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Refined Motivic Dimension

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Abstract. We define a refined motivic dimension for an algebraic variety by modifying the definition of motivic dimension by Arapura. We apply this to check and recheck the generalized Hodge conjecture for certain varieties, such as uniruled, rationally connected varieties and a rational surface fibration.

1 Introduction

Given a complex smooth projective variety X, singular cohomology with rational coefficients carries two natural filtrations, the *coniveau* filtration $N^\bullet$ and the *level* filtration $\mathcal{F}^\bullet$. Roughly speaking, $N^pH^i(X,\mathbb{Q})$ is the space of cycles supported on subvarieties of codimension at least p, and $\mathcal{F}^pH^i(X,\mathbb{Q})$ is the largest sub-Hodge structure of $H^i(X,\mathbb{Q})$ contained in $F^pH^i(X,\mathbb{C})\cap H^i(X,\mathbb{Q})$. They generalize the space of algebraic cycles and that of Hodge cycles; $N^pH^{2p}(X,\mathbb{Q}) = H^{2p}(X,\mathbb{Q})_{\mathrm{alg}}$ and $\mathcal{F}^pH^{2p}(X,\mathbb{Q}) = H^{2p}(X,\mathbb{Q})_{\mathrm{hodge}}$. We say that the generalized Hodge conjecture $\mathrm{GHC}(H^i(X,\mathbb{Q}),p)$ holds if $\mathcal{F}^pH^i(X,\mathbb{Q}) = N^pH^i(X,\mathbb{Q})$ [G]. If the two spaces coincide for all i and p, we simply say that the generalized Hodge conjecture (GHC) holds for X.

Arapura [A] introduced the notion of *motivic dimension* $\mu(X)$ for an algebraic variety X that can be a tool in checking the (G)HC of certain varieties. For a smooth projective variety X, this is the length of the coniveau filtration on $H^*(X,\mathbb{Q})$. In particular, $\mu(X) = 0$ for X when all the cohomology of X are generated by algebraic cycles. He observed that the Hodge conjecture holds for X if $\mu(X)\le 3$ and the GHC holds for X if $\mu(X)\le 2$ [A, corollary 4.10]. However, we note that the definition of motivic dimension is too strong to be used to check $\mathrm{GHC}(H^i(X,\mathbb{Q}),p)$ for specific i and p. For example, for a uniruled variety X, the motivic dimension provides an easy proof of the Hodge conjecture for X up to dimension four, but no information on $\mathrm{GHC}(H^{\dim X}(X,\mathbb{Q}),1)$ for X of dimension higher than four. The purpose of this note is to modify the definition of motivic dimension so that it can be used for checking a partial generalized Hodge conjecture. By applying Arapura's idea to a specific level $\mathcal{F}^mH^*(X,\mathbb{Q})$, we obtain a notion of the *m-th refined motivic dimension* $\mu_m(X)$ for a smooth projective complex variety X, and we show that $\mathrm{GHC}(H^i(X,\mathbb{Q}),m)$ holds if and only if $\mu_m(X)\le i-2m+1$. We apply this notion to recheck the (generalized) Hodge conjecture for a number of well-known examples. The last section contains a proof of $\mathrm{GHC}(H^{\dim X}(X,\mathbb{Q}),2)$ for a rational surface fibration $X\to Y$

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under assumption on the conjecture for the base. The decomposition of the diagonal class by Bloch and Srinivas [BS] is the key idea for this section. All varieties in this paper are defined over $\mathbb{C}$.

2 Refined Motivic Dimension

As we noted earlier, the p-th level filtration $\mathcal{F}^p H^i(X,\mathbb{Q})$ is defined to be the largest sub-Hodge structure of $H^i(X,\mathbb{Q})$ contained in $F^p H^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q})$, where $F^\bullet$ is the Hodge filtration on $H^i(X,\mathbb{C})$. Alternatively, $\mathcal{F}^p H^i(X,\mathbb{Q})$ is exactly the largest rational sub-Hodge structure of $H^i(X,\mathbb{Q})$ of level at most $i-2p$. Here, the *level* of a pure Hodge structure $H = \oplus H^{pq}$ is $\max\{|p-q| \mid \dim H^{pq} \neq 0\}$. The p-th coniveau filtration $N^p H^i(X,\mathbb{Q})$ is

$$\begin{aligned} N^p H^i(X,\mathbb{Q}) &= \sum_{\operatorname{codim}(S,X)\geq p} \ker\big[H^i(X,\mathbb{Q}) \to H^i(X-S,\mathbb{Q})\big] \\ &\cong \sum_{\operatorname{codim}(S,X)=q\geq p} \operatorname{im}\big[H^{i-2q}(\widetilde{S},\mathbb{Q}(-q)) \to H^i(X,\mathbb{Q})\big], \end{aligned}$$

where the sum is taken over all subvariety S of X of $\operatorname{codim}(S,X) \geq p$ and $\widetilde{S} \to S$ is a desingularization. The second description of the coniveau filtration, due to Deligne [D1], implies that $N^p H^i(X,\mathbb{Q}) \subset \mathcal{F}^p H^i(X,\mathbb{Q})$ since the Gysin map is a morphism of Hodge structures. The generalized Hodge conjecture states the two filtrations coincide [G, Le]:

$$\operatorname{GHC}\big(H^i(X,\mathbb{Q}), p\big) : N^p H^i(X,\mathbb{Q}) = \mathcal{F}^p H^i(X,\mathbb{Q}).$$

For a *fixed* integer m, we consider a non-negative integer $\mu_m(X)$, which is the smallest integer n, such that any $\alpha \in \mathcal{F}^m H^i(X,\mathbb{Q})$ vanishes on a Zariski closed set all of whose components have codimension at least $(i-n)/2$. When $m=0$, it is exactly Arapura's definition ([A]) of the motivic dimension $\mu(X)$.

Lemma 2.1 *Let X be a smooth projective variety of dimension d. For each $m \geq 0$,*

(i) $\mu_m(X) \geq \mu_{m+1}(X)$,

(ii) $\mu_m(X) \geq \ell_m \overset{\text{set}}{=} \operatorname{level}(\mathcal{F}^m H^*(X,\mathbb{Q})) \overset{\text{def}}{=} \max\{|p-q| \mid h^{pq} \neq 0,\ p \geq m\}$, *where the equality holds if* $\operatorname{GHC}(H^i(X,\mathbb{Q}), m)$ *holds for all* $i \geq 2m$.

Proof (i) follows immediately, because $\mathcal{F}^\bullet$ is a descending filtration. For (ii), let $\alpha \in \mathcal{F}^m H^i(X,\mathbb{Q})$ and let S be a Zariski closed set of $\operatorname{codim}(S,X) = p \geq (i-\mu_m(X))/2$ such that

$$\alpha \in \ker\big[H^i(X,\mathbb{Q}) \to H^i(X-S,\mathbb{Q})\big] \cong \operatorname{im}\big[H^{i-2p}\big(\widetilde{S},\mathbb{Q}(-p)\big) \to H^i(X,\mathbb{Q})\big]$$

where $\sigma\colon \widetilde{S} \to S$ is a desingularization. Since $i-2p \leq \mu_m(X)$, we have

$$\operatorname{level}\big(\mathcal{F}^m H^i(X,\mathbb{Q})\big) \leq \mu_m(X)$$

for each i, and we get the inequality in (ii). Now, suppose that $\operatorname{GHC}(H^i(X,\mathbb{Q}), m)$ holds for all $i \geq 2m$. Then for any $\alpha \in \mathcal{F}^m H^i(X,\mathbb{Q}) = N^m H^i(X,\mathbb{Q})$, there exists a

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Zariski closed set S of $\operatorname{codim}(S,X) = p \ge m$ such that

$$\alpha \in \ker[H^i(X,\mathbb{Q}) \to H^i(X-S,\mathbb{Q})].$$

This implies that $\ell_m \ge \operatorname{level}(\mathcal{F}^m H^i(X,\mathbb{Q})) \ge i - 2p$; equivalently, $p \ge (i-\ell_m)/2$. The definition of $\mu_m(X)$ implies $\ell_m \ge \mu_m(X)$. ∎

Let X be an algebraic variety of dimension d and let $H_i(X,\mathbb{Q})$ be the Borel–Moore homology. The isomorphism $H_i(X,\mathbb{Q}) \cong H_c^i(X,\mathbb{Q})^*$ endows a mixed Hodge structure on $H_i(X,\mathbb{Q})$ whose weights are concentrated on $[-i,0]$ ([D1]), and $W_{-i}H_i(X,\mathbb{Q})$ is a pure Hodge structure of weight $-i$. For a non-negative integer m, we denote

$$\mathcal{F}_m W_{-i}H_i(X,\mathbb{Q}) \stackrel{\text{def}}{=} \mathcal{F}^{-m} W_{-i}H_i(X,\mathbb{Q}),$$

i.e., the largest sub-Hodge structure of $W_{-i}H_i(X,\mathbb{Q})$ of level at most $|-i-2(-m)|$. Similarly in [A], we may consider $\bar{\mu}_m(X)$, the smallest non-negative integer n such that any $\alpha \in \mathcal{F}_m W_{-i}H_i(X,\mathbb{Q})$ lies in $\operatorname{im}[W_{-i}H_i(Y,\mathbb{Q}) \to W_{-i}H_i(X,\mathbb{Q})]$, where Y is a Zariski closed set and any irreducible component of Y has dimension at most $(i+n)/2$. In particular, if X is smooth and projective, we have a relation

$$\bar{\mu}_{d-m}(X) = \mu_m(X) \tag{2.1}$$

by dualities $\mathcal{F}^m H^i(X,\mathbb{Q}) \cong (\mathcal{F}^{m-d}H_{2d-i}(X,\mathbb{Q}))(d) = (\mathcal{F}_{d-m}H_{2d-i}(X,\mathbb{Q}))(d)$.

We have that $\bar{\mu}_m$ satisfies properties in [A, proposition 1.1] as well.

Proposition 2.2

(i) *If $f: X' \to X$ is proper and surjective, then $\bar{\mu}_m(X) \le \bar{\mu}_m(X')$.*
(ii) *If $Z \subset X$ is Zariski closed, then $\bar{\mu}_m(X) \le \max\{\bar{\mu}_m(Z), \bar{\mu}_m(X-Z)\}$.*
(iii) *If $\widetilde{X}$ is a desingularization of a partial compactification $\bar{X}$ of X, then $\bar{\mu}_m(X) \le \bar{\mu}_m(\bar{X}) \le \bar{\mu}_m(\widetilde{X})$.*
(iv) $\bar{\mu}_m(X_1 \times X_2) \le \max\{\bar{\mu}_t(X_1) + \bar{\mu}_s(X_2), t+s=m\}$.

Proof Statements (i) through (iii) can be proved by simple modification of arguments in [A, Proposition 1.1]. We present a proof of (ii) as an example. Let $d = \dim X$ and Z be a Zariski closed subset of X. Since the level filtration $\mathcal{F}^\bullet$ is an exact functor from the category of polarized Hodge structures with a fixed weight to itself, an exact sequence of pure Hodge structure of weight $-i$

$$\cdots \to W_{-i}H_i(Z,\mathbb{Q}) \to W_{-i}H_i(X,\mathbb{Q}) \to W_{-i}H_i(X-Z,\mathbb{Q}) \to 0,$$

gives rise to an exact sequence

$$\cdots \to \mathcal{F}_m W_{-i}H_i(Z,\mathbb{Q}) \to \mathcal{F}_m W_{-i}H_i(X,\mathbb{Q}) \xrightarrow{\phi} \mathcal{F}_m W_{-i}H_i(X-Z,\mathbb{Q}) \to 0.$$

Let $\alpha \in \mathcal{F}_m W_{-i}H_i(X,\mathbb{Q})$. Then there exists a Zariski closed set Y in $X-Z$ of dimension at most $(i+\bar{\mu}_m(X-Z))/2$ such that

$$\phi(\alpha) = g_*(\beta) \in \operatorname{im}\big[g_*: W_{-i}H_i(Y,\mathbb{Q}) \to W_{-i}H_i(X-Z,\mathbb{Q})\big],$$

where $g: Y \hookrightarrow X-Z$. Let $\bar{Y}$ be the Zariski closure of Y in X and $\bar{g}: \bar{Y} \to X$. A surjection $W_{-i}H_i(\bar{Y},\mathbb{Q}) \to W_{-i}H_i(Y,\mathbb{Q})$ extends β to a cycle $\bar{\beta} \in W_{-i}H_i(\bar{Y},\mathbb{Q})$. Since

$\phi(\bar{g}_*(\bar{\beta})) = g_*(\beta) = \phi(\alpha)$, we have

$$\alpha - \bar{g}_*(\bar{\beta}) \in \ker(\phi) = \operatorname{im}\big[\mathcal{F}_m W_{-i} H_i(Z,\mathbb{Q}) \to \mathcal{F}_m W_{-i} H_i(X,\mathbb{Q})\big],$$

and hence $\alpha - \bar{g}_*(\bar{\beta}) \in \operatorname{im}[W_{-i}H_i(Y_1,\mathbb{Q}) \to W_{-i}H_i(Z,\mathbb{Q}) \to W_{-i}H_i(X,\mathbb{Q})]$ where Y_1 is a closed set of $\dim Y_1 \le (i + \bar{\mu}_m(Z))/2$. By combining all, this implies that $\alpha \in \operatorname{im}[W_{-i}H_i(T,\mathbb{Q}) \to W_{-i}H_i(X,\mathbb{Q})]$ for some reducible Zariski closed set T of dimension at most $(i + \max\{\bar{\mu}_m(X-Z),\ \bar{\mu}_m(Z)\})/2$. This finishes a proof of (ii).

Now we prove (iv). By applying weight and level filtration to the Künneth formula, we have

$$\begin{aligned}\mathcal{F}_m W_{-i} H_i(X_1 \times X_2,\mathbb{Q}) &= \bigoplus_{j+\ell=i} \mathcal{F}_m\big(W_{-j}H_j(X_1,\mathbb{Q}) \otimes W_{-\ell}H_\ell(X_2,\mathbb{Q})\big)\\ &= \bigoplus_{j+\ell=i}\ \bigoplus_{t+s=m} \big(\mathcal{F}_t W_{-j}H_j(X_1,\mathbb{Q}) \otimes \mathcal{F}_s W_{-\ell}H_\ell(X_2,\mathbb{Q})\big).\end{aligned}$$

Hence, any $\alpha \in \mathcal{F}_m W_{-i}H_i(X_1 \times X_2,\mathbb{Q})$ has a decomposition

$$\alpha = \sum_{j,\ell,t,s} \alpha_{jt} \otimes \alpha_{\ell s}, \quad \text{where } \alpha_{jt} \otimes \alpha_{\ell s} \in \mathcal{F}_t W_{-j}H_j(X_1,\mathbb{Q}) \otimes \mathcal{F}_s W_{-\ell}H_\ell(X_2,\mathbb{Q}).$$

For each j, ℓ, t, and s satisfying $j + \ell = i$ and $t + s = m$, let $Y_{jt} \times Y_{\ell s}$ be a Zariski closed subset of $X_1 \times X_2$ such that

$$\begin{aligned}\alpha_{jt} &\in \operatorname{im}\big[W_{-j}H_j(Y_{jt},\mathbb{Q}) \to W_{-j}H_j(X_1,\mathbb{Q})\big],\\ \alpha_{\ell s} &\in \operatorname{im}\big[W_{-\ell}H_\ell(Y_{\ell s},\mathbb{Q}) \to W_{-\ell}H_\ell(X_2,\mathbb{Q})\big]\end{aligned}$$

and $\dim Y_{jt} \le (j + \bar{\mu}_t(X_1))/2$ and $\dim Y_{\ell s} \le (\ell + \bar{\mu}_s(X_2))/2$. Since

$$\dim(Y_{jt} \times Y_{\ell s}) \le \frac{j + \ell + \bar{\mu}_t(X_1) + \bar{\mu}_s(X_2)}{2} = \frac{i + \bar{\mu}_t(X_1) + \bar{\mu}_s(X_2)}{2},$$

we have $\bar{\mu}_m(X_1 \times X_2) \le \bar{\mu}_t(X_1) + \bar{\mu}_s(X_2)$ for $t + s = m$, and we are done. ∎

Proposition 2.3 *Let $\sigma\colon Y = \mathrm{Bl}_Z X \to X$ be the blow-up of a smooth projective variety X along a smooth center Z. Then*

$$\bar{\mu}_m(Y) \le \max\big\{\bar{\mu}_m(X),\ \bar{\mu}_m(Z)\big\},$$

or equivalently, if $c = \operatorname{codim}(Z,X)$,

$$\mu_m(Y) \le \max\big\{\mu_m(X),\ \mu_{m-c}(Z)\big\}. \tag{2.2}$$

Proof Let $\alpha \in \mathcal{F}^m H^i(Y,\mathbb{Q})$. Since

$$\mathcal{F}^m H^i(Y,\mathbb{Q}) = \sigma^*\big(\mathcal{F}^m H^i(X,\mathbb{Q})\big) + \iota_*\big(\mathcal{F}^{m-1}H^{i-2}(E,\mathbb{Q})(-1)\big),$$

where E is the exceptional divisor and $\iota\colon E \hookrightarrow Y$, there is a decomposition

$$\alpha = \sigma^*(\beta) + \iota_*(\gamma), \quad \text{where } \beta \in \mathcal{F}^m H^i(X,\mathbb{Q}),\ \gamma \in \mathcal{F}^{m-1}H^{i-2}(E,\mathbb{Q}).$$

Let S and T be Zariski closed sets such that $\beta \in \ker[H^i(X,\mathbb{Q}) \to H^i(X-S,\mathbb{Q})]$ and $\gamma \in \ker[H^{i-2}(E,\mathbb{Q}) \to H^{i-2}(E-T,\mathbb{Q})]$ and any irreducible component S' (resp. T') of S (resp. T) has codimension at least $(i - \mu_m(X))/2$ (resp. $(i - 2 - \mu_{m-1}(E))/2$) in

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X (resp. in E). Since E is a divisor of Y, codimension of an irreducible components of T in Y is at least $(i-2-\mu_{m-1}(E))/2+1=(i-\mu_{m-1}(E))/2$. This implies

$$\mu_m(Y)\le\max\big(\mu_m(X),\mu_{m-1}(E)\big). \tag{2.3}$$

In order to estimate $\mu_{m-1}(E)$, recall ([Le, proposition 8.23])

$$H^{i-2}(E,\mathbb{Q})=\bigoplus_{s=0}^{c-1}h^s\cup\sigma^*H^{i-2-2s}(Z,\mathbb{Q}),$$

where $h=c_1(\mathcal{O}_E(1)))$ is an algebraic cycle in $H^2(E,\mathbb{Q})$. By taking $\mathcal{F}^{m-1}$, we get

$$\mathcal{F}^{m-1}H^{i-2}(E,\mathbb{Q})=\bigoplus_{s=0}^{c-1}h^s\cup\sigma^*\big(\mathcal{F}^{m-1-s}H^{i-2-2s}(Z,\mathbb{Q})\big).$$

Hence, $\alpha\in\mathcal{F}^{m-1}H^{i-2}(E,\mathbb{Q})$ has a decomposition

$$\alpha=\sum_{s=0}^{c-1}\big(h^s\cup\sigma^*(\alpha_s)\big),\quad \text{where } \alpha_s\in\mathcal{F}^{m-1-s}H^{i-2-2s}(Z,\mathbb{Q}).$$

Let

$$\alpha_s\in\ker\big[H^{i-2-2s}(Z,\mathbb{Q})\to H^{i-2-2s}(Z-B,\mathbb{Q})\big]$$

for a Zariski closed subset B of Z such that any irreducible component B' of B has $\operatorname{codim}(B',Z)$ at least $(i-2-2s-\mu_{m-1-s}(Z))/2$. Since $h^s\cup\sigma^*(\alpha_s)$ is supported on a closed set $P=\sigma^{-1}(B)\cap H^s$ where H^s is a s-th iterated hyperplane section of E and $\sigma\colon E\to Z$ is a $\mathbb{P}^{c-1}$-bundle, dimension counting argument implies that any irreducible component of P has codimension equal to $\operatorname{codim}(B',Z)+s\ge(i-2-\mu_{m-1-s}(Z))/2$ in E. This observation and Lemma 2.1 implies

$$\mu_{m-1}(E)\le\max\{\mu_{m-1-s}(Z)\mid s=0,1,\dots,c-1\}=\mu_{m-c}(Z) \tag{2.4}$$

Now (2.3) and (2.4) imply the second estimate (2.2). Moreover, since X,Y,Z are all smooth and projective, relation (2.1) gives rise to an equivalent inequality

$$\bar\mu_{d-m}(Y)\le\max\big\{\bar\mu_{d-m}(X),\ \bar\mu_{(d-c)-(m-c)}(Z)\big\}=\max\{\bar\mu_{d-m}(X),\ \bar\mu_{d-m}(Z)\},$$

where $d=\dim X=\dim Y$. ∎

Corollary 2.4 *Given birationally equivalent smooth projective varieties X and Y, we have*

$$\bar\mu_m(Y)\le\max\big\{\bar\mu_m(X),\dim X-2\big\}\qquad\textit{for any } m.$$

Proof Let $f\colon X\to Y$ be a birational morphism. Then by resolution of singularities, there exist a smooth projective variety Z, a generically finite morphism $g\colon Z\to Y$ and $h\colon Z\to X$ a composition of finitely many blow ups along smooth centers of codimension at least 2. We may assume that $h\colon Z\to X$ is a single blow up along a smooth center B. Now Lemma 2.1 and Propositions 2.2 and 2.3 imply

$$\bar\mu_m(Y)\le\bar\mu_m(Z)\le\max\big\{\bar\mu_m(X),\bar\mu_m(B)\big\}\le\max\big\{\bar\mu_m(X),\mu(B)\big\}.$$

Now the claimed inequality follows from [A, corollary 0.2]. ∎

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3 Application to the Generalized Hodge Conjecture

In this section, after establishing useful lemmas, we use μ_m to recheck the generalized Hodge conjecture for well-known varieties such as uniruled and rationally connected varieties.

Lemma 3.1 $\mathrm{GHC}(H^i(X,\mathbb{Q}),m)$ *holds if and only if* $\mu_m(X) \le i - 2m + 1$.

Proof Suppose $\mu_m(X) \le i - 2m + 1$ and let $\alpha \in \mathcal{F}^m H^i(X,\mathbb{Q})$. Then α lies in $\ker[H^i(X,\mathbb{Q}) \to H^i(X - Z,\mathbb{Q})]$ for some Zariski closed set Z such that any component Z' of Z has $\mathrm{codim}(Z',X) \ge (i - \mu_m(X))/2 \ge m - 1/2$. Hence, $\alpha \in N^m H^i(X,\mathbb{Q})$ and $\mathrm{GHC}(H^i(X,\mathbb{Q}),m)$ holds. The converse holds, because $\mathrm{GHC}(H^i(X,\mathbb{Q}),m)$ implies that any irreducible $\alpha \in \mathcal{F}^m H^i(X,\mathbb{Q}) = N^m H^i(X,\mathbb{Q})$ lies in the Gysin image $[H^{i-2p}(Z,\mathbb{Q}(-p)) \to H^i(X,\mathbb{Q})]$ for a smooth Zariski closed set Z of $\mathrm{codim}(Z,X) = p \ge m = (i - (i-2m))/2$. Hence, $\mu_m(X) \le i - 2m < i - 2m + 1$. ∎

Lemma 3.2 ([Le, 13.6 Lemma]) *Let $f\colon X \to Y$ be a surjective holomorphic map of smooth projective varieties of the same dimension d. Then for any i and m,* $\mathrm{GHC}(H^i(Y,\mathbb{Q}),m)$ *holds if* $\mathrm{GHC}(H^i(X,\mathbb{Q}),m)$ *holds.*

Proof It follows from Proposition 2.2 and Lemma 3.1, since

$$\mu_m(Y) = \bar{\mu}_{d-m}(Y) \le \bar{\mu}_{d-m}(X) = \mu_m(X) \le i - 2m + 1. \qquad \blacksquare$$

Lemma 3.3 *Let $Y = \mathrm{Bl}_Z X$ be a blow up of a smooth projective variety X along a smooth subvariety Z of codimension c at least 2. Then for any i,* $\mathrm{GHC}(H^i(X,\mathbb{Q}),1)$ *holds if and only if* $\mathrm{GHC}(H^i(Y,\mathbb{Q}),1)$ *holds.*

Proof Suppose $\mathrm{GHC}(H^i(X,\mathbb{Q}),1)$ holds. Then Lemma 3.1 and (a proof of) Proposition 2.3 imply that any $\alpha \in \mathcal{F}^1 H^i(Y,\mathbb{Q})$ lies in $\ker[H^i(Y,\mathbb{Q}) \to H^i(Y - T,\mathbb{Q})]$ for a Zariski closed set T of codimension at least $(i - \max\{\mu_1(X),\ i-2\})/2$ in Y. Hence, $\alpha \in N^1 H^i(Y,\mathbb{Q})$ and $\mathrm{GHC}(H^i(Y,\mathbb{Q}),1)$ holds. The converse follows from Lemma 3.2. ∎

Corollary 3.4 *Let X and Y be birationally equivalent smooth projective varieties. Then for any i,* $\mathrm{GHC}(H^i(X,\mathbb{Q}),1)$ *holds if and only if* $\mathrm{GHC}(H^i(Y,\mathbb{Q}),1)$ *holds.*

Proof It is an immediate consequence of Lemmas 3.2 and 3.3. ∎

Lemma 3.5 ([S]) $\mathrm{GHC}(H^{2p}(X,\mathbb{Q}),p-1)$ *implies* $\mathrm{GHC}(H^{2p}(X,\mathbb{Q}),p)$.

Proof Suppose $\mathrm{GHC}(H^{2p}(X,\mathbb{Q}),p-1)$ holds and let $\alpha \in \mathcal{F}^p H^{2p}(X,\mathbb{Q})$ be a Hodge cycle. Since $\mathcal{F}^\bullet$ is a descending filtration, $\alpha \in \mathcal{F}^{p-1}H^{2p}(X,\mathbb{Q}) = N^{p-1}H^{2p}(X,\mathbb{Q})$. Hence there exists a Zariski closed set S of $\mathrm{codim}(S,X) = q \ge p-1$ such that $\alpha = g_*(\beta) \in \mathrm{im}[g_*\colon H^{2p-2q}(\widetilde{S},\mathbb{Q}(-q)) \to H^{2p}(X,\mathbb{Q})]$, where $\widetilde{S} \to S$ is a desingularization of S. If $q = p$, then we are done. If $q = p-1$, then $\beta \in H^2(\widetilde{S},\mathbb{Q}(-p+1))$ is a Hodge cycle, since the Hodge filtration is strictly preserved by a morphism of Hodge structure. Now the Lefschetz (1,1)-theorem implies the lemma. ∎

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Proposition 3.6 ([S]) *Let X be a uniruled d-fold. Then* $\mathrm{GHC}(H^d(X,\mathbb{Q}),1)$ *holds.*

Proof Since X is uniruled, there is a dominant rational map $f:\mathbb{P}^1\times Y \dashrightarrow X$, where Y is a smooth projective variety of dimension $d-1$. Then we have a smooth projective variety Z and morphisms $g: Z\to\mathbb{P}^1\times Y$ and $h: Z\to X$, where g is a composition of finitely many blow-ups and h is a generically finite morphism. By Proposition 2.2, Lemma 2.1 and [A, corollary 0.2],

$$\begin{aligned}\mu_1(Z) = \bar{\mu}_{d-1}(Z) &\le \max\big\{\bar{\mu}_{d-1}(\mathbb{P}^1\times Y),\ d-2\big\}\\ &\le \max\{d-1,d-2\} = d-1,\end{aligned}$$

since $\bar{\mu}_{d-1}(Y\times\mathbb{P}^1)\le\max\{\mu_0(\mathbb{P}^1)+\mu_1(Y),\ \mu_1(\mathbb{P}^1)+\mu_0(Y)\} = \mu_0(Y)\le\dim Y$. By Lemma 3.1 $\mathrm{GHC}(H^d(Z,\mathbb{Q}),1)$ holds; so does $\mathrm{GHC}(H^d(X,\mathbb{Q}),1)$ by Lemma 3.2. ∎

Corollary 3.7 ([CM]) *Hodge conjecture holds for uniruled 4-fold.*

Proof It follows from Lemma 3.5 and Proposition 3.6. ∎

Now we turn our attention to the generalized Hodge conjecture for a rationally connected variety. A key idea is using the decomposition of the diagonal class [B, BS, E]. We consider the following version [La, P] taken from [V].

Theorem 3.8 *Let X be a smooth complex projective variety of dimension d. Assume that for $k\le k_0$, the maps*

$$\mathrm{cl}:\mathrm{CH}_k(X)\otimes\mathbb{Q}\to H^{2d-2k}(X,\mathbb{Q})$$

are injective. Then there exist a decomposition

$$m\cdot\Delta_X = Z_0+\cdots+Z_{k_0}+Z'\in\mathrm{CH}^d(X\times X) \tag{3.1}$$

where $m\ne 0$ is an integer, Z_ℓ is supported in $W'_\ell\times W_\ell$ with $\dim W_\ell=\ell$ and $\dim W'_\ell = d-\ell$, and Z' is supported in $T\times X$, where $T\subset X$ is a closed algebraic subset of codimension $\ge k_0+1$.

Proof [V, theorem 10.29]. ∎

Proposition 3.9 *Let X be a smooth projective variety satisfying the assumption in Theorem 3.8, and let $\sigma_\ell:\widetilde{W}_\ell\to W_\ell$ (resp. $\tau:\widetilde{T}\to T$) be an desingularization of W_ℓ for $k\le k_0$ (resp. T). Then for each m,*

$$\mu_m(X)\le\max\{M_W,\ \mu_{m-c}(\widetilde{T})\}, \tag{3.2}$$

where

$$c=\mathrm{codim}(T,X)\ge k_0+1 \quad \text{and} \quad M_W=\max\{\mu_m(\widetilde{W}_\ell)+2\ell-2d \mid \ell=0,1,\dots,k_0\}.$$

Proof We use the same notation in Theorem 3.8. Since the diagonal class $[\Delta_X]$ in $\mathrm{CH}^d(X\times X)$ induces the identity map on cohomology, (3.1) induces a decomposition

$$H^i(X,\mathbb{Q}) = [\Delta_X]_*H^i(X,\mathbb{Q}) = (\iota\circ\tau)_*H^{i-2c}(\widetilde{T},\mathbb{Q}(-c)) + \sum_{\ell=0}^{k_0}[\widetilde{Z}_\ell]_*H^i(\widetilde{W}_\ell,\mathbb{Q})$$

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where $\widetilde{Z}_\ell = (1 \times \sigma_\ell)^{-1}(Z_\ell) \subset X \times \widetilde{W}_\ell$ and $\iota\colon T \hookrightarrow X$ is an inclusion. Now let

$$\alpha \in \mathcal{F}^m H^i(X, \mathbb{Q}) = (\iota \circ \tau)_* \big(\mathcal{F}^{m-c} H^{i-2c}(\widetilde{T}, \mathbb{Q}) \big)(-c) + \sum_{\ell=0}^{k_0} [\widetilde{Z}_\ell]_* (\mathcal{F}^m H^i(\widetilde{W}_\ell, \mathbb{Q})),$$

which can be decomposed $\alpha = (\iota \circ \tau)_*(\gamma) + \sum_{\ell=0}^{k_0} [\widetilde{Z}_\ell]_*(\beta_\ell)$, where γ (resp. β_ℓ) vanishes on the complement of a Zariski closed set C (resp. B_ℓ) in $\widetilde{T}$ (resp. $\widetilde{W}_\ell$) all of whose component has codimension at least $(i - 2c - \mu_{m-c}(\widetilde{T}))/2$ (resp. $(i - \mu_m(\widetilde{W}_\ell))/2$). Since $\operatorname{codim}(T, X) = c$ and $\operatorname{codim}(W_\ell, X) = d - \ell$, the dimension counting argument implies that α lies in the Gysin image of a Zariski closed set of codimension at least $(i - \max\{\mu_{m-c}(\widetilde{T}),\ M_W\})/2$, where

$$M_W = \max\{\mu_m(\widetilde{W}_\ell) + 2\ell - 2d \mid \ell = 0, 1, \cdots, k_0\}.$$

We have (3.2). ∎

Corollary 3.10 *Let X be a rationally connected d-fold. Then* $\mathrm{GHC}(H^i(X, \mathbb{Q}), 1)$ *holds for any i.*

Proof Since $k_0 = 0$ in the case of rationally connected varieties, $M_W = 0$ in the Proposition 3.9 and the identity map on $H^i(X, \mathbb{Q})$ factors through $H^{i-2c}(\widetilde{T}, \mathbb{Q}(-c))$ and $\operatorname{codim}(T, X) = c = (i - (i - 2c))/2 \geq 1$. Hence, $\mu_m(X) \leq i - 2c \leq i - 2$. Lemma 3.1 implies $\mathrm{GHC}(H^i(X, \mathbb{Q}), 1)$. ∎

Corollary 3.11 ([La]) *The Hodge conjecture holds for a rationally connected variety up to dimension five.*

Proof It follows immediately from Lemma 3.5 and Corollary 3.10. ∎

4 Application to the GHC of a Rational Surface Fibration

In this section, we consider a surjective morphism $f\colon X \to Y$ of smooth projective varieties whose fibers are rational surfaces. In this setting X is uniruled, so Proposition 3.6 implies that $\mathrm{GHC}(H^{\dim X}(X, \mathbb{Q}), 1)$ holds. The purpose of this section is to prove $\mathrm{GHC}(H^{\dim X}(X, \mathbb{Q}), 2)$ under the GHC assumption on the base Y. First we establish the following theorem.

Theorem 4.1 *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties such that fibers are rational surfaces. Then*

$$\bar{\mu}_m(X) \leq \max(\bar{\mu}_{m-1}(Y), \dim X - 3). \tag{4.1}$$

Proof Since rational surfaces are rationally connected, there exist a nonempty open set $V \subset Y$, a relative zero cycle $\xi \in Z_0(X \times_Y V/V)$, a proper closed subset $Z \subset X_V = X \times_Y V$, and a cycle Γ supported on $X \times_Y Z$ such that for any $y \in V$,

$$N \cdot \Delta|_{X_y \times X_y} = (\xi \times X)|_{X_y \times X_y} + \Gamma|_{X_y \times X_y} \in \mathrm{CH}_2(X_y \times X_y), \tag{4.2}$$

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where Δ is the relative diagonal in $X_V \times_V X_V$ and N is a nonzero integer [A1, Corollary 1] . Let $Y_1 = Y - V$ and $X_1 = f^{-1}(Y_1)$.

$$\begin{array}{ccccccc} X_1 = f^{-1}(Y_1) & \hookrightarrow & X & \hookleftarrow & X_V & \hookleftarrow & Z \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_V} & \swarrow & \\ Y_1 = Y - V & \hookrightarrow & Y & \hookleftarrow & V & & \end{array}$$

By Proposition 2.2, we have

$$\bar{\mu}_m(X) \leq \max(\bar{\mu}_m(X_V), \bar{\mu}_m(X_1)) \qquad \text{for each } m. \tag{4.3}$$

Claim 1: $\bar{\mu}_m(X_1) \leq d - 3$.

Let $\sigma\colon \widetilde{X}_1 \to X_1$ (resp. $\tau\colon \widetilde{Y}_1 \to Y_1$) be a desingularization of X_1 (resp. Y_1). Applying the base change by desingularization of Y_1 to the composition $f_1 \circ \sigma$, we get a surjective smooth morphism $h\colon \bar{X}_1 \to \widetilde{Y}_1$ of smooth projective varieties with rational surface fibers.

$$\begin{array}{ccc} \bar{X}_1 = \widetilde{X}_1 \times_{Y_1} \widetilde{Y}_1 & \longrightarrow & \widetilde{X}_1 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f_1 \circ \sigma} \\ \widetilde{Y}_1 & \xrightarrow{\ \tau\ } & Y_1 \end{array}$$

Then by (2.1), Proposition 2.2 and Lemma 2.1, (recall $\dim \bar{X}_1 = \dim X_1 = d - 1$)

$$\bar{\mu}_m(X_1) \leq \bar{\mu}_m(\bar{X}_1) = \mu_{d-1-m}(\bar{X}_1) \leq \mu(\bar{X}_1)$$

Since $h^{-1}(y)$ is a smooth rational surface, $\mu(h^{-1}(y)) = 0$ for any $y \in \widetilde{Y}_1$ [A, corollary 0.2] and [A, theorem 2.1] implies the claimed inequality:

$$\mu(\bar{X}_1) \leq \max_{y \in \widetilde{Y}_1(\mathbb{C})} \mu(h^{-1}(y)) + \dim \widetilde{Y}_1 \leq d - 3.$$

Claim 2: $\bar{\mu}_m(X_V) \leq \bar{\mu}_{m-1}(Y)$.

First consider the Leray spectral sequence associated with $f_V\colon X_V \to V$

$$E_2^{i,j} = H_c^i(V, R^j f_{V*}\mathbb{Q}) \Longrightarrow H_c^{i+j}(X_V, \mathbb{Q}).$$

Since it degenerates at E_2 [D] we have an exact sequence

$$0 \to H_c^{d-2}(V, R^2 f_{V*}\mathbb{Q}) \to \frac{H_c^d(X_V, \mathbb{Q})}{H_c^d(V, R^0 f_{V*}\mathbb{Q})} \to H_c^{d-4}(V, R^4 f_{V*}\mathbb{Q}) \to 0. \tag{4.4}$$

By the same argument in the proof of Proposition 3.9, (4.2) induces a decomposition

$$H^j(X_y, \mathbb{Q}) = [\Delta_{X_y}]_* H^j(X_y, \mathbb{Q}) = (i \circ \sigma)^* H^j(|\widetilde{\xi}|_y, \mathbb{Q}) + [\widetilde{\Gamma}_y]_* H^{j-2}(\widetilde{Z}_y, \mathbb{Q}(-1))$$

where $\sigma\colon |\widetilde{\xi}| \to |\xi|$ is a desingularization of the support $|\xi|$ of the zero cycle ξ, $|\widetilde{\xi}|_y = |\widetilde{\xi}|\big|_{X_y}$, and $\widetilde{\Gamma} = (1 \times \tau)^{-1}(\Gamma)$ with $\tau\colon \widetilde{Z} \to Z$ a desingularization of Z. Here we use the

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observation that $\operatorname{codim}(Z_y, X_y) = 1$ for each $y \in V$, since the fiber Z_y is a proper subset of a surface X_y and Z_y cannot be a zero cycle.

$$H^j(X_y, \mathbb{Q}) = \operatorname{im}[H^{j-2}(\widetilde{Z}_y, \mathbb{Q}(-1)) \to H^j(X_y, \mathbb{Q})] \quad \text{for each } y \in V \text{ and for } j > 0.$$

By using the semi-simplicity of monodromy action [D1], we get a split surjection of local systems

$$R^{j-2} g_* \mathbb{Q}(-1) \twoheadrightarrow R^j f_{V*} \mathbb{Q}$$

where $g: \widetilde{Z} \to Z \to V$, and a surjection on cohomology with compact support

$$H_c^i(V, R^{j-2} g_* \mathbb{Q}(-1)) \twoheadrightarrow H_c^i(V, R^j f_{V*} \mathbb{Q}) \qquad \text{for } j > 0. \tag{4.5}$$

By combining (4.4), (4.5) and duality, we get an injection

$$\mathcal{F}_m W_{-d} H_d(X_V, \mathbb{Q}) \hookrightarrow \mathcal{F}_m W_{-d} \begin{pmatrix} H_d(V, \mathbb{Q}) \oplus \big(H_{d-2}(V, \mathbb{Q}) \otimes H_2(\widetilde{Z}_y, \mathbb{Q}) \big) \\ \oplus \big(H_{d-4}(V, \mathbb{Q}) \otimes H_4(X_y, \mathbb{Q}) \big) \end{pmatrix}$$

Via a surjection $H_d(Y, \mathbb{Q}) \twoheadrightarrow W_{-d} H_d(V, \mathbb{Q})$, any class in $W_{-d} H_d(V, \mathbb{Q})$ can be lifted to a class in $H_d(Y, \mathbb{Q}) \cong H^{d-4}(Y, \mathbb{Q}(d-2)) \cong H^d(Y, \mathbb{Q}(d)) \cong H_{d-4}(Y, \mathbb{Q}(2))$ by the Poincaré duality and the Lefschetz isomorphism. By restricting the cycle to $W_{-d+4} H_{d-4}(V, \mathbb{Q})$, we can combine the first and the third spaces in the summand:

$$\mathcal{F}_m W_{-d} H_d(X_V, \mathbb{Q}) \hookrightarrow \begin{pmatrix} \mathcal{F}_{m-1} W_{-d+2} H_{d-2}(V, \mathbb{Q}) \otimes H_2(\widetilde{Z}_y, \mathbb{Q}) \\ \oplus \\ \mathcal{F}_{m-2} W_{-d+4} H_{d-4}(V, \mathbb{Q}) \otimes H_4(X_y, \mathbb{Q}) \end{pmatrix}$$

This shows that any class $\alpha \in \mathcal{F}_m W_{-d} H_d(X_V, \mathbb{Q})$ lies in an image of $W_{-d} H_d(T, \mathbb{Q}) \to W_{-d} H_d(X_V, \mathbb{Q})$ for some Zariski closed set T such that

$$\begin{aligned} \dim T &\le \max\Big(\frac{(d-2) + \bar{\mu}_{m-1}(V)}{2} + 1, \quad \frac{(d-4) + \bar{\mu}_{m-2}(V)}{2} + 2 \Big) \\ &\le \max\Big(\frac{d + \bar{\mu}_{m-1}(Y)}{2}, \quad \frac{d + \bar{\mu}_{m-2}(Y)}{2} \Big) = \frac{d + \bar{\mu}_{m-1}(Y)}{2} \end{aligned}$$

since Y is a smooth compactification of V (Proposition 2.2) and $\bar{\mu}_m \le \bar{\mu}_{m+1}$ by $\bar{\mu}_m(Y) = \mu_{\dim Y - m}(Y)$ and Lemma 2.1. Therefore $\bar{\mu}_m(X_V) \le \bar{\mu}_{m-1}(Y)$, and the theorem follows from (4.3), and Claims 1 and 2. ∎

Corollary 4.2 *Given a surjective morphism $f: X \to Y$ of smooth projective varieties with rational surface fibers,* $\mathrm{GHC}(H^d(X, \mathbb{Q}), 2)$ *holds if* $\mathrm{GHC}(H^{d-2}(Y, \mathbb{Q}), 1)$ *holds where $d = \dim X$.*

Proof Since X (resp. Y) is a smooth projective variety of dimension d (resp. $d-2$), (4.1) is equivalent to

$$\mu_{d-m}(X) \le \max\big(\mu_{(d-2)-(m-1)}(Y), \ d-3 \big) = \max\big(\mu_{d-m-1}(Y), \ d-3 \big).$$

Now $\mathrm{GHC}(H^{d-2}(Y, \mathbb{Q}), 1)$ implies $\mathrm{GHC}(H^d(X, \mathbb{Q}), 2)$ immediately by Lemma 3.1, since $\mu_2(X) \le \max(\mu_1(Y), \ d-3) = d-3$. ∎

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