left\{ \varphi_n(c_s, \vec{a}_{s\upharpoonright(n+1)}) \wedge \bigwedge_{m<s(n)} \neg \varphi_n(c_s, \vec{a}_{s\upharpoonright n,m}) : n < \omega, s \in \omega^\omega \right\}.$$



(iii) *For all κ, $\kappa^\omega \leqq f_T(\kappa)$.*

THEOREM 3. (Shelah (1971)). (i) $\to$ (ii) $\to$ (iii).

(i) For some κ, $\kappa^\omega < f_T(\kappa)$.

(ii) *There is a basic formula $\varphi(x, \vec{y})$ such that T is consistent with*

$$\left\{ \begin{array}{lll} \varphi(c_m, \vec{a}_n) & \text{if} & n < m \\ \neg\varphi(c_m, \vec{a}_n) & \text{if} & m \leqq n \end{array} : m, n < \omega \right\}$$

(This suggests the name "ordered").

(iii) *For all κ*, ded $\kappa \leqq f_T(\kappa)$

The theorem below is the one new step.

THEOREM 4. (i) $\to$ (ii) $\to$ (iii).

(i) *For some κ*, ded $\kappa < f_T(\kappa)$.

(ii) *There are basic formulas $\varphi_n(x, \vec{y})$, $n < \omega$, such that T is consistent with*

$$\left\{ \begin{array}{lll} \varphi_k(c_s, \vec{a}_{kn}) & \text{if} & n < s(k) \\ \neg\varphi_k(c_s, \vec{a}_{kn}) & \text{if} & s(k) \leqq n \end{array} : n, k < \omega, s \in \omega^\omega \right\}$$

(This suggests the name "multiply ordered").

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \varphi_1(x, \vec{a}_{10}) & \varphi_1(x, \vec{a}_{11}) & \varphi_1(x, \vec{a}_{12}) \cdots \\ \varphi_0(x, \vec{a}_{00}) & \varphi_0(x, \vec{a}_{01}) & \varphi_0(x, a_{02}) \cdots \end{array}$$

https://doi.org/10.1017/S1446788700018541 Published online by Cambridge University Press

(iii) *For all* κ, $(\operatorname{ded}\kappa)^\omega \leqq f_T(\kappa)$.

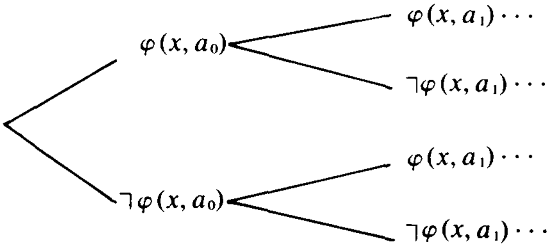
THEOREM 5. (Shelah (1971)). (i)→(ii)→(iii).

(i) *For some* κ, $(\operatorname{ded}\kappa)^\omega < f_T(\kappa)$.

(ii) *There is a basic formula* $\varphi(x, \vec{y})$ *such that T is consistent with*

$$\left\{ \begin{matrix} \varphi(c_s, \vec{a}_m) & \text{if} & m \in s \\ & & \\ \neg\varphi(c_s, \vec{a}_m) & \text{if} & m \notin s \end{matrix} \quad : m \in \omega, s \subseteq \omega \right\}$$

(This suggests the name "independent".)



(iii) *For all* κ, $f_T(\kappa) = 2^\kappa$.

In each of Theorems 1–5, the proof of (ii)→(iii) is a fairly easy application of the compactness and Löwenheim-Skolem theorems. The idea is to have the a's be elements of a model $\mathfrak{A}$ of power κ and the c's be elements with different types in an extension of $\mathfrak{A}$. The proofs of (i)→(ii) are harder combinational arguments. Shelah used a result of Erdös and Makkai (1966) in the proof of Theorem 3, and the same result is used a second time in the proof of Theorem 4.

3. Elementary Stability

In the preceding sections we considered the Stone space $S(\mathfrak{A}, T)$ built up from basic formulas. In the literature one often finds instead the elementary Stone space, which is built up from arbitrary formulas. In this section we clarify the relation between the two approaches. It turns out that Theorem A implies the analogous theorem for elementary Stone spaces.

Let $\mathfrak{B}$ be an elementary extension of $\mathfrak{A}, \mathfrak{A} < \mathfrak{B}$, and let $b \in B$. The *elementary type* of b over $\mathfrak{A}$ is the set of all formulas $\varphi(x)$ of L_A such that $\mathfrak{B} \models \varphi[b]$, Morley (1965). The *elementary Stone space* of $\mathfrak{A}, S(\mathfrak{A})$, is the set of all elementary types over $\mathfrak{A}$.

Note that the Stone space $S(\mathfrak{A}, T)$ depends on both $\mathfrak{A}$ and T, while the Stone space $S(\mathfrak{A})$ depends only on $\mathfrak{A}$. It follows from the compactness theorem that for every model $\mathfrak{A}$ there is an elementary extension $\mathfrak{B} > \mathfrak{A}$ such that every

https://doi.org/10.1017/S1446788700018541 Published online by Cambridge University Press

type in the elementary Stone space $S(\mathfrak{A})$ is realized by some element of $\mathfrak{B}$.

The *elementary stability function* of T, g_T, is defined by

$$g_T(\kappa) = \sup\{|S(\mathfrak{A})| : \mathfrak{A} \text{ is a model of } T \text{ of power } \kappa\}.$$

We now show that the elementary Stone space is a special case of the Stone space. Let L^* be the expansion of L formed by adding a new relation symbol $R_\varphi(x_1 \cdots x_n)$ for each formula $\varphi(x_1 \cdots x_n)$, let T^* be the theory with axioms $\varphi \leftrightarrow R_\varphi$, and let $\mathfrak{A}^*$ be the unique expansion of $\mathfrak{A}$ to a model of T^*. Thus $\mathfrak{A} < \mathfrak{B}$ iff $\mathfrak{A}^* \subset \mathfrak{B}^*$.

PROPOSITION 3. *For any model $\mathfrak{A}$ for L, $S(\mathfrak{A})$ is homeomorphic to $S(\mathfrak{A}^*, T^*)$. For every theory T of L, $g_T = f_{T \cup T^*}$.*

From Theorem A and Proposition 3 above, we immediately obtain the analogue of Theorem A for the elementary stability function.

THEOREM A′. *For every countable theory T with an infinite model, the elementary stability function of T is one of the following:*

$$\kappa, \kappa + 2^\omega, \kappa, \text{ded } \kappa, (\text{ded } \kappa)^\omega, 2^\kappa.$$

For specific examples it is usually easier to compute the stability function than the elementary stability function of a theory. The following proposition gives a case where the two stability functions are the same.

We call T a *submodel completion* of T_0 if every model of T_0 can be extended to a model of T and vice versa, and every formula is equivalent to a quantifier-free formula with respect to T.

PROPOSITION 4. *If T is a submodel completion of T_0 then the elementary stability function of T is equal to the stability functions of both T and T_0,*

$$g_T(\kappa) = f_T(\kappa) = f_{T_0}(\kappa).$$

4. Examples

4.1. *Examples of ω-stable theories, $f_T(\kappa) = \kappa$.*
Integral domains
Fields
Differential fields of characteristic zero. (Blum (1968)).
Equivalence relations
Abelian groups
A function of one variable
Vector spaces over fields
Projective geometries

The following examples are submodel completions of theories from the above list, and thus are elementarily ω-stable, $g_T(\kappa) = f_T(\kappa) = \kappa$.

Algebraically closed fields.
Differentially closed fields of characteristic zero.
Doubly infinite equivalence relations.

4.2. *Examples of strictly superstable theories*, $f_T(\kappa) = \kappa + 2^\omega$.
Countably many unary relations.
One unary relation and one unary function.

The theory of countably many independent unary relations is a submodel completion and is elementarily strictly superstable, $g_T(\kappa) = f_T(\kappa) = \kappa + 2^\omega$.

4.3. *Examples of strictly stable theories*, $f_T(\kappa) = \kappa^\omega$.
Countably many equivalence relations.
One equivalence relation and one unary function.
Two unary functions.
Countable decreasing chains of abelian groups.
Theory of R-modules where R is a fixed countable ring (for most R).
Differential fields of characteristic p, with a symbol for $\sqrt[p]{\;}$.
(Shelah (1973)).

The theory of countably many independent equivalence relations is a submodel completion and is elementarily strictly stable, $g_T(\kappa) = f_T(\kappa) = \kappa^\omega$.

4.4. *Examples of ordered theories*, $f_T(\kappa) = \operatorname{ded} \kappa$.
Linear order
Ordered fields
Ordered abelian groups
Fields with valuation

The following examples are submodel completions which are elementarily ordered, $g_T(\kappa) = f_T(\kappa) = \operatorname{ded} \kappa$.

Dense linear order without endpoints.
Real closed ordered fields.
Divisible ordered abelian groups.

4.5. *Examples of multiply ordered theories*, $f_T(\kappa) = (\operatorname{ded} \kappa)^\omega$.
Countably many linear orderings.
Ordered R-modules (for most fixed countable ordered rings R).
One linear ordering and one unary function.

The following theory is a submodel completion and is elementarily multiply ordered,

$$g_T(\kappa) = f_T(\kappa) = (\operatorname{ded} \kappa)^\omega.$$

Countably many independent dense linear orderings without endpoints.

I.e. countably many dense linear orderings without endpoints such that if X_m is an interval in the *mth* ordering, $m = 0, 1, \cdots, n$, then $X_0 \cap \cdots \cap X_n \neq 0$.

4.6. *Examples of independent theories,* $f_T(\kappa) = 2^\kappa$.
Boolean algebras
Groups
Inner product spaces
Integral domains with a symbol for $a \mid b$
Division rings
Algebraically closed fields with a unary relation which is a subfield.

The theory of atomless boolean algebras is a submodel completion which is elementarily independent, $g_T(\kappa) = f_T(\kappa) = 2^\kappa$.

5. Open problems

PROBLEM 1. *Are there any extensions of group theory (in the same language) which are strictly superstable, strictly stable, ordered, or multiply ordered?*

One can ask a similar question for division rings, etc.

The next problem is whether the six functions in Theorem A can all be different. The functions

$$\kappa, \kappa + 2^\omega, \kappa^\omega, \text{ded } \kappa$$

are, of course, all different. However, if the GCH holds then

$$\text{ded } \kappa, (\text{ded } \kappa)^\omega, 2^\kappa$$

are all the same. Thus under the GCH there are only four stability functions. This already followed from the results of Shelah (1971)

It was shown by Mitchell (1972) that if ZFC is consistent, so is

$$\text{ZFC} + \text{ded } \omega_1 = \aleph_{\omega_1} + 2^{\omega_1} = \aleph_{\omega_1+1}.$$

In this model of ZFC we have

$$\text{ded } \omega_1 = (\text{ded } \omega_1)^\omega < 2^{\omega_1},$$

and there are at least five stability functions.

The open problem is,

PROBLEM 2. *Is* $\text{ZFC} + \exists\kappa\, (\text{ded } \kappa < (\text{ded } \kappa)^\omega)$ *consistent?*

Kunen remarked that if $\kappa = \kappa^\omega$ then $\text{ded } \kappa = (\text{ded } \kappa)^\omega$. (The proof uses a countable ultrapower of $(\text{ded } \kappa, \kappa)$). Thus $\text{ded } \kappa$ can differ from $(\text{ded } \kappa)^\omega$ only when $\kappa < \kappa^\omega < \text{ded } \kappa < 2^\kappa$.

Theorems 1–5 always give syntactical characterizations of the stability function of T, but the characterization will depend on which of the six functions are different in a given model of set theory.

Theorem A does not completely solve the problem as stated by Shelah (1971). For one thing, he asked for stability functions for uncountable as well as countable theories.

PROBLEM 3. (Shelah). *Which stability functions are possible for theories T of power λ?*

Partial results are known but the general problem seems hard.

We say that the stability function $f_T(\kappa)$ is *attained* if there is a pair of models $\mathfrak{A} \subset \mathfrak{B}$ of T with $|A| = \kappa$ and $|S(\mathfrak{A}, \mathfrak{B})| = f_T(\kappa)$, i.e., the supremum is a maximum. Shelah actually considered the slightly different stability function

$$k_T(\kappa) = \begin{cases} f_T(\kappa)^+ & \text{if } f_T(\kappa) \text{ is attained} \\ f_T(\kappa) & \text{otherwise} \end{cases}$$

PROBLEM 4. (Shelah). *Is $f_T(\kappa)$ always attained? If not, which stability functions $k_T(\kappa)$ are possible?*

The functions $\kappa, \kappa + 2^\omega, \kappa^\omega$, and 2^κ are always attained. If the GCH holds, the answer to problem 4 is trivially yes. Using the compactness theorem we see that if $cf(\text{ded})\,\kappa)^\omega) \leqq \kappa$ then $(\text{ded}\,\kappa)^\omega$ is attained. Because we may put together κ orders of power κ. So problem 4 is interesting only for ded κ and $(\text{ded}\,\kappa)^\omega$, and only when their cofinalities are greater than κ.

We conclude with two general problems.

PROBLEM 5. *What classification of theories do we get by considering the set of values $F_T(\kappa) = \{|S(\mathfrak{A}, T)| : \mathfrak{A}$ is a model of T of power $\kappa\}$ instead of the supremum $f_T(\kappa)$?*

PROBLEM 6. *Develop a sharper stability theory by making use of topological properties of Stone spaces instead of only their cardinalities.*

Added in proof: J. Baldwin and J. Saxl (1976) have recently announced a partial solution of Problem 1. Namely, every extension of group theory closed under direct products is either ω-stable or independent. G. Sabbagh has given an example of a group whose theory T has the elementary stability function $g_T(\kappa) = \text{ded}\,\kappa$: the group of invertible n by n matices $(n > 1)$ over an ordered field.

References

L. Blum (1968), 'Generalized algebraic theories: a model-theoretic approach.' *Ph.D. Thesis, M.I.T.*

C. C. Chang and H. J. Keisler (1973), 'Model Theory'. *North Holland.*

P. Erdös and M. Makkai (1966), 'Some remarks on set theory, X.' *Stud. Sci. Math. Hung*, **1**, 157–159.

https://doi.org/10.1017/S1446788700018541 Published online by Cambridge University Press

H. J. Keisler, (1974), 'The number of types in a first order theory', *Notices Amer. Math. Soc.*, **21**, A-316.

W. Mitchell (1972), 'Aronsajn trees and the independence of the transfer property', *Ann. Math. Logic* **5**, 21–46.

M. Morely (1965), 'Categoricity in power', *Trans. Amer. Math. Soc.* **144,** 514–538.

G. Sacks (1972), 'Saturated Model Theory', *Benjamin.*

S. Shelah (1971), 'Stability, the finite cover property, and superstability', *Annls. of Math. Logic* **3**, 271–362.

S. Shelah (1973), 'Differentially closed fields', *Notices Amer. Math. Soc.* **20**, A–444.

J. Baldwin and J. Saxl (1976), 'Logical Stability in Group Theory', *J. Austral. Math. Soc.* **21** (Series A), 139–149.

Mathematics Department
University of Wisconsin
Van Vleck Hall
480 Lincoln Drive
Madison, Wisconsin 53706
U.S.A.