HIRING AND FIRING – A SIGNALING GAME

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Abstract

We study a signaling game between an employer and a potential employee, where the employee has private information regarding their production capacity. At the initial stage, the employee communicates a salary claim, after which the true production capacity is gradually revealed to the employer as the unknown drift of a Brownian motion representing the revenues generated by the employee. Subsequently, the employer has the possibility to choose a time to fire the employee in case the estimated production capacity falls short of the salary. In this setup, we use filtering and optimal stopping theory to derive an equilibrium in which the employee provides a randomized salary claim and the employer uses a threshold strategy in terms of the conditional probability for the high production capacity. The analysis is robust in the sense that various extensions of the basic model can be solved using the same methodology, including cases with positive firing costs, incomplete information about an individual's own type, as well as an additional interview phase.

Keywords: Signaling game; asymmetric information; optimal stopping

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1. Introduction

Incomplete information is a key ingredient in many hiring processes, where full knowledge about the true capacity of a potential employee is rarely available to the employer at the hiring time. Instead, if the candidate is hired, such information is gradually revealed to the employer over time. On the other hand, the potential employee would typically possess more accurate information, and would use this additional information when providing their salary claim. Naturally, a high salary is costly for the employer, and thus increases the risk for the employee of being fired. Therefore, there is a trade-off in the choice between a high salary claim to increase personal income and a small salary claim to decrease the risk of being fired.

To model one possible instance of the strategic interaction between an employer and a potential employee, we set up and study a stochastic game with asymmetric information between two players. The game is informally described as follows.

(i) The capacity μ of the employee (Player 1) is a random variable with a known two-point distribution.

(ii) At time $t = 0$, Player 1 learns about the realization of the random variable μ, and presents to the employer (Player 2) a non-negotiable salary claim C; the salary can only take two values.

(iii) At time 0, Player 2 observes the salary claim, and subsequently also noisy observations of μ, based upon which a choice is made of a stopping time τ to terminate the employment; here, $\tau = 0$ corresponds to a case in which the salary claim is not accepted (no hiring), $0 < \tau < \infty$ corresponds to an accepted salary claim, but with firing in finite time, and $\tau = \infty$ to an accepted salary claim, with no firing taking place.

(iv) Up to the termination time τ, Player 1 receives compensation at the chosen rate C per unit of time. Player 2, on the other hand, earns a net payment stream consisting of increments of a stochastic process $\mu t + \sigma W_t - Ct$, where W is a Brownian motion which accounts for random fluctuations about the mean rate $\mu - C$.

This is a *signaling game*, with two possible types of Player 1, corresponding to the two possible values of μ, and where Player 1 sends a signal by choosing the salary level C. As such, there is an incomplete and asymmetric information structure since the players have different knowledge about μ.

Variants of such signaling games with asymmetric information have a long history in the literature on hiring of staff and salary formation. A classical reference is [23], where an example with a job seeker that can have two different types is studied. The job seeker knows their own type and chooses an education level, where the cost of education depends on the type, thereby conveying information to the employer. In the setup of [23], all information is conveyed at the initial time, and the signal that consists of the chosen education level does not influence the actual ability of the employee. Extensions are studied in [1, 6] with a type-dependent continuation value for the job seeker, thus allowing for a future change in the salary level. In particular, the setup includes cases of employer learning based on an additional report (such as an interview phase) or based on the employee's on-the-job performance. For related studies allowing for uncertainty about an individual's own type, see [24], and for more possible types of job-seeker and competition between employers, see [7]. Also, a signaling game outside the job market is explored in [5], where an owner of a company and several potential buyers are considered. The seller holds private information of the company type, and buyers learn gradually from noisy observations of the unknown type and from the actions (lack of actions) of the seller. The hiring problem that we consider is also related to so-called principal-agent problems (see, e.g., [4, 15, 21]), in which a principal seeks to set up a compensation scheme to control the effort of the agent.

The methods we use to derive an equilibrium in the signaling game rely on a combination of stochastic filtering and stochastic control theory. For stochastic filtering of the drift of a diffusion process we refer to [19], and for a classical application to sequential hypothesis testing of the drift of Brownian motion, where filtering and optimal stopping theory is combined, see [22, Chapter 4]. For other studies of combined filtering and control, see, e.g., [8, 10, 17] for single-agent problems, and [3, 9, 13] for strategic setups.

In contrast to the setup of [23] and many subsequent papers, the current study does not use education level as a signal. Instead, we equip the employee with the right to provide a salary claim, which is then paid out continuously until a possible firing time. This allows us to study the trade-off between a high personal income and the risk of becoming a burden for

the company. In line with the literature on signaling games as above (see also [12, 14, 20]) we use the concept of *perfect Bayesian equilibrium* (PBE) as a solution concept. We show the existence of a semi-separating PBE, in which the strong type always chooses the high salary, whereas the weak type randomizes between the low and the high salary.

In Section 2, a precise formulation of our hiring-and-firing game is presented, and Section 3 recalls some standard results from filtering theory. In Section 4 we argue heuristically to derive a candidate equilibrium of strategies; the candidate equilibrium is then verified to in fact be a PBE. While our Bayesian game setup is rather simplistic, with only two possible types of employee and two possible salary claims, it may serve as a benchmark for more involved problems. This is illustrated in Section 5, where a few such extensions are briefly discussed. For these extensions, the solution of the benchmark case presented in Sections 2–4 is utilized.

2. Setup

To describe the game in further detail, let W be a standard Brownian motion, and let μ be a modified Bernoulli-distributed random variable independent of W with $\mathbb{P}(\mu = \mu_1) = p = 1 - \mathbb{P}(\mu = \mu_0)$, where μ_0, μ_1, and p are known constants with $\mu_0 < \mu_1$ and $p \in (0, 1)$. We assume that the employee (Player 1) generates a payment stream to the employer (Player 2) modeled as the increments of a process $X_t = \mu t + \sigma W_t$, where σ is a positive constant. The random variable μ will be referred to as the *capacity* of the employee.

Player 1 knows their own type, i.e. the realization of their capacity μ, and gives at the initial time $t = 0$ a salary claim C in the set $\{c_0, c_1\}$, where $0 < c_0 < c_1$. More precisely, allowing for randomized strategies, a strategy of Player 1 consists of a pair $a = (a_0, a_1) \in \mathcal{P}$, where $\mathcal{P} = [0, 1]^2$ is the unit square. Here, a_i represents the conditional probability of choosing $C = c_1$ given that Player 1 is of type i, $i = 0, 1$.

To describe the possible strategies of Player 2, denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$ the augmentation of the filtration generated by the process X, and by $\mathbb{F} = (\mathcal{F}_t^{X,C})_{t \geq 0}$ the augmentation of the filtration generated by the process X and the random variable C. Also, let $\mathcal{T}^X$ be the collection of $\mathbb{F}^X$-stopping times, and $\mathcal{T}$ be the collection of $\mathbb{F}$-stopping times. Clearly, since C only takes two possible values, any stopping time $\tau \in \mathcal{T}$ can be decomposed as

$$\tau = \begin{cases} \tau_0 & \text{on } \{C = c_0\}, \\ \tau_1 & \text{on } \{C = c_1\}, \end{cases}$$

where $(\tau_0, \tau_1) \in \mathcal{T}^X \times \mathcal{T}^X$. Conversely, defining τ in this way for a given pair $(\tau_0, \tau_1) \in \mathcal{T}^X \times \mathcal{T}^X$ yields that $\tau \in \mathcal{T}$. Thus we may identify $\mathcal{T}$ with $\mathcal{T}^X \times \mathcal{T}^X$, and we therefore write $\tau = (\tau_0, \tau_1)$.

In addition to a pair $(a, \tau) \in \mathcal{P} \times \mathcal{T}$ of strategies, the definition of a perfect Bayesian equilibrium (given below) also requires the specification of a belief system $\Pi_0 = (\Pi_0^0, \Pi_0^1) \in \mathcal{P}$. Here, Π_0^i represents the probability that Player 2 assigns to the event $\{\mu = \mu_1\}$ conditional on the signal $C = c_i$, $i = 0, 1$.

The payoff structure of the game is now described as follows. Up to the stopping time τ, Player 1 receives compensation for their work at rate C per unit of time. Player 2, on the other hand, receives increments of the net payment stream

$$X_t - Ct = (\mu - C)t + \sigma W_t \tag{1}$$

per unit of time. Both players seek to maximize their expected discounted future payoff. More precisely, for a given triple $(a, \tau, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ with $a = (a_0, a_1)$, $\tau = (\tau_0, \tau_1)$, and $\Pi_0 =$

$\left(\Pi_0^0, \Pi_0^1\right)$, and for a given discount rate $r > 0$, define

$$J_1^0(a, \tau) = (1 - a_0)\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} c_0 \, dt \mid \mu = \mu_0\right] + a_0\mathbb{E}\left[\int_0^{\tau_1} e^{-rt} c_1 \, dt \mid \mu = \mu_0\right],$$

$$J_1^1(a, \tau) = (1 - a_1)\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} c_0 \, dt \mid \mu = \mu_1\right] + a_1\mathbb{E}\left[\int_0^{\tau_1} e^{-rt} c_1 \, dt \mid \mu = \mu_1\right].$$

Then J_1^i represents the expected payoff for Player 1 given the capacity $\mu = \mu_i$, $i = 0, 1$. Similarly, define

$$J_2^0(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^0}\left[\int_0^{\tau_0} e^{-rt}(\mu - c_0) \, dt\right], \qquad J_2^1(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1} e^{-rt}(\mu - c_1) \, dt\right],$$

so that J_2^i is the expected payoff for Player 2 given that $C = c_i$, $i = 0, 1$. Here, the subindex in the expected value indicates that the expected value is calculated using the belief system Π_0 as the initial probability of the type $\mu = \mu_1$.

Remark 1. Note that Player 1 is equipped with randomized strategies, whereas Player 2 is not. The intuitive reason for this is as follows. Player 1 acts at time 0 (revealing the realization of C), and once this is done, the game collapses to a single-player optimization problem of choosing a non-anticipative stopping strategy for Player 2. In such a Markovian optimal stopping problem (cf. Section 4.1), however, the optimal value is attained for a hitting time (a pure strategy), and there are typically no other strategies (e.g. mixed strategies) that are optimal.

Remark 2. While a key ingredient in our setup is asymmetric information about the capacity μ, we point out that the setup itself, including the numerical values of all parameters p, μ_0, μ_1, σ, r, c_0, and c_1, is common knowledge to both players.

We now introduce the solution concept that we will use.

Definition 1. *(Perfect Bayesian equilibrium.)* We call a triplet $(a^*, \tau^*, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ a perfect Bayesian equilibrium (PBE) if the following conditions are satisfied.

(A) Rationality: $J_1^i(a^*, \tau^*) \geq J_1^i(a, \tau^*)$ for $i = 0, 1$, and $J_2^i(\tau^*, \Pi_0) \geq J_2^i(\tau, \Pi_0)$, $i = 0, 1$, for all pairs $(a, \tau) \in \mathcal{P} \times \mathcal{T}$.

(B) Bayesian updating: If $\min\{a_0^*, a_1^*\} < 1$, then

$$\Pi_0^0 = \frac{p(1 - a_1^*)}{p(1 - a_1^*) + (1 - p)(1 - a_0^*)},$$

and if $\max\{a_0^*, a_1^*\} > 0$, then

$$\Pi_0^1 = \frac{pa_1^*}{pa_1^* + (1 - p)a_0^*}.$$

Remark 3. In Definition 1, condition (A) requires the strategy triplet to form an equilibrium in the usual sense that neither of the players wants to unilaterally deviate from it. Condition (B) requires Player 2 to update their belief system using Bayes' rule on events with positive probability. If $a_0^* = a_1^* = 1$ then Player 1 always chooses $C = c_1$, and if $a_0^* = a_1^* = 0$, then Player 1 always chooses $C = c_0$, and in these cases the belief system can be chosen with no restriction.

Remark 4. If $a_0^* = a_1^* \in \{0, 1\}$, then the PBE is of *pooling type*; if $a_0^* = 1 - a_1^* \in \{0, 1\}$, then the PBE is *separating*; finally, if $a_i^* \in \{0, 1\}$ and $a_{1-i}^* \notin \{0, 1\}$, then the PBE is *semi-separating*.

3. Filtering

From the perspective of Player 2, the problem is a two-source learning problem: at time $t = 0$, the salary claim C is observed and the prior distribution of μ is updated in accordance with the specified belief system; at subsequent times $t > 0$, the posterior distribution is updated using observations of X.

Given $\pi \in [0, 1]$, define the process $\tilde{\Pi} := \tilde{\Pi}^\pi$ by $\tilde{\Pi}_t := \mathbb{P}_\pi(\mu = \mu_1 \mid \mathcal{F}_t^X)$, where the index π indicates that the conditional probability is calculated using an initial estimate π for the event $\{\mu = \mu_1\}$. Thus, $\tilde{\Pi}$ is the probability that $\mu = \mu_1$ conditioned merely on observations of X, and calculated with an initial belief π. It is well known from filtering theory (see [19]) that the conditional probability $\tilde{\Pi}$ satisfies $d\tilde{\Pi}_t = \omega \tilde{\Pi}_t (1 - \tilde{\Pi}_t) \, d\hat{W}_t$, where $\omega := (\mu_1 - \mu_0)/\sigma$ is the *signal-to-noise ratio* and the *innovations process*

$$\hat{W}_t := \frac{1}{\sigma} \left(X_t - \int_0^t (\mu_0 + (\mu_1 - \mu_0)\tilde{\Pi}_s) \, ds \right)$$

is an $\mathcal{F}^X$-Brownian motion. In particular, the process $\tilde{\Pi}$ is strong Markov.

Now, given a belief system $\Pi_0 = (\Pi_0^0, \Pi_0^1) \in \mathcal{P}$, we define the conditional probability process

$$\Pi_t := \begin{cases} \tilde{\Pi}_t^{\Pi_0^0} & \text{on } \{C = c_0\}, \\ \tilde{\Pi}_t^{\Pi_0^1} & \text{on } \{C = c_1\}. \end{cases} \tag{2}$$

If the Bayesian updating property (B) holds, then Π_t coincides with $\mathbb{P}(\mu = \mu_1 \mid \mathcal{F}_t^{X,C})$ on the event $\{C = c_i\}$ provided that $\mathbb{P}(C = c_i) > 0$.

Lemma 1. *Let* $(\tau, \Pi_0) \in \mathcal{T} \times \mathcal{P}$. *Then, for* $i = 0, 1$,

$$J_2^i(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt}(\mu_0 - c_i + (\mu_1 - \mu_0)\Pi_t) \, dt \right].$$

Proof. By conditioning,

$$\mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt} \mu \, dt \right] = \mathbb{E}_{\Pi_0^i} \left[\mu \frac{1 - e^{-r\tau_i}}{r} \right] = \mathbb{E}_{\Pi_0^i} \left[(\mu_0 + (\mu_1 - \mu_0)\Pi_{\tau_i}) \frac{1 - e^{-r\tau_i}}{r} \right],$$

where $\Pi_t := \mathbb{P}_{\Pi_0^i}(\mu = \mu_1 \mid \mathcal{F}_t^X)$. Moreover, by an application of Itô's formula and optional sampling,

$$\mathbb{E}_{\Pi_0^i} \left[(\mu_0 + (\mu_1 - \mu_0)\Pi_{\tau_i}) \frac{1 - e^{-r\tau_i}}{r} \right] = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt}(\mu_0 + (\mu_1 - \mu_0)\Pi_t) \, dt \right].$$

Consequently,

$$J_2^i(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt}(\mu - c_i) \, dt \right] = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt}(\mu_0 - c_i + (\mu_1 - \mu_0)\Pi_t) \, dt \right]. \qquad \square$$

4. A semi-separating PBE

Note that if $c_0 \geq \mu_1$, then the net drift $\mu - C$ in (1) is non-positive, and Player 2 should choose immediate firing ($\tau = 0$). Similarly, if $\mu_0 \geq c_1$, then $\tau = \infty$ would always be optimal. Thus, to rule out degenerate cases, a minimal assumption is that $\mu_0 < c_1 < \mu_1$. Moreover, we will make the additional assumption that $c_0 \leq \mu_0$ so that the net drift $\mu - C$ in (1) is non-negative on the event $\{C = c_0\}$. We thus impose the parameter ordering

$$0 < c_0 \leq \mu_0 < c_1 < \mu_1. \tag{3}$$

It is straightforward to see that there is no PBE of separating type. Indeed, in a separating equilibrium with $a^* = (0, 1)$, the strong type would never be fired, and therefore the weak type would have an incentive to deviate and choose c_1; similarly, if $a^* = (1, 0)$, then the weak type would be fired immediately, and thus again have an incentive to deviate. The aim of the current section is to derive a perfect Bayesian equilibrium of semi-separating type under the assumption (3). In Sections 4.1 and 4.2 we use intuitive arguments to derive a candidate equilibrium, which is then verified in Section 4.3.

4.1. The employer's perspective

Under the assumption (3), the lower salary level c_0 is smaller than the capacity μ with probability 1. Thus, if (3) holds, it is clear that if Player 1 chooses $C = c_0$, then an optimal response for Player 2 should be to choose $\tau_0 = \infty$.

On the other hand, on the event $\{C = c_1\}$, Player 2 would stop if there is sufficient evidence that $\mu = \mu_0$. More precisely, we expect a boundary level b such that

$$\tau_1 := \inf\{t \geq 0 \colon \Pi_t \leq b\} \tag{4}$$

is an optimal response for Player 2. To determine b, standard optimal stopping theory based on the dynamic programming principle (see, e.g., [22]) suggests that the pair (V, b), where

$$V(\pi) := \sup_\tau \mathbb{E}\left[\int_0^\tau e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\tilde{\Pi}_t^\pi)\, \mathrm{d}t \right]$$

solves the free-boundary problem

$$\begin{cases} \mathcal{L}V + \mu_0 - c_1 + (\mu_1 - \mu_0)\pi = 0, & \pi \in (b, 1), \\ V(b) = 0, \\ V_\pi(b) = 0, \\ V(1-) = (\mu_1 - c_1)/r, \end{cases} \tag{5}$$

where

$$\mathcal{L} = \frac{1}{2}\omega^2\pi^2(1-\pi)^2\frac{\mathrm{d}^2}{\mathrm{d}\pi^2} - r.$$

Here, the two boundary conditions at b constitute the so-called condition of smooth fit, and the boundary condition at $\pi = 1$ corresponds to receiving (discounted) payments at rate $\mu_1 - c_1$ until time $\tau = \infty$. Note that the ordinary differential equation (ODE) in (5) is of second order, so there are two degrees of freedom; additionally, the boundary b is unknown. On the other hand, there are three boundary conditions, so we would expect that (5) is well-posed.

To solve the free-boundary problem (5), we readily verify that the general solution of the ODE is given by

$$V(\pi) = A_1(1 - \pi)\left(\frac{\pi}{1 - \pi}\right)^{\gamma_1} + A_2(1 - \pi)\left(\frac{\pi}{1 - \pi}\right)^{\gamma_2} + \frac{\mu_0 - c_1 + (\mu_1 - \mu_0)\pi}{r},$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the solutions of the quadratic equation

$$\gamma^2 - \gamma - \frac{2r}{\omega^2} = 0, \tag{6}$$

and A_1 and A_2 are arbitrary constants. Imposing the boundary condition at $\pi = 1$, we must have $A_2 = 0$, and so the two remaining boundary conditions yield

$$\begin{cases} A_1(1 - b)\left(\dfrac{b}{1 - b}\right)^{\gamma_1} + \dfrac{\mu_0 - c_1 + (\mu_1 - \mu_0)b}{r} = 0, \\ A_1(\gamma_1 - b)\left(\dfrac{b}{1 - b}\right)^{\gamma_1} + \dfrac{(\mu_1 - \mu_0)b}{r} = 0. \end{cases}$$

Eliminating A_1, we find that

$$b = \frac{-(c_1 - \mu_0)\gamma_1}{\mu_1 - c_1 - (\mu_1 - \mu_0)\gamma_1}. \tag{7}$$

Thus, the candidate optimal response for Player 2 when Player 1 chooses the higher salary $C = c_1$ is to stop when the conditional probability process Π falls below the constant boundary b in (7). Moreover, the candidate value for the employer is then

$$V(\pi) = \begin{cases} \dfrac{c_1 - \mu_0 - (\mu_1 - \mu_0)b}{r}\left(\dfrac{\pi(1 - b)}{b(1 - \pi)}\right)^{\gamma_1}\dfrac{1 - \pi}{1 - b} + \dfrac{\mu_0 - c_1 + (\mu_1 - \mu_0)\pi}{r}, & \pi > b, \\ 0, & \pi \leq b. \end{cases}$$

For a graphical illustration of the function V and the threshold b, see Figure 1.

4.2. The employee's perspective

We now take the perspective of Player 1. We will construct an equilibrium in which Player 1 always chooses $C = c_1$ on the event $\{\mu = \mu_1\}$, and on the event $\{\mu = \mu_0\}$ uses a strategy such that $\mathbb{P}(C = c_1 \mid \mu = \mu_0) = a_0 = 1 - \mathbb{P}(C = c_0 \mid \mu = \mu_0)$ for some $a_0 \in [0, 1]$ to be determined. Thus, in the notation of Section 2, we consider the strategy $a = (a_0, 1) \in \mathcal{P}$.

As noted above, on the event $\{C = c_0\}$, Player 2 would use $\tau_0 = \infty$. By the *indifference principle* in game theory (see, e.g., [11] or [18]), to have an equilibrium with a strategy pair (a^*, τ^*) in which Player 1 uses a mixed strategy $a^* = (a_0, 1)$ with $a_0 \in (0, 1)$ and Player 2 uses $\tau^* = (\infty, \tau_1)$ with τ_1 as in (4), we need the expected payoffs $J_1^0((0, 1), \tau^*)$ and $J_1^0((1, 1), \tau^*)$ to coincide. Clearly, choosing $C = c_0$ gives the expected payoff $J_1^0((0, 1), (\infty, \tau_1)) = c_0/r$ for Player 1.

To determine the expected payoff $J_1^0((1, 1), \tau^*)$ for Player 1 on the event $\{C = c_1\}$, note that on the event $\{C = c_1\}$, Player 2 would first re-evaluate the probability that Player 1 has the larger capacity $\mu = \mu_1$ according to the specified belief system Π_0. Moreover, by the Bayesian updating requirement of the belief system, we have

$$\Pi_0^1 = \mathbb{P}(\mu = \mu_1 \mid C = c_1) = \frac{\mathbb{P}(\mu = \mu_1, C = c_1)}{\mathbb{P}(C = c_1)} = \frac{p}{p + (1 - p)a_0}.$$
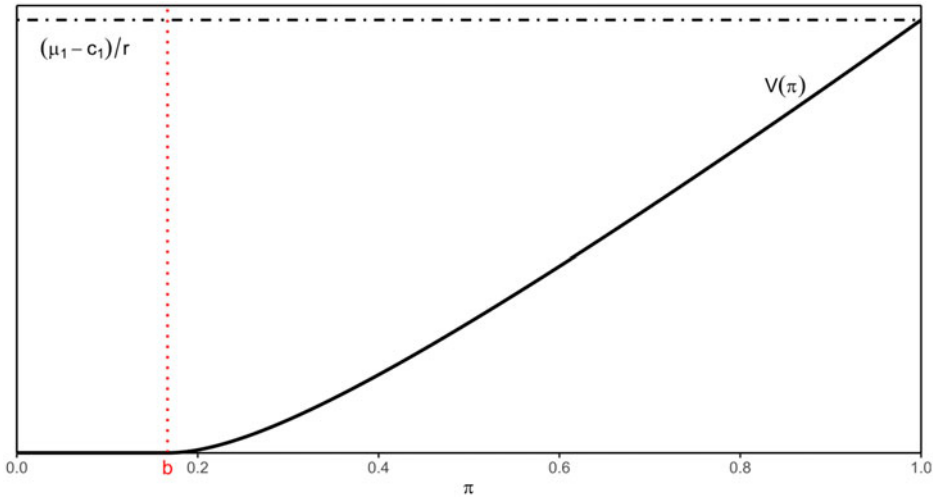
FIGURE 1. The value function $V(\pi)$ of the employer on the event $\{C = c_1\}$. The parameter values chosen for this example figure are $c_1 = 1.5$, $\mu_0 = 1.4$, $\mu_1 = 1.7$, $r = 0.05$, and $\sigma = 1$. The value function attains positive values only after the boundary level $b \approx 0.167$, and it approaches its maximum value $(\mu_1 - c_1)/r$ for π close to 1.

Thus, Π_t makes an initial jump from $\Pi_{0-} = p$ up to

$$\Pi_0^1 = \frac{p}{p + (1-p)a_0} \geq p,$$

and then it diffuses with dynamics $d\Pi_t = \omega\Pi_t(1 - \Pi_t)\,d\hat{W}_t$. From the perspective of Player 1, however, $\hat{W}$ is not a Brownian motion since Player 1 knows the true value of μ. Instead,

$$d\Pi_t = \omega\Pi_t(1 - \Pi_t)\,d\hat{W}_t = -\omega^2\Pi_t^2(1 - \Pi_t)\,dt + \omega\Pi_t(1 - \Pi_t)\,dW_t.$$

Consequently, if Player 1 chooses $C = c_1$, then their value is $U(\Pi_0^1)$, where

$$U(\pi) := \mathbb{E}_\pi\left[\int_0^{\tau_1} e^{-rt}c_1\,dt \mid \mu = \mu_0\right] \tag{8}$$

solves

$$\begin{cases} \dfrac{\omega^2\pi^2(1-\pi)^2}{2}U_{\pi\pi} - \omega^2\pi^2(1-\pi)U_\pi - rU + c_1 = 0, & \pi \in (b, 1), \\ U(b) = 0, \\ U(1-) = c_1/r. \end{cases}$$

This ODE has the general solution

$$U(\pi) = B_1\left(\frac{\pi}{1-\pi}\right)^{\gamma_1} + B_2\left(\frac{\pi}{1-\pi}\right)^{\gamma_2} + c_1/r,$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the solutions of (6) as before, and B_1 and B_2 are arbitrary constants. As before, the boundary condition at $\pi = 1$ yields $B_2 = 0$, and then the boundary condition at $\pi = b$ gives

$$B_1 = \frac{-c_1}{r} \left(\frac{b}{1-b} \right)^{-\gamma_1},$$

so

$$U(\pi) = \begin{cases} \dfrac{c_1}{r} \left(1 - \left(\dfrac{\pi(1-b)}{(1-\pi)b} \right)^{\gamma_1} \right), & \pi > b, \\ 0, & \pi \le b. \end{cases}$$

Now recall that by the indifference principle we are looking for $a_0 \in (0, 1)$ such that

$$\frac{c_0}{r} = U\left(\Pi_0^1 \right) = U\left(\frac{p}{p + (1-p)a_0} \right).$$

This is possible only if $U(p) < c_0/r$, i.e. if

$$\frac{p}{1-p} < \frac{b}{1-b} \left(1 - \frac{c_0}{c_1} \right)^{1/\gamma_1}.$$

Equivalently, we need to have

$$p < \hat{p} := \frac{b(1 - c_0/c_1)^{1/\gamma_1}}{1 - b + b(1 - c_0/c_1)^{1/\gamma_1}}, \tag{9}$$

where $\hat{p}$ is the indifference point such that $U(\hat{p}) = c_0/r$. Moreover, in that case, a_0 should be chosen so that

$$\frac{p}{p + (1-p)a_0} = \frac{b(1 - c_0/c_1)^{1/\gamma_1}}{1 - b + b(1 - c_0/c_1)^{1/\gamma_1}},$$

i.e.

$$a_0 = \frac{p(1-b)}{(1-p)b(1 - c_0/c_1)^{1/\gamma_1}}.$$

That is, the candidate optimal strategy for Player 1 is as follows. If $\mu = \mu_1$, then the high salary is chosen with probability 1 ($a_1 = 1$). On the other hand, if Player 1 is of the weak type ($\mu = \mu_0$), then the high salary c_1 is chosen with probability a_0, where

$$a_0 = \begin{cases} \dfrac{p(1-b)}{(1-p)b(1 - c_0/c_1)^{1/\gamma_1}}, & p < \hat{p}, \\ 1, & p \ge \hat{p}. \end{cases}$$

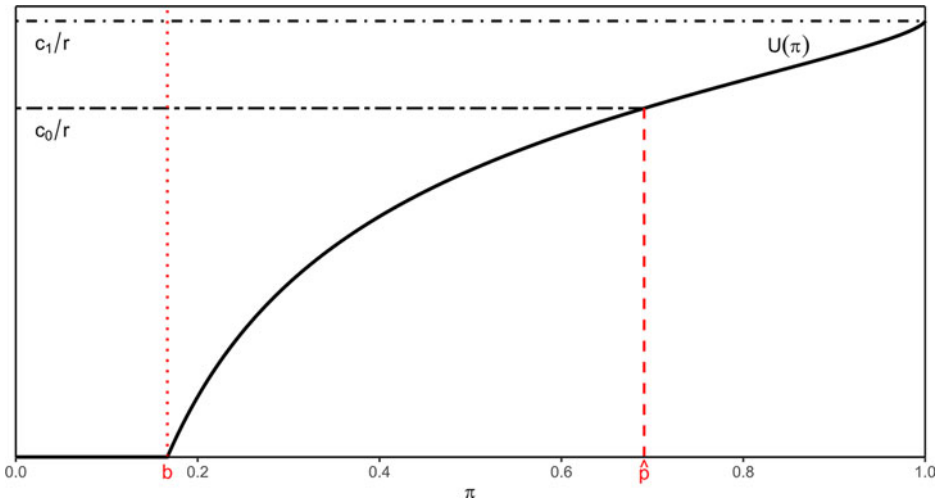For a graphical illustration of the value function U and the indifference point $\hat{p}$, see Figure 2.

FIGURE 2. The value function $U(\pi)$ for the weak-type employee on the event $\{C = c_1\}$. The parameter values of c_1, μ_0, μ_1, r, and σ are the same as in Figure 1, and $c_0 = 1.2$. Here, $\hat{p}$ is the unique value such that $U(\hat{p}) = c_0/r$.

4.3. Verification of equilibrium

We now summarize the strategies described above in Theorem 1. We then verify that these strategies together constitute a perfect Bayesian equilibrium.

Let b be defined as in (7), and define the strategy $a^* = (a_0^*, a_1^*)$ of Player 1 by

$$a_0^* = \begin{cases} \dfrac{p(1-b)}{(1-p)b(1-c_0/c_1)^{1/\gamma_1}}, & p < \hat{p}, \\ 1, & p \geq \hat{p} \end{cases}$$

and $a_1^* = 1$, where $\hat{p}$ is as in (9). Moreover, let $\Pi_0 := \left(\Pi_0^0, \Pi_0^1\right) = (0, \hat{p} \vee p)$, and let $\tau^* = (\tau_0^*, \tau_1^*)$ be defined by $\tau_0^* := \infty$ and $\tau_1^* := \inf\left\{t \geq 0 \colon \tilde{\Pi}_t^{\Pi_0^1} \leq b\right\}$.

Theorem 1. *Assume that* (3) *holds. Then the triplet* (a^*, τ^*, Π_0) *specified above is a perfect Bayesian equilibrium. Moreover, if $p < \hat{p}$, the equilibrium is semi-separating; if $p \geq \hat{p}$, then the equilibrium is of pooling type.*

Proof. We first note that, by construction, the belief system Π_0 satisfies the Bayesian updating property. The proof of rationality is divided into two parts.

Optimality of τ^*. First note that $\Pi_0^0 = 0$ yields that

$$J_2^0(\tau, \Pi_0) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(\mu_0 - c_0)\, dt \mid \mu = \mu_0\right] \leq \frac{\mu_0 - c_0}{r} = J_2^0(\tau^*, \Pi_0)$$

for any $\tau \in \mathcal{T}$, so $\tau_0^* = \infty$ is a rational response to $C = c_0$.

Next, if the employer observes the event $\{C = c_1\}$, then the stopping time $\tau_1^* := \inf\{t \geq 0 : Z_t \leq b\}$ is used, where $Z_t := \tilde{\Pi}_t^{\Pi_0^1} = \mathbb{P}_{\Pi_0^1}(\mu = \mu_1 \mid \mathcal{F}_t^X)$, cf. (2). By Section 3, $dZ_t = \omega Z_t(1 - Z_t)\, d\hat{W}_t$, so an application of Itô's formula together with (5) shows that

$$Y_t := e^{-rt}V(Z_t) + \int_0^t e^{-rs}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_s)\, ds$$

is a bounded supermartingale. For any stopping time $\tau' = (\tau_0', \tau_1') \in \mathcal{T}$, optional sampling therefore gives that

$$V(\Pi_0^1) \geq \mathbb{E}\left[e^{-r(T \wedge \tau_1')}V(Z_{T \wedge \tau_1'}) + \int_0^{T \wedge \tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right]$$

$$\geq \mathbb{E}\left[\int_0^{T \wedge \tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right] \to \mathbb{E}\left[\int_0^{\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right]$$

as $T \to \infty$ by bounded convergence. Since

$$\mathbb{E}\left[\int_0^{\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right] = J_2^1(\tau', \Pi_0)$$

by Lemma 1, we find that

$$J_2^1(\tau', \Pi_0) \leq V(\Pi_0^1) \tag{10}$$

for all $\tau' \in \mathcal{T}$.

Furthermore, for τ^*, the stopped process $Y_{t \wedge \tau_1^*}$ is a martingale, so optional sampling and bounded convergence give

$$V(\Pi_0^1) = \mathbb{E}\left[e^{-r(T \wedge \tau_1^*)}V(Z_{T \wedge \tau_1^*}) + \int_0^{T \wedge \tau_1^*} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right]$$

$$\to \mathbb{E}\left[\int_0^{\tau_1^*} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)Z_t)\, dt \right] = J_2^1(\tau^*, \Pi_0)$$

as $T \to \infty$, which together with (10) implies that τ_1^* is an optimal response to $C = c_1$.

Optimality of a^*. We have that

$$J_1^0(a, \tau^*) = (1 - a_0)\frac{c_0}{r} + a_0 \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt}c_1\, dt \mid \mu = \mu_0 \right]$$

$$= (1 - a_0)\frac{c_0}{r} + a_0 U(\hat{p} \vee p) \leq (1 - a_0^*)\frac{c_0}{r} + a_0^* U(\hat{p} \vee p) = J_1^0(a^*, \tau^*),$$

where the inequality holds since if $p > \hat{p}$ then we have $U(\hat{p} \vee p) = U(p) \geq c_0/r$ and $a_0^* = 1 \geq a_0$, and if $p \leq \hat{p}$ then we have $U(\hat{p} \vee p) = U(\hat{p}) = c_0/r$.

Similarly,

$$J_1^1(a, \tau^*) = (1 - a_1)\frac{c_0}{r} + a_1 \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt}c_1\, dt \mid \mu = \mu_1 \right]$$

$$\leq \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt}c_1\, dt \mid \mu = \mu_1 \right] = J_1^1(a^*, \tau^*),$$

where the inequality follows from the inequalities

$$\mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1 \, dt \mid \mu = \mu_1\right] \geq \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1 \, dt \mid \mu = \mu_0\right] = U(\Pi_0^1) \geq c_0/r.$$

Remark 5. As is often the case for signaling games, there is no uniqueness of PBEs. Indeed, consider the strategy pair (a, τ), where $a = (0, 0)$ and $\tau = (\infty, 0)$; in words, Player 1 always chooses $C = c_0$ (regardless of their type) and Player 2 never stops if $C = c_0$ and stops immediately if $C = c_1$. Then (a, τ, Π_0) with $\Pi_0 = (p, \Pi_0^1)$ is also a perfect Bayesian equilibrium (of pooling type) provided the belief Π_0^1 is chosen small enough (e.g. $\Pi_0^1 \leq b$).

There have been substantial efforts in the literature to refine the notion of PBE (cf. [2, 16]) in order to rule out some non-intuitive equilibria. Rather than taking that path, however, we simply note that in the pooling equilibrium both types have the same equilibrium value c_0/r, which is dominated by the corresponding equilibrium values in Theorem 1, so the semi-separating PBE is preferred by the first-mover of our game.

Remark 6. We have analyzed the game under the assumption (3) that $c_0 \leq \mu_0 < c_1 < \mu_1$. In the alternative ordering $\mu_0 < c_0 < c_1 < \mu_1$, the smaller salary c_0 provides a negative running reward for the employer in the case of a weak-type employee, and a semi-separating (or separating) equilibrium is not feasible. As in Remark 5, we can construct a pooling PBE with $a = (0, 0)$, but we also obtain a pooling equilibrium with $a = (1, 1)$, supported by a sufficiently small belief Π_0^0. While semi-separating and separating PBEs are not feasible, it remains an open question whether mixing between the two pooling equilibria is possible in this parameter regime.

5. Extensions

In this section we briefly discuss a few extensions of the basic setup we have presented. All of these extensions can be easily solved using the methods of the current article, thus demonstrating the robustness of the benchmark case studied. For the sake of brevity, we merely outline the solutions and leave out the full arguments.

5.1. Firing cost

In this section we consider the specification (i)–(iv) in the introduction, but with the addition that

(v) At the firing time τ, Player 2 pays a firing cost ϵ, where $\epsilon \in (0, (c_1 - \mu_0)/r)$.

Note that the assumption $\epsilon < (c_1 - \mu_0)/r$ implies that the firing cost is smaller than the maximal possible loss for Player 2. Adding assumption (v), the expected payoff for Player 2 is now

$$J_2^{\epsilon, i}(\tau, \Pi_0) := \mathbb{E}_{\Pi_0^i}\left[\int_0^{\tau_i} e^{-rt}(\mu - c_i) \, dt - \epsilon e^{-r\tau_i} \mathbf{1}_{\{0 < \tau_i < \infty\}}\right]$$

$$= \mathbb{E}_{\Pi_0^i}\left[\int_0^{\tau_i} e^{-rt}(\mu_0 - c_i + (\mu_1 - \mu_0)\Pi_t) \, dt - \epsilon e^{-r\tau_i} \mathbf{1}_{\{0 < \tau_i < \infty\}}\right]$$

by Lemma 1. Note that the choice $\tau_i = 0$ gives rise to no firing cost, with the interpretation that no hiring takes place.

Replacing the first boundary condition in the free-boundary problem (5) with $V(b) = -\epsilon$, wer obtain a stopping boundary

$$b := b^\epsilon := \frac{-(c_1 - \mu_0 - \epsilon r)\gamma_1}{\mu_1 - c_1 + r\epsilon - (\mu_1 - \mu_0)\gamma_1},$$

where γ_1 is the negative solution to the quadratic equation (6). Due to the parameter ordering in (3), we can verify that indeed $b^\epsilon \in (0, 1)$ when $\epsilon \in (0, (c_1 - \mu_0)/r)$. Arguing as in Section 4.2, we find that the indifference point $\hat{p}$ at which $U(\hat{p}) = c_0/r$ is given by

$$\hat{p} := \hat{p}^\epsilon := \frac{b^\epsilon (1 - c_0/c_1)^{1/\gamma_1}}{1 - b^\epsilon + b^\epsilon (1 - c_0/c_1)^{1/\gamma_1}},$$

which leads to the candidate strategy $a^* = (a_0^*, 1)$ with

$$a_0^* = \begin{cases} \dfrac{p(1 - b^\epsilon)}{(1 - p)b^\epsilon (1 - c_0/c_1)^{1/\gamma_1}}, & p < \hat{p}, \\ 1, & p \geq \hat{p}. \end{cases}$$

Thus we specify a triplet $(a^*, \tau^*, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ by $a^* = (a_0^*, 1)$, where $\Pi_0 = (0, \hat{p} \vee p)$ and $\tau^* = (\infty, \tau_1)$, with $\tau_1^* = \inf\left\{t \geq 0 \colon \tilde{\Pi}_t^{\Pi_0^1} \leq b^\epsilon\right\}$. It is then straightforward to verify that if $V(\hat{p} \vee p) > 0$, then (a^*, τ^*, Π_0) is a PBE. (On the other hand, if $V(\hat{p} \vee p) \leq 0$, then $((0, 0), (0, 0), (p, p))$, corresponding to always choosing $C = c_0$ and no hiring, is an equilibrium.)

The addition of a firing cost $\epsilon \in (0, (c_1 - \mu_0)/r)$, together with $V(\hat{p} \vee p) > 0$, then constitutes a PBE with a lower stopping boundary b^ϵ for Player 2 and a higher randomizing probability a_0^* for a low-type Player 1 compared to the corresponding values in the benchmark case of Section 4.

5.2. Uncertainty about type

In this section we consider the same setup as specified in (i)–(iv), but where (ii) is replaced by

(ii′) At time $t = 0$, Player 1 first receives a *noisy observation* of their capacity μ, and then presents to Player 2 a salary claim $C \in \{c_0, c_1\}$.

In this way, Player 1 also has incomplete information about their own capacity (cf. [24]). More precisely, assume that the noisy signal is either of the two events 'strong belief' and 'weak belief', where $p_1 := \mathbb{P}(\text{strong belief})$, $q := \mathbb{P}(\mu = \mu_1 \mid \text{strong belief})$, and

$$\mathbb{P}(\mu = \mu_1 \mid \text{weak belief}) = 0. \tag{11}$$

With this notation, the probability (denoted p in the previous sections) that Player 1 has the large capacity is $p = p_1 q$. Note that when $q = 1$, this extension collapses to our original benchmark model. Also note that a consequence of (11) is that Player 1 has a tendency to overestimate their capacity: if Player 1 has weak belief, then they are always of the weak type, whereas if they have strong belief, then they may still be of the weak type. We adapt the strategy $a = (a_0, a_1)$ so that a_0 and a_1 now denote the conditional probabilities that Player 1 chooses $C = c_1$ given 'weak belief' and 'strong belief', respectively.

For simplicity, assume that $q > \hat{p}$, where $\hat{p}$ is defined in (9). Now specify a triplet $(a^*, \tau^*, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ by $a^* = (a_0^*, 1)$, $\Pi_0 = (0, \hat{p} \vee (p_1 q))$, and $\tau^* = (\infty, \tau_1^*)$, where

$$a_0^* = \begin{cases} \dfrac{p_1(q - \hat{p})}{\hat{p}(1 - p_1)}, & p_1 q < \hat{p}, \\ 1, & p_1 q \geq \hat{p}, \end{cases}$$

$$\tau_1^* = \inf\left\{t \geq 0 \colon \tilde{\Pi}_t^{\Pi_0^1} \leq b\right\},$$

with b as in (7). It is then straightforward to check that (a^*, τ^*, Π_0) constitutes a PBE.

5.3. Adding an interview phase

As a last extension of the setup in (i)–(iv), consider a situation where (iii) is replaced with

(iii′) At time 0, Player 2 observes the salary claim *together with the result 'weak result' or 'strong result' of a test*, and subsequently also noisy observations of μ, based upon which a choice is made of a stopping time τ to terminate the employment.

Thus, further to the salary claim C, Player 2 also receives information from an additional test result (cf. [1, 6]) with two possible outcomes. For definiteness, we assume that $\mathbb{P}(\text{strong result} \mid \mu = \mu_0) = q \in (c_0/c_1, 1)$ and

$$\mathbb{P}(\text{strong result} \mid \mu = \mu_1) = 1, \tag{12}$$

which corresponds to a situation in which strong types always perform well in the interview, and weak types sometimes do. Note that the outcome of the interview test is not available for Player 1, which is similar to the situation studied in [1, 6].

A belief system and strategy for Player 2 can now be described as the quadruples $\Pi_0 = (\Pi_0^{0,\text{weak}}, \Pi_0^{0,\text{strong}}, \Pi_0^{1,\text{weak}}, \Pi_0^{1,\text{strong}})$ and $\tau = (\tau_0^{\text{weak}}, \tau_0^{\text{strong}}, \tau_1^{\text{weak}}, \tau_1^{\text{strong}})$, where for $i = 0, 1$, $\Pi_0^{i,\text{weak}}, \Pi_0^{i,\text{weak}}, \tau_i^{\text{strong}}$, and τ_i^{strong} are the beliefs and stopping times used provided $C = c_i$ and the test result 'weak result' or 'strong result' are observed, respectively.

By the assumption in (12), if Player 2 observes 'weak result' in the test, then Player 1 is automatically of the weak type ($\Pi_0^{i,\text{weak}} = 0$), and immediate firing ($\tau_1^{\text{weak}} = 0$) would then be optimal in the case $C = c_1$. Therefore, choosing $C = c_1$ is risky if Player 1 is of the weak type, and the obtained value from choosing $C = c_1$ (for the weak-type player) is $qU(\Pi_0^{1,\text{strong}})$, with U as in (8). We thus define the indifference point $\hat{P}$ via the indifference principle (cf. Section 4.2) so that $qU(\hat{P}) = c_0/r$ (note that $\hat{P}$ is well-defined since $q > c_0/c_1$ by assumption), and let

$$a^* = \left(1, \frac{p(1 - \hat{P})}{\hat{P}(1 - p)q} \wedge 1\right), \qquad \Pi_0 = (0, 0, 0, \Pi_0^{1,\text{strong}}),$$

with

$$\Pi_0^{1,\text{strong}} = \hat{P} \vee \frac{p}{p + (1 - p)q}, \qquad \tau^* = (\infty, \infty, 0, \tau_b), \qquad \tau_b = \inf\left\{t \geq 0 \colon \tilde{\Pi}_t^{\Pi_0^{1,\text{strong}}} \leq b\right\}.$$

It is then straightforward to check that (a^*, τ^*, Π_0) is a PBE.

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