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EMBEDDINGS AND C^*-ENVELOPES OF EXACT OPERATOR SYSTEMS

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Abstract

We prove a necessary and sufficient condition for embeddability of an operator system into $\mathcal{O}_2$. Using Kirchberg's theorems on a tensor product of $\mathcal{O}_2$ and $\mathcal{O}_\infty$, we establish results on their operator system counterparts $\mathcal{S}_2$ and $\mathcal{S}_\infty$. Applications of the results, including some examples describing C^*-envelopes of operator systems, are also discussed.

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1. Introduction

Operator systems with universal generators for some well-studied C^*-envelopes have attracted considerable interest in recent years. Zheng [21] introduced the operator system $\mathcal{S}_n$ generated by Cuntz isometries and, later, in [17], Paulsen and Zheng explored tensor products and nuclearity for this operator system.

In 1977, Cuntz [5] introduced the C^*-algebras $\mathcal{O}_n$ $(1 \le n \le \infty)$. These were the first explicit examples of simple infinite separable C^*-algebras. Cuntz proved that his algebras are simple and purely infinite and independent of the choice of generators.

These algebras played an important role in the classification theory of purely infinite, simple, separable and nuclear C^*-algebras, by Kirchberg and Philips. The classification theory for separable C^*-algebras with certain properties in terms of the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ was given by Kirchberg and Rørdam (see [19]).

Kirchberg established three fundamental theorems: the embedding of separable exact C^*-algebras into the Cuntz algebra $\mathcal{O}_2$ and the tensor product theorems for $\mathcal{O}_2$ and $\mathcal{O}_\infty$. Many generalisations were later proved by Kirchberg and Rørdam. Recently, Lupini [15] established an operator system analogue of Kirchberg's nuclear embedding theorem involving the Gurarij operator system $\mathbb{GS}$.

For $1 \le n \le \infty$, $\mathcal{O}_n$ is a simple C^*-algebra, so $\mathcal{O}_n$ is the C^*-envelope of $\mathcal{S}_n$ (see [21]). This motivates our study of Kirchberg's theorems on $\mathcal{O}_n$ $(2 \le n \le \infty)$ in terms of the C^*-envelopes of operator systems.

After collecting prerequisites in Section 2, we prove an embedding theorem for operator systems motivated by Kirchberg's exact embedding theorem in Section 3. It gives a necessary and sufficient condition for embedding an operator system into $\mathcal{O}_2$ in terms of exactness of its C^*-envelopes. We extend these embeddability conditions to finite *minimal* tensor products of operator systems. We also discuss some nuclearity properties of operator systems which embed into $\mathcal{O}_2$.

In Section 4, we prove results on the embedding of operator systems of the form $\mathcal{S} \otimes_{\min=c} \mathcal{S}_2$ into $\mathcal{O}_2$ and obtain some equivalent conditions for their C^*-envelopes to be $*$-isomorphic either to $\mathcal{O}_2$ or to a C^*-subalgebra of $\mathcal{O}_2$. We also prove results on operator systems of the form $\mathcal{S} \otimes_{\min=c} \mathcal{S}_\infty$.

Finally, in Section 5, as an application of our results, we check the embeddability into $\mathcal{O}_2$ of some operator systems whose C^*-envelopes are already calculated. We describe the C^*-envelopes of some operator systems with tensor product factor $\mathcal{S}_2$ or $\mathcal{S}_\infty$, adding more operator systems to the short list with known C^*-envelopes.

2. Preliminaries

2.1. Cuntz algebras and Kirchberg's theorems. The *Cuntz algebra* $\mathcal{O}_n$, where $2 \le n < \infty$, is the universal unital C^*-algebra generated by isometries $s_1, s_2, \ldots, s_n$ satisfying $s_1 s_1^* + s_2 s_2^* + \cdots + s_n s_n^* = 1$. The *Cuntz algebra* $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by an infinite sequence of isometries $s_1, s_2, s_3, \ldots$ with mutually orthogonal range projections $s_j s_j^*$ which add up to the identity. (See [5].)

A finite set $\{t_j\}_{j=1}^n$ of isometries in a unital C^*-algebra A is said to satisfy the *Cuntz relation* if $t_1 t_1^* + t_2 t_2^* + \cdots + t_n t_n^* = 1$. A sequence $\{t_j\}_{j=1}^\infty$ of isometries satisfies the Cuntz relation if their range projections $\{t_j t_j^*\}_{j=1}^\infty$ are mutually orthogonal. The Cuntz algebras are independent of the choice of generating isometries.

A self-contained survey of the theorems stated below can be found in [19].

THEOREM 2.1 [5]. *For each $n \in \mathbb{N}$ and for $n = \infty$, the Cuntz algebra $\mathcal{O}_n$ is unital, separable, simple, nuclear and purely infinite.*

THEOREM 2.2 [4]. *The C^*-algebras $\mathcal{O}_2 \otimes_{C^*\text{-}\min} \mathcal{O}_2$ and $\mathcal{O}_2$ are isomorphic.*

THEOREM 2.3 [12, Theorem 2.8]. *A unital separable C^*-algebra A is exact if and only if it admits a unital embedding into $\mathcal{O}_2$.*

THEOREM 2.4 [12, Theorem 3.7]. *The tensor product $A \otimes_{C^*\text{-}\min} \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ if and only if A is unital, simple, separable and nuclear.*

THEOREM 2.5 [12, Theorem 7.2.6]. *For a simple, nuclear and separable C^*-algebra A, $A \cong A \otimes_{C^*\text{-}\min} \mathcal{O}_\infty$ if and only if A is purely infinite.*

THEOREM 2.6 ([19, Theorem 6.1.10], [20, Corollary 4.21]).

(i) *Every C^*-subalgebra of an exact C^*-algebra is again exact.*
(ii) *Every quotient of an exact C^*-algebra is again exact.*
(iii) *If A and B are exact, then so is $A \otimes_{C^*\text{-}\min} B$.*
(iv) *If A and B are simple C^*-algebras, then $A \otimes_{C^*\text{-}\min} B$ is also simple.*

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2.2. Operator systems. For further details on operator systems and their tensor products, see [9–11].

A concrete operator system is a unital self-adjoint subspace of $B(H)$ for some Hilbert space H. A C^**-cover* [8, Section 2] of an operator system $\mathcal{S}$ is a pair (A, i) consisting of a unital C^*-algebra A and a complete order embedding $i : \mathcal{S} \to A$ such that $i(A)$ generates the C^*-algebra A. The C^**-envelope*, $C_e^*(\mathcal{S})$, of an operator system $\mathcal{S}$ is a C^*-cover defined as the C^*-algebra generated by $\mathcal{S}$ in its injective envelope $I(\mathcal{S})$. From [8, Corollary 4.2], the C^*-envelope $C_e^*(\mathcal{S})$ has the following universal 'minimality' property: *identifying $\mathcal{S}$ with its image in $C_e^*(\mathcal{S})$, for any C^*-cover (A, i) of $\mathcal{S}$, there is a unique surjective unital $*$-homomorphism $\pi : A \to C_e^*(\mathcal{S})$ such that $\pi(i(s)) = s$ for every s in $\mathcal{S}$.*

Remark 2.7. If an operator system $\mathcal{S}$ has a simple C^*-cover (A, i), then using this minimality property π is injective and A is $*$-isomorphic to the C^*-envelope of $\mathcal{S}$.

From [21], for the Cuntz algebra $\mathcal{O}_n$ with generators $s_1, s_2, \ldots, s_n$ $(n \geq 2)$, the Cuntz operator system $\mathcal{S}_n$ is the operator system generated by $s_1, s_2, \ldots, s_n$, that is,

$$\mathcal{S}_n = \text{span}\{I, s_1, s_2, \ldots, s_n, s_1^*, s_2^*, \ldots, s_n^*\} \subset \mathcal{O}_n,$$

where I is the identity. Similarly, for the generators $s_1, s_2, \ldots$ of $\mathcal{O}_\infty$,

$$\mathcal{S}_\infty = \text{span}\{I, s_1, s_2, \ldots, s_1^*, s_2^*, \ldots\} \subset \mathcal{O}_\infty.$$

The following well-known fact follows directly from Remark 2.7 and Theorem 2.1.

Proposition 2.8 [21]. *$C_e^*(\mathcal{S}_n) = \mathcal{O}_n$ for $1 \leq n \leq \infty$.*

Kavruk *et al.* [10] introduced a lattice of tensor products of operator systems admitting a natural partial order: $\min \leq e \leq el, er \leq c \leq \max$. A natural operator system tensor product 'ess' arising from the enveloping C^*-algebras, namely, $\mathcal{S} \otimes_{\text{ess}} \mathcal{T} \subseteq \mathrm{C}_\mathrm{e}^*(\mathcal{S}) \otimes_{\max} \mathrm{C}_\mathrm{e}^*(\mathcal{T})$, was defined in [6].

Kavruk *et al.* [11] formalised the notion of quotient for operator systems, leading to the notion of *exactness*. An operator system $\mathcal{S}$ is said to be exact if for every unital C^*-algebra A and a closed ideal I in A, we have the exact sequence

$$0 \longrightarrow \mathcal{S} \hat{\otimes}_{\min} I \longrightarrow \mathcal{S} \hat{\otimes}_{\min} A \to \mathcal{S} \hat{\otimes}_{\min} (A/I) \to 0.$$

Given two operator system tensor products α and β, an operator system $\mathcal{S}$ is said to be (α, β)*-nuclear* if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$, that is, $\mathcal{S} \otimes_\alpha \mathcal{T} = \mathcal{S} \otimes_\beta \mathcal{T}$. An operator system $\mathcal{S}$ is said to be C^**-nuclear* if $\mathcal{S} \otimes_{\min} A = \mathcal{S} \otimes_{\max} A$ for all unital C^*-algebras A. For a C^*-algebra A, $A \otimes_\mathrm{c} \mathcal{S} = A \otimes_{\max} \mathcal{S}$ for every operator system $\mathcal{S}$ [10, Theorem 6.7], that is, A is $(\min, \max)$-nuclear if and only if it is $(\min, \mathrm{c})$-nuclear. Exactness is one of the few intrinsic properties of operator systems that has been used as a tool in characterising nuclearity properties of operator systems (see Kavruk [9]).

Theorem 2.9 [17, Proposition 1.1 and Corollary 2.8]. *$\mathcal{S}_n$ is $(\min, \mathrm{c})$-nuclear but not $(\min, \max)$-nuclear.*

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Remark 2.10. Since $C_e^*(\mathcal{S}_n) = \mathcal{O}_n$ is C^*-nuclear (Theorem 2.1), by [7, Proposition 4.2], $\mathcal{S}_n$ is (min, ess)-nuclear. By [7, Proposition 5.2], for $1 \le n < \infty$, $\mathcal{S}_n$ is not (ess, max)-nuclear. This gives an alternative proof that $\mathcal{S}_n$ is not (min, max)-nuclear.

Kirchberg and Wassermann [13, Section 3] introduced the universal C^*-algebra $C_u^*(\mathcal{S})$, which has the following universal 'maximality' property: *every unital completely positive map $\phi : \mathcal{S} \to A$, where A is a unital C^*-algebra, extends uniquely to a unital $*$-homomorphism $\pi : C_u^*(\mathcal{S}) \to A$.*

A subspace J of an operator system $\mathcal{S}$ is said to be a *kernel* if it is the kernel of some unital completely positive map from $\mathcal{S}$ into some operator system $\mathcal{T}$. From [11, Corollary 3.8], a subspace J of an operator system $\mathcal{S}$ is a kernel of $\mathcal{S}$ if and only if J is an intersection of a closed two-sided ideal in $C_u^*(\mathcal{S})$ with $\mathcal{S}$. We note that $C_u^*(\mathcal{S})$ is never simple.

Proposition 2.11. *For an operator system $\mathcal{S}$ with* $\dim(\mathcal{S}) > 1$, *$C_u^*(\mathcal{S})$ is not simple.*

Proof. Let $J \subset \mathcal{S}$ be a kernel in $\mathcal{S}$. By [11, Corollary 3.8], $J = I \cap \mathcal{S}$ for some closed two-sided ideal I in $C_u^*(\mathcal{S})$. If $C_u^*(\mathcal{S})$ is simple, then either $J = (0)$ or $J = \mathcal{S}$. But, by [9, Corollary 6.12], any operator system with dimension greater than 1 has a nontrivial kernel, which is a contradiction. □

By the minimality property of C^*-envelopes, there is a surjective $*$- homomorphism $\sigma_{\mathcal{S}} : C_u^*(\mathcal{S}) \to C_e^*(\mathcal{S})$ that fixes $\mathcal{S}$. The simplicity of $C_u^*(S)$ implies simplicity of $C_e^*(\mathcal{S})$ (Remark 2.7). Therefore, an operator system kernel has no relation with the simplicity of its C^*-envelope.

An operator system $\mathcal{S}$ for which $\sigma_{\mathcal{S}}$ is a $*$-isomorphism is said to be *universal* [13]. In particular, this property implies that if $\sigma_S : \mathcal{S} \to A$ is any C^*-cover of S, then $A \cong C_u^*(\mathcal{S}) \cong C_e^*(\mathcal{S})$. From Proposition 2.11, we have the following corollary.

Corollary 2.12. *There does not exist any universal operator system $\mathcal{S}$ with simple C^*-cover unless $\mathcal{S} = \mathbb{C}$.*

In general, the isomorphism between operator systems need not extend to their C^*-covers, but the following result from [3] is quite useful.

Theorem 2.13 [3, Theorem 2.2.5]. *For $\mathcal{S} \subseteq C_e^*(\mathcal{S})$ and $\mathcal{T} \subseteq C_e^*(\mathcal{T})$ and for any complete order isomorphism ϕ of $\mathcal{S}$ onto $\mathcal{T}$, there exists a $*$-isomorphism $\hat{\phi}$ from $C_e^*(\mathcal{S})$ onto $C_e^*(\mathcal{T})$ with $\hat{\phi}|_{\mathcal{S}} = \phi$.*

An operator subsystem $\mathcal{S}$ of a unital C^*-algebra A is said *to contain enough unitaries of A* if the unitaries in $\mathcal{S}$ generate A as a C^*-algebra [11, Section 9]. The next lemma is folklore.

Lemma 2.14. *For operator systems $\mathcal{S}$ and $\mathcal{T}$ with either both $C_e^*(\mathcal{S})$ and $C_e^*(\mathcal{T})$ simple, or both $\mathcal{S}$ and $\mathcal{T}$ having enough unitaries of $C_e^*(\mathcal{S})$ and $C_e^*(\mathcal{T})$, respectively, the inclusion of $\mathcal{S} \otimes_{\min} \mathcal{T}$ into $C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{T})$ extends to a $*$-isomorphism between $C_e^*(\mathcal{S} \otimes_{\min} \mathcal{T})$ and $C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{T})$, that is,*

$$C_e^*(\mathcal{S} \otimes_{\min} \mathcal{T}) \cong C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{T}).$$

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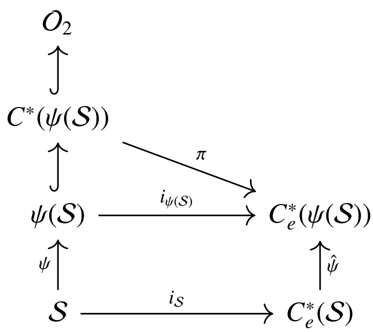
Proof. Consider the natural inclusions $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow C_e^*(\mathcal{S})$ and $i_{\mathcal{T}} : \mathcal{T} \hookrightarrow C_e^*(\mathcal{T})$. Then $i_{\mathcal{S}} \otimes i_{\mathcal{T}} : \mathcal{S} \otimes_{\min} \mathcal{T} \hookrightarrow C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{T})$ is a C^*-cover of $\mathcal{S} \otimes_{\min} \mathcal{T}$. If $C_e^*(\mathcal{S})$ and $C_e^*(\mathcal{T})$ are simple, so is the cover by Proposition 2.6(iv) and the statement follows by Remark 2.7. For the case of enough unitaries, $\mathcal{S} \otimes_{\min} \mathcal{T}$ has enough unitaries of $C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{T})$. By [11, Proposition 5.6], $C_e^*(\mathcal{S} \otimes_{\min} \mathcal{T}) = C_e^*(\mathcal{S}) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T})$ up to $*$-isomorphism that fixes $\mathcal{S} \otimes_{\min} \mathcal{T}$. □

3. Embedding of exact operator systems into O_2

The relationship between an operator system and its C^*-envelope is a mysterious one. In [13], Kirchberg and Wassermann gave an example of a universal separable exact operator system $\mathcal{S}$ with nonexact C^*-envelope. Another interesting example was recently constructed by Lupini in [14], namely, the Gurarij operator system $\mathbb{GS}$, which is exact but does not admit any complete order embedding into any exact C^*-algebra. Thus, in general, unlike C^*-algebras, separable exact operator systems need not embed into O_2. But, in the next theorem we prove an embedding theorem that shows that it is the exactness of the C^*-envelope, rather than that of the operator system, that makes an operator system embeddable into O_2.

Theorem 3.1. *For a separable operator system $\mathcal{S}$, the C^*-envelope $C_e^*(\mathcal{S})$ is exact if and only if there exists a unital complete order embedding of $\mathcal{S}$ into O_2.*

Proof. For the if part, let $\psi : \mathcal{S} \to O_2$ be a complete order embedding. By Theorem 2.13, ψ can be extended to a $*$-isomorphism on the C^*-envelope of $\mathcal{S}$, say, $\hat{\psi} : C_e^*(\mathcal{S}) \to C_e^*(\psi(\mathcal{S}))$, such that $\hat{\psi}|_{\mathcal{S}} = \psi$. Consider the C^*-algebra generated by $\psi(\mathcal{S}) \subset O_2$, $C^*(\psi(\mathcal{S})) \subseteq O_2$. By Theorem 2.3, $C^*(\psi(\mathcal{S}))$, being a C^*-subalgebra of O_2, is exact. By the universal (minimality) property of C^*-envelopes of operator systems, there is a surjective $*$-homomorphism $\pi : C^*(\psi(\mathcal{S})) \to C_e^*(\psi(\mathcal{S}))$ such that the following diagram commutes:



Here $i_{\mathcal{S}}$ and $i_{\psi(\mathcal{S})}$ denote the natural complete order inclusions of $\mathcal{S}$ and $\psi(\mathcal{S})$ into their respective C^*-envelopes. Thus, $C_e^*(\psi(\mathcal{S}))$ is the $*$-homomorphic image of an exact C^*-algebra $C^*(\psi(\mathcal{S}))$ and so exact by Theorem 2.6(ii). Therefore, $\hat{\psi}^{-1}(C_e^*(\psi(\mathcal{S}))) = C_e^*(\mathcal{S})$ is exact.

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Conversely, let $C_e^*(\mathcal{S})$ be exact. From Kirchberg's embedding theorem, there is a complete order embedding ϕ of $C_e^*(\mathcal{S})$ into O_2. Then $\phi \circ i_{\mathcal{S}} : \mathcal{S} \to O_2$ is the required unital complete order embedding of $\mathcal{S}$ into O_2, where $i_{\mathcal{S}}$ denotes the natural complete order inclusion of $\mathcal{S}$ into $C_e^*(\mathcal{S})$. □

Corollary 3.2. *For an exact separable operator system $\mathcal{S}$ containing enough unitaries of its C^*-envelope, $\mathcal{S}$ embeds into O_2.*

Proof. By [11, Proposition 10.12], for the case when $\mathcal{S}$ contains enough unitaries of $C_e^*(\mathcal{S})$, exactness of $\mathcal{S}$ is equivalent to exactness of $C_e^*(\mathcal{S})$. Therefore, the result follows from Theorem 3.1. □

Proposition 3.3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be separable operator systems. If $C_e^*(\mathcal{T}_1)$ and $C_e^*(\mathcal{T}_2)$ are exact, then the operator system $\mathcal{T}_1 \otimes_{min} \mathcal{T}_2$ embeds into O_2. The converse holds if either both $C_e^*(\mathcal{T}_1)$ and $C_e^*(\mathcal{T}_2)$ are simple, or both $\mathcal{T}_1$ and $\mathcal{T}_2$ contain enough unitaries of $C_e^*(\mathcal{T}_1)$ and $C_e^*(\mathcal{T}_2)$, respectively.*

Proof. Let $C_e^*(\mathcal{T}_1)$ and $C_e^*(\mathcal{T}_2)$ be exact C^*-algebras. By Kirchberg's embedding theorem (Theorem 3.1), there exist complete order embeddings $\phi_1 : C_e^*(\mathcal{T}_1) \hookrightarrow O_2$ and $\phi_2 : C_e^*(\mathcal{T}_2) \hookrightarrow O_2$. Since C^*- min is injective, we have the complete order isomorphism

$$\phi_1 \otimes_{\min} \phi_2 : C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2) \hookrightarrow O_2 \otimes_{C^*\text{-}\min} O_2. \tag{3.1}$$

The operator system min tensor product is injective [10, Theorem 4.6], so, using the natural complete order inclusions $i_{\mathcal{T}_1}$ and $i_{\mathcal{T}_2}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ into their respective C^*-envelopes, gives the complete order isomorphism $i_{\mathcal{T}_1} \otimes i_{\mathcal{T}_2}$ of $\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2$ into $C_e^*(\mathcal{T}_1) \otimes_{\min} C_e^*(\mathcal{T}_2)$. Since the operator system min tensor product of C^*-algebras embeds complete order isomorphically into their C^*- min tensor product [10, Corollary 4.10], the complete order isomorphism can be considered as

$$i_{\mathcal{T}_1} \otimes_{\min} i_{\mathcal{T}_2} : \mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \hookrightarrow C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2). \tag{3.2}$$

The isomorphism $O_2 \otimes_{C^*\text{-}\min} O_2 \cong O_2$ (Theorem 2.2) and the composition of complete order isomorphisms in (3.1) and (3.2) give the required complete order isomorphism

$$\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \hookrightarrow O_2.$$

Conversely, suppose that there is an embedding of $\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2$ into O_2. If $C_e^*(\mathcal{T}_i)$ is simple for $i = 1, 2$ or $\mathcal{T}_i$, $i = 1, 2$, contains enough unitaries of $C_e^*(\mathcal{T}_i)$, then, by Lemma 2.14,

$$C_e^*(\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2) \cong C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2),$$

which is separable (being the minimal C^*-tensor product of separable C^*-algebras). By Theorem 3.1, $C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2)$ is exact and, for each i, the C^*-subalgebras $C_e^*(\mathcal{T}_i)$ (through the injective $*$-homomorphisms of $C_e^*(\mathcal{T}_1)$ and $C_e^*(\mathcal{T}_2)$ given by $C_e^*(\mathcal{T}_1) \ni a_1 \mapsto a_1 \otimes 1 \in C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2)$, $C_e^*(\mathcal{T}_2) \ni a_2 \mapsto 1 \otimes a_2 \in C_e^*(\mathcal{T}_1) \otimes_{C^*\text{-}\min} C_e^*(\mathcal{T}_2)$) are exact (Theorem 2.6(iii)). □

Since min is associative [10, Theorem 4.6], the preceding proposition can be extended to a finite tensor product.

COROLLARY 3.4. *Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ be separable operator systems. If $C_e^*(\mathcal{T}_i)$ is exact for $1 \le i \le m$, then the operator system $\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}_m$ embeds into $\mathcal{O}_2$. The converse holds if either $C_e^*(\mathcal{T}_i)$ is simple for all i or each $\mathcal{T}_i$ contains enough unitaries of $C_e^*(\mathcal{T}_i)$ for $1 \le i \le m$.*

PROOF. Suppose that $C_e^*(\mathcal{T}_i)$ is exact for all $i = 1, 2, \ldots, m$. Then, using the associativity of min and C^*- min and the complete order isomorphism $\bigotimes_{i=1}^m \mathcal{O}_2 \cong \mathcal{O}_2$ [19, Corollary 5.2.4] in Proposition 3.3, we have the required complete order isomorphism

$$\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}_m \hookrightarrow \mathcal{O}_2.$$

For the converse, the associativity of min and C^*- min, extends Lemma 2.14 to finitely many factors, so that if either $C_e^*(\mathcal{T}_i)$ is simple for all i or each $\mathcal{T}_i$ contains enough unitaries of $C_e^*(\mathcal{T}_i)$ for $1 \le i \le m$,

$$C_e^*(\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}_m) \cong C_e^*(\mathcal{T}_1) \otimes_{C^*\text{- min}} C_e^*(\mathcal{T}_2) \otimes_{C^*\text{- min}} \cdots \otimes_{C^*\text{- min}} C_e^*(\mathcal{T}_m).$$

Therefore, the embedding $\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}_m$ into $\mathcal{O}_2$ implies the exactness of $C_e^*(\mathcal{T}_1 \otimes_{\min} \mathcal{T}_2 \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}_m)$ (Theorem 3.1) and hence of each of its C^*-subalgebras $C_e^*(\mathcal{T}_i)$ for $i = 1, 2, \ldots, m$. □

Nuclearity properties of operator systems have been characterised in terms of various intrinsic properties (see [9]), and the relation with the nuclearity of their C^*-envelope was studied in [7]). *The double commutant expectation property (DCEP)* was introduced in [11] as a generalisation of the *weak expectation property (WEP)*. An operator system $\mathcal{S}$ has the DCEP if for every complete order embedding $\mathcal{S} \subset B(H)$ there exists a completely positive map $\varphi : B(H) \to \mathcal{S}''$ fixing $\mathcal{S}$. A C^*-algebra has DCEP if and only it has WEP. In the next corollary, we give some nuclearity properties of operator systems embeddable into $\mathcal{O}_2$.

COROLLARY 3.5. *For a separable operator system $\mathcal{S}$ having an embedding into $\mathcal{O}_2$:*

(i) *$\mathcal{S}$ is exact and hence* (min, el)*-nuclear;*

(ii) *$C_e^*(\mathcal{S})$ is nuclear if and only if $C_e^*(\mathcal{S})$ has the DCEP and then $\mathcal{S}$ is* (min, ess)*-nuclear;*

(iii) *if $\mathcal{S}$ has enough unitaries of $C_e^*(\mathcal{S})$, $\mathcal{S}$ is* (min, ess)*-nuclear if and only if $C_e^*(\mathcal{S})$ has the DCEP (or WEP).*

PROOF. (i) Since exactness passes to operator subsystems [11, Corollary 5.8] and (min, el)-nuclearity is equivalent to exactness of the operator system [11, Theorem 5.7], we have (i) from Theorem 3.1.

(ii) A unital C^*-algebra is nuclear if and only if it is exact and has DCEP (see [18, Section 17] and [11, Section 7]) and nuclearity of the C^*-envelope implies (min, ess)-nuclearity of the operator system [7, Proposition 4.2]. Thus, Theorem 3.1 implies the result.

(iii) For an operator system having enough unitaries in $C_e^*(\mathcal{S})$, (min, ess)-nuclearity is equivalent to nuclearity of $C_e^*(\mathcal{S})$ [7, Theorem 4.3], so (iii) follows from (ii). □

4. Tensor product with $\mathcal{S}_2$ and $\mathcal{S}_\infty$

Next we give a characterisation of those operator systems which are absorbed by tensoring them finitely many times with $\mathcal{S}_2$, in terms of their C^*-envelopes.

PROPOSITION 4.1. *Suppose that $\mathcal{S}$ is a separable operator system with simple C^*-envelope. Then $C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2) \cong \mathcal{O}_2$ if and only if $C_e^*(\mathcal{S})$ is a nuclear C^*-algebra.*

PROOF. Since $C_e^*(\mathcal{S})$ is simple, by Lemma 2.14,

$$C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{S}_2) \cong C_e^*(\mathcal{S} \otimes_{\min} \mathcal{S}_2) \cong \mathcal{O}_2$$

and so $C_e^*(\mathcal{S})$ is nuclear by Theorem 2.4. Conversely, if $C_e^*(\mathcal{S})$ is a nuclear C^*-algebra, then, by Lemma 2.14, $C_e^*(\mathcal{S} \otimes_{\min} \mathcal{S}_2) \cong C_e^*(\mathcal{S}) \otimes_{C^*\text{-min}} C_e^*(\mathcal{S}_2)$ and then, by Theorem 2.4 and Proposition 2.8, $C_e^*(\mathcal{S} \otimes_{\min} \mathcal{S}_2) \cong C_e^*(\mathcal{S}_2) \cong \mathcal{O}_2$. □

We have given the proof for an operator system of the form $\mathcal{S} \otimes_{\min} \mathcal{S}_2$, but it can be generalised to $\mathcal{S} \otimes_{\min} \bigotimes_{i=1}^m \mathcal{S}_2$, using the identification $\bigotimes_{i=1}^m \mathcal{O}_2 = \mathcal{O}_2$ [19, Corollary 5.2.4].

COROLLARY 4.2. *For any simple, unital, separable and nuclear C^*-algebra A, we have $C_e^*(A \otimes_{\text{min=max}} \mathcal{S}_2) \cong \mathcal{O}_2$.*

PROOF. The assertion follows directly from Proposition 4.1 and the fact that $C_e^*(A) = A$ [7, Proposition 2.3]. □

Let A and B be unital C^*-algebras. Two completely positive maps $\phi, \psi : A \to B$ are said to be *unitarily equivalent*, denoted by $\phi \sim_u \psi$, if there is a unitary u in B such that $u\psi(a)u^* = \phi(a)$ for all $a \in A$ [19, Definition 1.1.15]. If, for every $\varepsilon > 0$ and for every finite subset F of A, there is a unitary u in B with $\|u\psi(a)u^* - \phi(a)\| \leq \varepsilon$ for all $a \in F$, then ϕ and ψ are said to be *approximately unitarily equivalent*, denoted by $\phi \approx_u \psi$. Approximate unitary equivalence of completely positive maps has been used extensively in [19, Theorems 5.1.1 and 6.3.8] to prove various isomorphisms of C^*-algebras involving $\mathcal{O}_2$.

COROLLARY 4.3. *For a separable operator system $\mathcal{S}$ with simple C^*-envelope, the following statements are equivalent:*

(i) *$C_e^*(\mathcal{S})$ is exact;*
(ii) *$\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2$ embeds into $\mathcal{O}_2$;*
(iii) *$C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2)$ is exact;*
(iv) *$C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2)$ can be embedded into $\mathcal{O}_2$ as a C^*-subalgebra.*

Moreover, if any one of the above holds, then there exist injective $*$-homomorphisms, $\rho : \mathcal{O}_2 \to C_e^*(\mathcal{S}) \otimes_{C^*\text{- min}} \mathcal{O}_2 \cong C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2)$ and $\gamma : C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2) \to \mathcal{O}_2$, such that $\gamma \circ \rho \approx_u \mathrm{id}_{\mathcal{O}_2}$ and, in addition, if $\rho \circ \gamma \approx_u \mathrm{id}_{C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2)}$, then $C_e^*(\mathcal{S})$ is nuclear and $C_e^*(\mathcal{S} \otimes_{\text{min=c}} \mathcal{S}_2) \cong \mathcal{O}_2$.

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Proof. Since $C_e^*(\mathcal{S})$ and $C_e^*(\mathcal{S}_2) = \mathcal{O}_2$ are both simple and exact, by the converse of Proposition 3.3, $\mathcal{S} \otimes_{\min} \mathcal{S}_2$ embeds into $\mathcal{O}_2$. Also, if $\mathcal{S} \otimes_{\min} \mathcal{S}_2$ embeds into $\mathcal{O}_2$, then $C_e^*(\mathcal{S})$ is exact by Proposition 3.3. Thus, (i) and (ii) are equivalent. Theorem 3.1 implies that (ii) and (iii) are equivalent. Equivalence of (iii) and (iv) follows using Kirchberg's exact embedding theorem (Theorem 2.3).

Now, suppose that $\mathcal{S}$ satisfies any of these equivalent assertions and let ρ and γ be the injective $*$-homomorphisms given by (ii) and (iv). By [19, Theorem 5.1.1], any injective $*$-homomorphism from $\mathcal{O}_2$ into $\mathcal{O}_2$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$, so $\gamma \circ \rho \approx_u \mathrm{id}_{\mathcal{O}_2}$. If, further, $\rho \circ \gamma \approx_u \mathrm{id}_{C_e^*(\mathcal{S} \otimes_{\min=c} \mathcal{S}_2)}$, then $C_e^*(\mathcal{S} \otimes_{\min=c} \mathcal{S}_2)$ is isomorphic to $\mathcal{O}_2$ by [19, Theorem 6.3.8](ii) and so $C_e^*(\mathcal{S})$ is nuclear by Proposition 4.1. □

Remark 4.4. If the complete order embedding of $\mathcal{S} \otimes_{\min=c} \mathcal{S}_2$ into $\mathcal{O}_2$ in Corollary 4.3(ii) is such that $\mathcal{O}_2$ is a C^*-cover, then trivially $C_e^*(\mathcal{S} \otimes_{\min=c} \mathcal{S}_2) = \mathcal{O}_2$ (Remark 2.7).

A simple C^*-algebra A is said to be *purely infinite* if A is not isomorphic to $\mathbb{C}$ and, for every pair of nonzero elements a and b in A, there exists x in A such that $b = x^*ax$; [19, Proposition 4.1.1] gives six equivalent conditions for a unital and simple C^*-algebra to be purely infinite.

We now characterise those operator systems whose C^*-envelopes remain unaffected by tensoring finitely many times with $\mathcal{S}_\infty$. The proof follows that of Proposition 4.1, using Kirchberg's characterisation of simple, purely infinite and nuclear C^*-algebras (Theorem 2.5).

Proposition 4.5. *Let $\mathcal{S}$ be a separable operator system with simple C^*-envelope $C_e^*(\mathcal{S})$. Then $C_e^*(\mathcal{S})$ is a nuclear and purely infinite C^*-algebra if and only if $C_e^*(\mathcal{S} \otimes_{\min=c} \mathcal{S}_\infty)$ is isomorphic to $C_e^*(\mathcal{S})$.*

Again, we stated the last proposition for operator systems of the form $\mathcal{S} \otimes_{\min=c} \mathcal{S}_\infty$, but it can be generalised to operator systems of the form $\mathcal{S} \otimes_{\min=c} \bigotimes_{i=1}^m \mathcal{S}_\infty$, using the identification $\mathcal{O}_\infty = \bigotimes_{i=1}^m \mathcal{O}_\infty$ [19, Theorem 7.2.6].

Corollary 4.6. *For a separable operator system $\mathcal{S}$ with simple, nuclear and purely infinite C^*-envelope, there exists a complete order embedding of $\mathcal{S} \otimes_{\min=c} \mathcal{S}_\infty$ into $C_e^*(\mathcal{S})$.*

Corollary 4.7. *Let A be a unital, simple, nuclear, separable and purely infinite C^*-algebra. Then $C_e^*(A \otimes_{\min=\max} \mathcal{S}_\infty) \cong A$.*

Proof. For a unital C^*-algebra, $C_e^*(A) = A$ [7, Proposition 2.3]. □

Corollary 4.8. *For a separable operator system $\mathcal{S}$ with simple and nuclear C^*-envelope $C_e^*(\mathcal{S})$, if $\mathcal{S} \cong \mathcal{S} \otimes_{\min} \mathcal{S}_\infty$, then $\mathcal{S}$ is infinite dimensional and $C_e^*(\mathcal{S})$ is purely infinite.*

Proof. If $\mathcal{S} \otimes_{\min} \mathcal{S}_\infty \cong \mathcal{S}$, the statement follows from Theorem 2.13 and Proposition 4.5. □

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REMARK 4.9. The converse of Corollary 4.8 is not known. Note that $\mathcal{S}_n$ is a finite-dimensional operator system with purely infinite and simple $C_e^*(\mathcal{S}_n) = \mathcal{O}_n$ (Theorem 2.1), but $\mathcal{S}_n \not\cong \mathcal{S}_n \otimes_{\min} \mathcal{S}_\infty$.

5. Applications

Our results can be applied to operator systems with known C^*-envelopes to check their embeddability into $\mathcal{O}_2$, and to describe the C^*-envelopes obtained after tensoring with $\mathcal{S}_2$ or $\mathcal{S}_\infty$.

In [6], an operator system $\mathcal{S}(\mathfrak{u})$ is associated to $C^*(G)$, the full group C^*-algebra of the group G for a countable discrete group G with generating set $\mathfrak{u}$, by setting $\mathcal{S}(\mathfrak{u}) := \mathrm{span}\{1, u, u^* : u \in \mathfrak{u}\} \subset C^*(G)$. It was shown in [6, Proposition 2.2] that $C_e^*(\mathcal{S}(\mathfrak{u})) = C^*(G)$. On similar lines in [7], another natural operator system was associated to a reduced group C^*-algebra by $\mathcal{S}_r(\mathfrak{u}) := \mathrm{span}\{1, u, u^* : u \in \mathfrak{u}\} \subset C_r^*(G)$. Further, $C_e^*(\mathcal{S}_r(\mathfrak{u})) = C_r^*(G)$ [7, Proposition 2.9].

Kavruk *et al.* [10] associated an operator system to a finite graph G with n vertices: $\mathcal{S}_G = \mathrm{span}\{\{E_{i,j} : (i, j) \in G\} \cup \{E_{i,i} : 1 \leq i \leq n\}\}$ is a finite-dimensional operator subsystem of $M_n(\mathbb{C})$, where $\{E_{i,j}\}$ is the standard system of matrix units in $M_n(\mathbb{C})$ and (i, j) denotes (an unordered) edge in G. For a connected graph G on n vertices, $C_e^*(\mathcal{S}_G) = M_n$ [16, Theorem 3.2].

EXAMPLE 5.1. From Theorem 3.1, the following operator systems embed into $\mathcal{O}_2$.

(i) $\mathcal{S}(\mathfrak{u}) \subseteq C^*(G)$, where G is a finitely generated discrete amenable group.
(ii) $\mathcal{S}_r(\mathfrak{u}) \subseteq C_r^*(G)$, where G is any exact discrete group. In particular, for $G = F_n$, the free group on n generators, $\mathcal{S}_r(\mathfrak{u}_n) \subset C_r^*(F_n)$, embeds into $\mathcal{O}_2$.
(iii) $\mathcal{S}_G \subset M_n$, where G is a connected graph on n vertices, embeds into $\mathcal{O}_2$.

On the other hand, $\mathcal{S}(\mathfrak{u}_n) \subseteq C^*(F_n)$ does not embed into $\mathcal{O}_2$.

EXAMPLE 5.2. From Theorems 2.1 and 2.9 and Corollary 4.2, $C_e^*(\mathcal{S}_n \otimes_{\min=c} \mathcal{S}_2) \cong \mathcal{O}_2$ for $2 \leq n \leq \infty$.

EXAMPLE 5.3. From Propositions 4.5 and 2.8 and Theorem 2.1, $C_e^*(\mathcal{S}_n \otimes_{\min=c} \mathcal{S}_\infty) \cong \mathcal{O}_n$ for $2 \leq n \leq \infty$ and $C_e^*(M_n \otimes_{\min=c} \mathcal{S}_2) \cong \mathcal{O}_2$ for all $n \in \mathbb{N}$.

We know that $C^*(G)$ is never simple unless $G = \mathbb{C}$, as it always has a one-dimensional quotient coming from the trivial representation of G and has an ideal of co-dimension 1, called *the augmented ideal*. But, for $n \geq 2$, $C_r^*(F_n)$ (the reduced group algebra of a free group with n generators) is always simple.

EXAMPLE 5.4. Consider $\mathcal{S}_r(\mathfrak{u}_n) \subseteq C_r^*(F_n)$ for $n \geq 2$. Then $C_e^*(\mathcal{S}_r(\mathfrak{u}_n)) = C_r^*(F_n)$ is simple, separable, unital and exact but not nuclear, and $C_e^*(\mathcal{S}_r(\mathfrak{u}_n) \otimes_{\min} \mathcal{S}_n) \cong C_r^*(F_n) \otimes_{\min} \mathcal{O}_2$ is a proper C^*-subalgebra of $\mathcal{O}_2$.

EXAMPLE 5.5. Consider $\mathcal{S}_G$, the graph operator system of a connected graph G on n vertices. Then $C_e^*(\mathcal{S}_G \otimes_{\min} \mathcal{S}_2) \cong M_n \otimes_{C^*\text{-}\min} \mathcal{O}_2 \cong \mathcal{O}_2$.

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Argerami and Farenick [1, 2] defined operator systems generated by a single bounded linear operator T acting on a complex Hilbert space $\mathcal{H}$ as the unital self-adjoint subspace $\mathcal{OS}(T) = \text{span}\{1, T, T^*\} \subset B(\mathcal{H})$.

EXAMPLE 5.6. For $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in (\mathbb{C}^\times)^d$, the irreducible weighted unilateral shift with weights $\xi_1, \xi_2, \ldots, \xi_d$ is the operator $W(\xi)$ on $\mathbb{C}^{d+1}$ given by the matrix

$$W(\xi) = \begin{bmatrix} 0 & & & & 0 \\ \xi_1 & 0 & & & \\ & \xi_2 & \ddots & & \\ & & \ddots & 0 & \\ & & & \xi_d & 0 \end{bmatrix}$$

and $C_e^*(\mathcal{OS}(W(\xi))) = M_{d+1}(\mathbb{C})$ [1, Proposition 3.2]. By Corollary 4.3, $\mathcal{OS}(W(\xi))$ and $\mathcal{OS}(W(\xi)) \otimes_{\min} \mathcal{S}_2$ embed into $\mathcal{O}_2$ and $C_e^*(\mathcal{OS}(W(\xi)) \otimes_{\min} \mathcal{S}_2) \cong \mathcal{O}_2$ by Proposition 4.1.

An operator J on an n-dimensional Hilbert space $\mathcal{H}$ is a basic Jordan block if there is an orthonormal basis of $\mathcal{H}$ for which J has a matrix representation of the form

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & \lambda \end{bmatrix}$$

for some $\lambda \in \mathbb{C}$.

EXAMPLE 5.7. For $J = \bigoplus_{k=1}^{\infty} J_{m_k}(\lambda) \in B(l^2(\mathbb{N}))$ with $m := \sup\{m_k : k \in \mathbb{N}\} < \infty$, $C_e^*(\mathcal{OS}(J)) = M_m(\mathbb{C})$ [2, Proposition 2.2]. Thus, $\mathcal{OS}(J)$ and $\mathcal{OS}(J) \otimes_{\min=c} \mathcal{S}_2$ embed into $\mathcal{O}_2$ and $C_e^*(\mathcal{OS}(J) \otimes_{\min=c} \mathcal{S}_2) \cong \mathcal{O}_2$ by Proposition 4.1 and Corollary 4.3.

EXAMPLE 5.8. Suppose that $J = \bigoplus_{k=1}^{n} (J_{m_k}(\lambda_k) \otimes I_{d_k})$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ real and $\max\{m_2, \ldots, m_{n-1}\} \leq \min\{m_1, m_n\}$. By [2, Corollary 2.12], $C_e^*(\mathcal{OS}(J))$ is a nuclear, simple, separable C^*-algebra for the cases $m_1 = 1, m_n \geq 2, |\lambda_1 - \lambda_n| \leq \cos(\pi/(m_n + 1))$ and $m_1 \geq 2, m_n = 1, |\lambda_1 - \lambda_n| \leq \cos(\pi/(m_1 + 1))$. Therefore, for these cases, $\mathcal{OS}(J)$ and $\mathcal{OS}(J) \otimes_{\min} \mathcal{S}_2$ embed into $\mathcal{O}_2$ and $C_e^*(\mathcal{OS}(J) \otimes_{\min} \mathcal{S}_2) \cong \mathcal{O}_2$.

EXAMPLE 5.9. By Corollary 4.7 and Theorem 2.1, $C_e^*(\mathcal{O}_n \otimes_{\min=\max} \mathcal{S}_\infty) \cong \mathcal{O}_n$ for n with $2 \leq n \leq \infty$.

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