A NOTE ON THE CONSTRUCTION OF PROJECTIVE PLANES FROM GROUPS

P. B. KIRKPATRICK

1. Introduction. André (**1**) gave a construction for translation planes from abelian groups possessing "congruences" of subgroups. Schwerdtfeger (**3**) constructed the plane over a field F from the group $\mathscr{G}_F$ of substitutions $x \rightarrow ax + b$ $(a, b \in F; a \neq 0)$. In this note we describe a construction (inspired by Schwerdtfeger's work), from groups, of planes which are duals of near-field planes.

If a plane is (l, m)-transitive (cf. **2**, p. 67) for some pair of distinct lines l, m, then the central collineations ϕ with axis m and centre on l may be identified with the "proper" points (that is, points not on l or m) of the plane once an origin O is chosen (not on l or m):

$$\phi \leftrightarrow O^{\phi}.$$

Thus, the "proper" part of the plane may be considered as a group, isomorphic to the group of substitutions $x \rightarrow ax + b$ $(a \neq 0)$ over the system K^0 obtained by reversing multiplication in the near-field K attached to the dual plane.

Every (l, m)-transitive plane $(l \neq m)$, except the trivial plane of order 2, may be obtained by the construction to be described in § 2; (l, l)-transitive planes are, of course, translation planes.

2. A construction for (l, m)-transitive planes. Schwerdtfeger (**3**) constructed the plane over a field F as follows: he took as points the elements of $\mathscr{G}_F$ and as lines cosets of centralizers $\mathscr{C}(X)$ of elements X of $\mathscr{G}_F$. A projective plane with two lines removed is obtained. The plane is completed by taking as new points classes of "left-parallel" lines [left cosets of a line $\mathscr{C}(X)$] and classes of "right-parallel" lines [right cosets].

Provided $F \not\cong \mathrm{GF}(2)$, $\mathscr{G}_F$ satisfies the following condition on a group $\mathscr{G}$:

(*) $\mathscr{G}$ contains two non-trivial subgroups $\mathscr{H}$ and $\mathscr{K}$ such that

(i) $\mathscr{H} \triangleleft \mathscr{G}$,

(ii) $\mathscr{H} \cap \mathscr{K} = 1$,

(iii) $\mathscr{H} \cup (\bigcup_{H \in \mathscr{H}} H^{-1}\mathscr{K}H) = \mathscr{G}$,

(iv) for all $A \notin \mathscr{K}$, $B \notin \mathscr{K}$, $(A\mathscr{K}B) \cap \mathscr{K}$ contains exactly one element.

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To see this, take $\mathscr{H}$ to be the normal subgroup of substitutions $x \to x + b$, and $\mathscr{K}$ the centralizer in $\mathscr{G}_F$ of some element $A \notin \mathscr{H}$.

Since the centralizer of any element of $\mathscr{G}_F$ outside $\mathscr{H}$ is conjugate to $\mathscr{K}$, the lines of the incomplete plane are the sets $A\mathscr{K}B$ $(A, B \in \mathscr{G}_F)$ and the cosets of $\mathscr{H}$. $\mathscr{H}$ is the centralizer of any $H \in \mathscr{H}$, $H \neq 1$. The lines $A\mathscr{K}B$ and $C\mathscr{K}D$ are left-parallel if $A\mathscr{K} = C\mathscr{K}$, right-parallel if $\mathscr{K}B = \mathscr{K}D$. The cosets of $\mathscr{H}$ are both left-parallel and right-parallel to each other.

Note that $\mathscr{H}\mathscr{K} = \mathscr{G}$ is an immediate consequence of (i) and (iii).

Now let $\mathscr{G}$ be any group satisfying (*). It is easily verified that a projective plane with two lines removed is obtained if we take as points the elements of $\mathscr{G}$ and as lines the sets $A\mathscr{K}B$ $(A, B \in \mathscr{H})$ and the cosets of $\mathscr{H}$. The line joining points P and Q $(P \neq Q)$ is found as follows: if $QP^{-1} \in \mathscr{H}$, the line is $P\mathscr{H}$; otherwise (by (iii)) $QP^{-1} \in H^{-1}\mathscr{K}H$ for some $H \in \mathscr{H}$, and the required line is $(H^{-1}\mathscr{K}H)P$.

We adjoin a line l_0 whose points are the left-parallel classes, and a line l_∞ whose points are the right-parallel classes.

The resulting projective plane Π is (l_0, l_∞)-transitive. For, let X be a fixed element of $\mathscr{G}$. Then the permutation $G \to XG$ on $\mathscr{G}$ induces a collineation of Π which is central, having l_∞ as axis and $l_0 \cap (H^{-1}\mathscr{K}H)$ $[X \notin \mathscr{H}]$ or $l_0 \cap \mathscr{H}$ $[X \in \mathscr{H}]$ as centre, where $H^{-1}\mathscr{K}H$ is, when $X \notin \mathscr{H}$, the line joining **1** and X.

It follows that the plane dual to Π is (L, M)-transitive for some pair of distinct points L, M; that is, can be coordinatized by an associative V-W system (near-field) K (cf. **2**, p. 103), if we write the equation of a line with slope m as $y = mx + b$. Π can therefore be coordinatized with the system K^0 obtained by defining a new multiplication $*$ thus: $a * b = ba$. Instead of the right distributive law $(x + y)z = xz + yz$ of K we have in K^0 the left distributive law.

Taking $OY = l_0$, $XY = l_\infty$, any collineation with axis l_∞ and centre on l_0 is induced by a map

$$(x, y) \to (\sigma x, \sigma y + \rho)$$

for some $\sigma, \rho \in K^0$, with $\sigma \neq 0$. Therefore, $\mathscr{G}$ is isomorphic to the group of substitutions $y \to \sigma y + \rho$ $(\sigma \neq 0)$ over K^0.

Now let Π' be any plane, except the plane of order 2, which is (l, m)-transitive for some pair of distinct lines l, m. Let $\mathscr{G}'$ be the group of collineations with axis m and centre on l, $\mathscr{H}'$ the subgroup consisting of the $(l \cap m, m)$-collineations, $\mathscr{K}'$ the subgroup consisting of the (L, m)-collineations, where L is any point not equal to $l \cap m$ on l. Then $\mathscr{G}'$ satisfies condition (*). For the plane of order 2, $\mathscr{H}'$ and $\mathscr{K}'$ are trivial subgroups of $\mathscr{G}'$ $(\mathscr{K}' = 1, \mathscr{H}' = \mathscr{G}')$, and hence condition (*) is not satisfied. Thus our construction yields all (l, m)-transitive planes $(l \neq m)$ except the trivial plane of order 2; and we have incidentally identified the groups satisfying condition (*).

References

1. J. André, *Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe*, Math. Z. *60* (1954), 156–186.
2. G. Pickert, *Projektive Ebenen* (Springer, Berlin, 1955).
3. H. Schwerdtfeger, *Projective geometry in the one-dimensional affine group*, Can. J. Math. *16* (1964), 683–700.

University of Sydney,
Sydney, Australia

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