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TENSOR STRUCTURE ON $k\mathcal{C}$-MOD AND COHOMOLOGY

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Dedicated to Bob Oliver and Ron Solomon on the occasion of their 60th birthdays

Abstract Let $\mathcal{C}$ be a finite category and let k be a field. We consider the category algebra $k\mathcal{C}$ and show that $k\mathcal{C}$-mod is closed symmetric monoidal. Through comparing $k\mathcal{C}$ with a co-commutative bialgebra, we exhibit the similarities and differences between them in terms of homological properties. In particular, we give a module-theoretic approach to the multiplicative structure of the cohomology rings of small categories. As an application, we prove that the Hochschild cohomology rings of a certain type of finite category algebras are finitely generated.

Keywords: category algebra; closed symmetric monoidal category; ordinary cohomology ring; Hochschild cohomology ring; transporter categories; finite generation

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1. Introduction

Let $\mathcal{C}$ be a finite category (that is, $\mathrm{Mor}\,\mathcal{C}$ is a finite set), let k be a field and let Vect_k be the category of finite-dimensional k-vector spaces. There are three well-defined mathematical subjects: the category algebra $k\mathcal{C}$, the functor category $\mathrm{Vect}_k^{\mathcal{C}}$ and the classifying space $B\mathcal{C}$. These seemingly different things are closely related in terms of their homological properties, as in the special case where $\mathcal{C} = G$ is a group considered as a category with a single object. Recall that the category algebra $k\mathcal{C}$ [**23, 24**] as a k-vector space has as a basis the set of all morphisms in $\mathcal{C}$, and the product of any two base elements is their composition in $\mathcal{C}$, or zero if they are not composable. By a result of Mitchell [**14**], there is an isomorphism $\mathrm{Vect}_k^{\mathcal{C}} \cong k\mathcal{C}$-mod, where $k\mathcal{C}$-mod is the category of finitely generated $k\mathcal{C}$-modules. This gives $k\mathcal{C}$-mod a symmetric monoidal category structure, although a category algebra is usually not a bialgebra. Despite the fact that $k\mathcal{C}$ is not a bialgebra, we show it does behave like a co-commutative bialgebra. In fact, the canonical diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ induces a co-multiplication on $k\mathcal{C}$, still denoted by $\Delta\colon k\mathcal{C} \to k\mathcal{C} \otimes_k k\mathcal{C}$, and thus gives $k\mathcal{C}$ some sort of 'co-algebra' structure. Moreover, this diagonal functor can be used to define an internal hom on $k\mathcal{C}$-mod. Hence, $k\mathcal{C}$-mod is

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closed in the sense of [**13**]. We shall see in this paper that the multiplication and co-multiplication on $k\mathcal{C}$ are compatible in a way (see Proposition 2.2) with the symmetric monoidal category structure on $k\mathcal{C}$-mod. As a consequence, a category algebra and a co-commutative bialgebra share many common homological properties. Notably, let $\boldsymbol{k}$ be the tensor identity of $k\mathcal{C}$-mod. We can introduce the ordinary cohomology ring of $k\mathcal{C}$ as $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$, in which the cup product is defined by the tensor product on $k\mathcal{C}$-mod. From this definition we can easily identify the ordinary cohomology ring of $k\mathcal{C}$ with $H^*(B\mathcal{C}, k)$ where the cup product is described via the Alexander–Whitney map. The new construction gives us a module-theoretic way to understand the cohomology ring structure, complementing the existing simplicial method (see, for instance, [**23**, **25**]). Moreover, it allows further applications. After our description of the cohomology ring action on Ext groups, we continue to show that the Hochschild cohomology rings of a certain type of finite categories are finitely generated.

The paper is organized as follows. Section 2 is devoted to describing the closed symmetric monoidal category structure on $k\mathcal{C}$-mod and to listing properties of $k\mathcal{C}$ that are comparable to a co-commutative bialgebra. Section 3 is mainly devoted to demonstrating how one can define the cup product on the cohomology ring of $\mathcal{C}$ using the tensor structure on $k\mathcal{C}$-mod. We prove that this cup product is identified with the one defined by simplicial methods. Then we use this new definition of cup product to reinterpret certain homomorphisms among several cohomology rings. In § 4, we prove the finite generation of certain Hochschild cohomology rings. Finally, we include the construction of the Grothendieck spectral sequence in Appendix A.

2. Tensor structure on $k\mathcal{C}$-mod

Let us fix a field k and a finite category $\mathcal{C}$ (that is, $\mathrm{Mor}\,\mathcal{C}$ is a finite set). The category algebra $k\mathcal{C}$ [**23**, **24**] is defined as a k-vector space with a basis the set of morphisms in $\mathcal{C}$. The multiplication is given on base elements by $\alpha * \beta = \alpha \circ \beta$ if $\alpha, \beta \in \mathcal{C}$ are composable, or zero otherwise. The category algebra $k\mathcal{C}$ is a finite dimensional associative algebra with an identity $1_{k\mathcal{C}} = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} 1_x$. In this section, we aim to describe the co-multiplicative structure on $k\mathcal{C}$ and furthermore the closed symmetric monoidal category structure on $k\mathcal{C}$-mod, the category of finitely generated $k\mathcal{C}$-modules. We do so through comparing them with a finite-dimensional co-commutative bialgebra A, together with its module category.

The category Vect_k is a symmetric monoidal category equipped with a tensor product $\otimes_k$ and a tensor identity k. A key property which makes a co-commutative k-bialgebra A interesting is that A-mod inherits the tensor product and the tensor identity. Moreover, the tensor identity k plays the role of both the unit and co-unit of A. In what follows, we begin with a description about how Vect_k gives rise to a symmetric monoidal category structure on $k\mathcal{C}$-mod, which is well known. Then we characterize the structure of $k\mathcal{C}$ using some structure maps comparable to those of a co-commutative bialgebra. In this way we demonstrate why a category algebra and a co-commutative bialgebra, as well as their module categories, are similar yet different. It is the intrinsic structure of a category algebra that makes it a natural and interesting subject of investigation. We note that

Vect_k itself is the module category of the k-category algebra of the trivial category, based on the result of Mitchell stated in § 1.

Fix a finite category $\mathcal{C}$. The so-called internal product on $\mathrm{Vect}_k^{\mathcal{C}} \cong k\mathcal{C}$-mod, in which the tensor product is denoted by $\hat{\otimes}_k$, is defined by $(M \hat{\otimes}_k N)(x) = M(x) \otimes_k N(x)$ for any $M, N \in k\mathcal{C}$-mod $\cong \mathrm{Vect}_k^{\mathcal{C}}$ and $x \in \mathrm{Ob}\,\mathcal{C}$. The module structure of $M \hat{\otimes} N$ can be viewed as being given by the co-multiplication $\Delta\colon k\mathcal{C} \to k\mathcal{C} \otimes_k k\mathcal{C}$, induced by the canonical diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ whose action on each $\alpha \in \mathrm{Mor}\,\mathcal{C}$ is $\Delta(\alpha) = \alpha \otimes \alpha$. One can easily verify that the constant functor $\boldsymbol{k}$, which takes the tensor identity k of Vect_k as its value at each object, is the tensor identity with respect to $\hat{\otimes}_k$. Sometimes we also call $\boldsymbol{k}$ the trivial $k\mathcal{C}$-module, because when $\mathcal{C}$ is a group, $\boldsymbol{k} = k$ is exactly the trivial module of the group algebra. For the sake of simplicity, we shall write $\otimes$ for $\otimes_k$, and $\hat{\otimes}$ for $\hat{\otimes}_k$, throughout this paper. We note that in the literature (see, for example, [**7**]), the symbol $\otimes$ is often used instead of $\hat{\otimes}$ for the internal tensor product of functors. The new notation $\hat{\otimes}$ is introduced because we need to distinguish between $M \otimes N$ and $M \hat{\otimes} N$. In fact, as k-vector spaces, the inclusion $M \hat{\otimes} N \subset M \otimes N$, for any $k\mathcal{C}$-modules M and N, is often strict.

As we mentioned above, the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ induces a co-multiplication on the category algebra $k\mathcal{C}$. The co-multiplication $\Delta\colon k\mathcal{C} \to k\mathcal{C} \otimes k\mathcal{C}$ gives us an almost co-algebra structure on $k\mathcal{C}$ except that there is not a suitable choice of a map $k\mathcal{C} \to k$ which would serve as the co-unitary map. However, we have the following natural construction.

Lemma 2.1. *There exists a surjective linear map*

$$k\mathcal{C} \overset{\epsilon}{\twoheadrightarrow} k\,\mathrm{Ob}\,\mathcal{C},$$

defined on base elements by

$$\epsilon(\alpha) = t(\alpha),$$

where $t(\alpha)$ is the target of α. It gives $k\,\mathrm{Ob}\,\mathcal{C}$ a $k\mathcal{C}$-module structure such that $k\,\mathrm{Ob}\,\mathcal{C} \cong \boldsymbol{k}$, the trivial $k\mathcal{C}$-module and the tensor identity in $k\mathcal{C}$-mod with respect to $\hat{\otimes}$.

Proof. The surjective map induces a $k\mathcal{C}$-module structure on $k\,\mathrm{Ob}\,\mathcal{C}$ as follows. If $x \in \mathrm{Ob}\,\mathcal{C}$ and $\beta \in \mathrm{Hom}_{\mathcal{C}}(x', y)$, then $\beta x = y$ when $x = x'$, and $\beta x = 0$ otherwise. Using Mitchell's equivalence described in the first paragraph of § 1, one can easily verify that this is isomorphic to $\boldsymbol{k}$. □

From now on, we shall identify $k\,\mathrm{Ob}\,\mathcal{C}$ with $\boldsymbol{k}$, and write $k\mathcal{C} \xrightarrow{\epsilon} \boldsymbol{k}$ always. When $\mathcal{C}$ is a group, ϵ is exactly the augmentation map to k.

If we return to Δ, we realize that the image of it really lies in $k\mathcal{C} \hat{\otimes} k\mathcal{C}$, a subspace of $k\mathcal{C} \otimes k\mathcal{C}$, and moreover $\Delta\colon k\mathcal{C} \to k\mathcal{C} \hat{\otimes} k\mathcal{C}$ becomes a $k\mathcal{C}$-map, since $k\mathcal{C} \hat{\otimes} k\mathcal{C}$, unlike $k\mathcal{C} \otimes k\mathcal{C}$, is a well-defined $k\mathcal{C}$-module. The above observations hint that, in order to get a sound 'co-algebra' structure, one needs to use $\hat{\otimes}$ rather than $\otimes$ to define the structure maps. This motivates us to write down the following maps, which resemble almost all of the structure maps for a co-commutative bialgebra. Abusing terminology, we adopt the same names for the structure maps of a category algebra, such as co-multiplication and co-unit, as their counterparts for a bialgebra. In a coalgebra, the augmentation map and co-unit are the same, so by analogy the map ϵ in Lemma 2.1 will occasionally be called the co-unit. We emphasize that the unit, given by the natural inclusion map $k \cong k \cdot 1_{k\mathcal{C}} \xrightarrow{\iota} k\mathcal{C}$, is different from the co-unit.

Proposition 2.2. *Let $k\mathcal{C}$ be the category algebra of a finite category $\mathcal{C}$. Then we have the following k-linear maps: a* co-multiplication *$\Delta\colon k\mathcal{C} \to k\mathcal{C} \otimes k\mathcal{C}$, defined by*

$$\Delta\left(\sum_{\alpha} \lambda_\alpha \alpha\right) = \sum_{\alpha} \lambda_\alpha \alpha \otimes \alpha,$$

a co-unit *$\epsilon\colon k\mathcal{C} \to \boldsymbol{k}$ defined as above, and a twist map $\tau\colon k\mathcal{C} \otimes k\mathcal{C} \to k\mathcal{C} \otimes k\mathcal{C}$ defined on base elements by $\tau(\alpha \otimes \alpha') = \alpha' \otimes \alpha$, such that the following diagrams are commutative:*

(1) *co-associativity*

$$\begin{tikzcd}
k\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & k\mathcal{C} \otimes k\mathcal{C} \arrow[d, "\Delta\otimes\mathrm{Id}"] \\
k\mathcal{C} \otimes k\mathcal{C} \arrow[r, "\mathrm{Id}\otimes\Delta"'] & k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C}
\end{tikzcd}$$

(2) *co-unitary property*

$$\begin{tikzcd}
& k\mathcal{C} \arrow[dl, "\cong"'] \arrow[d, "\Delta"'] \arrow[dr, "\cong"] & \\
\boldsymbol{k} \hat{\otimes} k\mathcal{C} & k\mathcal{C} \hat{\otimes} k\mathcal{C} \arrow[l, "\epsilon\otimes\mathrm{Id}"] \arrow[r, "\mathrm{Id}\otimes\epsilon"'] & k\mathcal{C} \hat{\otimes} \boldsymbol{k}
\end{tikzcd}$$

(3) *co-commutativity*

$$\begin{tikzcd}
& k\mathcal{C} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
k\mathcal{C} \otimes k\mathcal{C} \arrow[rr, "\tau"'] & & k\mathcal{C} \otimes k\mathcal{C}
\end{tikzcd}$$

If we denote the multiplication by μ, then we have three further commutative diagrams:

(4) *multiplication and co-multiplication*

$$\begin{tikzcd}
k\mathcal{C} \otimes k\mathcal{C} \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & k\mathcal{C} \arrow[r, "\Delta"] & k\mathcal{C} \otimes k\mathcal{C} \\
k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \arrow[rr, "\mathrm{Id}\otimes\tau\otimes\mathrm{Id}"'] & & k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}$$

(5) *unit and co-multiplication*

$$\begin{tikzcd}
& k \otimes k \arrow[dl, "\mu"'] \arrow[dr, "\iota\otimes\iota"] & \\
k\mathcal{C} \arrow[rr, "\Delta"'] & & k\mathcal{C} \hat{\otimes} k\mathcal{C}
\end{tikzcd}$$

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(6) *unit and co-unit*

$$\begin{tikzcd} & k \arrow[dl, "\iota"'] \arrow[dr, "\eta"] & \\ k\mathcal{C} \arrow[rr, "\epsilon"'] & & \boldsymbol{k} \end{tikzcd}$$

where the k-linear map $\eta\colon k \to \boldsymbol{k}$ is defined by $1 \mapsto \sum_{x \in \mathrm{Ob}\,\mathcal{C}} x$.

We note that the only missing diagram is the compatibility of multiplication and co-unit. The reason for this is that one usually cannot give $\boldsymbol{k}$ a meaningful algebra structure, so $\epsilon\colon k\mathcal{C} \to \boldsymbol{k}$ becomes an algebra homomorphism. The existence of the above tensor structure $\hat{\otimes}$ is well known. Notably, it has been used to define the internal product in functor homology theory (see, for example, [**7**]). The earlier considerations on $\hat{\otimes}$ in the literature are quite different in nature from what we are about to do, and, in particular, it has not been formulated in such a way that one may compare a category algebra with a co-commutative bialgebra. Before we switch to cohomology theory, we state some facts that illustrate the differences between a category algebra and a co-commutative Hopf algebra.

Remark 2.3. There are significant differences between a category algebra and a co-commutative Hopf algebra. Most importantly, usually one *cannot* define an antipode for a category algebra. This causes a problem when one attempts to define $\mathrm{Hom}_k(M, N)$ as a $k\mathcal{C}$-module (the internal hom). We shall give a remedy below. Another relevant fact is that the product of two projective $k\mathcal{C}$-modules is in general *not* projective. These make many homological properties, such as the cohomology ring structure, of a category algebra different from those of a co-commutative Hopf algebra.

Proposition 2.4. *Let $M, N \in k\mathcal{C}$-mod. Then we can define an internal hom $\mathfrak{hom}(M, N) \in k\mathcal{C}$-mod.*

Proof. Consider the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}\times\mathcal{C}$. It naturally induces a functor, the restriction along Δ, $\mathrm{Res}_\Delta\colon k(\mathcal{C}\times\mathcal{C})$-mod $\to k\mathcal{C}$-mod, which has a right adjoint, the right Kan extension $RK_\Delta\colon k\mathcal{C}$-mod $\to k(\mathcal{C}\times\mathcal{C})$-mod (see [**13**]). Because $k(\mathcal{C}\times\mathcal{C}) \cong k\mathcal{C}\otimes k\mathcal{C}$, for $L, M, N \in k\mathcal{C}$-mod, $L \otimes M$ is a $k(\mathcal{C}\times\mathcal{C})$-module, and

$$\begin{aligned} \mathrm{Hom}_{k\mathcal{C}}(L \mathbin{\hat{\otimes}} M, N) &= \mathrm{Hom}_{k\mathcal{C}}(\mathrm{Res}_\Delta(L\otimes M), N) \\ &\cong \mathrm{Hom}_{k(\mathcal{C}\times\mathcal{C})}(L\otimes M, RK_\Delta N) \\ &\cong \mathrm{Hom}_{k\mathcal{C}\otimes k\mathcal{C}}(L\otimes M, RK_\Delta N) \\ &\cong \mathrm{Hom}_{k\mathcal{C}}(L, \mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)). \end{aligned}$$

Here in $\mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)$, $RK_\Delta N \in k\mathcal{C}\otimes k\mathcal{C}$-mod is considered as a $(k\cdot 1_{k\mathcal{C}})\otimes k\mathcal{C}$-module and hence a $k\mathcal{C}$-module. The $k\mathcal{C}\otimes k\mathcal{C}$-module structure, or rather the $k\mathcal{C}\otimes(k\cdot 1_{k\mathcal{C}})$-module structure, on $RK_\Delta N$ provides a $k\mathcal{C}$-module structure on $\mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)$. Then we define $\mathfrak{hom}(M, N) = \mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)$ and one can easily verify that it is an internal hom. $\square$

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Swenson built the internal hom in a different way in his 2009 PhD thesis [**21**, § 12.3], and hence we attribute this construction to him. In his definition, he defines $\mathcal{H}om(M,N) \in k\mathcal{C}$-mod by $\mathcal{H}om(M,N)(x) = \mathrm{Hom}_{k\mathcal{C}}({}^xF \hat{\otimes} M, N)$, where ${}^xF = k\mathcal{C} \cdot 1_x$ for each $x \in \mathrm{Ob}\,\mathcal{C}$. Since, using the method in our proof,

$$\begin{aligned}
\mathrm{Hom}_{k\mathcal{C}}({}^xF \hat{\otimes} M, N) &= \mathrm{Hom}_{k\mathcal{C}}(\mathrm{Res}_\Delta({}^xF \otimes M), N) \\
&\cong \mathrm{Hom}_{k\mathcal{C}\otimes k\mathcal{C}}({}^xF \otimes M, RK_\Delta N) \\
&\cong \mathrm{Hom}_{k\mathcal{C}}({}^xF, \mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)) \\
&\cong \mathrm{Hom}_{k\mathcal{C}}(M, RK_\Delta N)(x) \\
&= \mathfrak{hom}(M,N)(x),
\end{aligned}$$

Swenson's construction is identified with ours. When $\mathcal{C} = G$ is a group, it is straightforward from Swenson's definition that $\mathfrak{hom}(M,N) \cong \mathrm{Hom}_k(M,N)$ as kG-modules. From our definition, if one notes that $RK_\Delta \cong \uparrow_G^{G\times G}$, then, by applying the Mackey decomposition formula, one gets the same isomorphism.

Example 2.5. Let k be a field of characteristic 2 and let $\mathcal{C}$ be the following category:

$$\begin{tikzcd}
x \arrow[loop left, "\{1_x\}"] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & y \arrow[loop right, "\{1_y, g\}"]
\end{tikzcd}$$

with $g^2 = 1_y$, $g\alpha = \alpha$ and $g\beta = \beta$. One can describe the indecomposable projectives and simples of $k\mathcal{C}$ using general methods given in [**23**, **24**]. Indeed, the algebra $k\mathcal{C}$ has two (one-dimensional) simples $S_{x,k}$, $S_{y,k}$ and their projective covers are $P_{x,k} = k\{1_x, \alpha, \beta\}$ and $P_{y,k} = k\{1_y, g\}$, respectively. The product $P_{x,k} \hat{\otimes} P_{x,k} \cong P_{x,k} \oplus S_{y,k}^2$ is not projective because $S_{y,k} \neq P_{y,k}$.

Remark 2.6. Using the tensor product, one can introduce a 'representation ring' of $k\mathcal{C}$, namely $a(k\mathcal{C})$, which consists of $\mathbb{Z}$-linear combinations of symbols such as $[M]$, representing an isomorphism class of a $k\mathcal{C}$-module M. For any two elements $[M]$ and $[N]$, the multiplication is defined by $[M] \cdot [N] = [M \hat{\otimes} N]$. However, this product does not exist in $K_0(k\mathcal{C})$, which is spanned over the set of isomorphism classes of indecomposable projectives.

Remark 2.7. If we remove the finiteness condition on $\mathcal{C}$, then there is no identity in the algebra $k\mathcal{C}$. Nevertheless, many of the constructions are still valid, although in this case we are forced to use the full subcategory $\mathrm{Vect}_k^{\mathcal{C}}$ because $k\mathcal{C}$-mod does not have a tensor structure.

Remark 2.8. If $\mathcal{C}$ happens to be a poset, then there is another co-multiplication, which one can find in [**20**]. Let $\alpha \in \mathrm{Mor}\,\mathcal{C}$. One then has $\bar{\Delta}\colon k\mathcal{C} \to k\mathcal{C} \otimes k\mathcal{C}$ such that

$$\bar{\Delta}(\alpha) = \sum_{\{\beta,\gamma \mid \beta\gamma = \alpha\}} \beta \otimes \gamma.$$

Remark 2.9. Let A be an algebra. Although A-mod is not monoidal in general, the module category A^e-mod is always equipped with a tensor product $\otimes_A$ such that A is the tensor identity.

When $A = k\mathcal{C}$ is a category algebra, $(k\mathcal{C})^e \cong k\mathcal{C}^e$ becomes the category algebra of the category $\mathcal{C}^e := \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, and hence there are two distinct monoidal structures on $k\mathcal{C}^e$-mod. The monoidal structure given by $\otimes_{k\mathcal{C}}$ is more interesting to us, and we shall deal with it in the next section.

3. Cup products

The homology and cohomology of small categories with coefficients in functors have been studied since the 1960s [**1**, **8**], but mainly using simplicial methods. Among well-known results is that $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \cong H^*(B\mathcal{C}, k)$ as graded k-vector spaces (see, for example, [**15**]). This is indeed an algebra isomorphism. Here, using the tensor structure on $k\mathcal{C}$-mod, we provide a module-theoretic description to the ring $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$, and then it follows that the above-mentioned algebra isomorphism naturally exists. In the meantime we pave the way for studying the ring action of $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ on various Ext groups. We comment that since $(k\mathcal{C}\text{-mod}, \hat{\otimes}, \boldsymbol{k})$ is a monoidal category with an exact tensor product, it gives rise to a suspended monoidal category $(D^-(k\mathcal{C}), \hat{\otimes}, \boldsymbol{k})$, and then, following a general statement [**19**] on the endomorphisms of the identity in a suspended monoidal category, $\mathrm{End}_{D^-(k\mathcal{C})}(\boldsymbol{k})$ is a graded commutative ring. It will be clear in this section that this endomorphism ring is isomorphic to what we call the ordinary cohomology ring $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$.

Let $M, M', N, N' \in k\mathcal{C}$-mod. We will define the cup product to be

$$\bigcup\colon \mathrm{Ext}^i_{k\mathcal{C}}(M, N) \otimes \mathrm{Ext}^j_{k\mathcal{C}}(M', N') \to \mathrm{Ext}^{i+j}_{k\mathcal{C}}(M \mathbin{\hat{\otimes}} M', N \mathbin{\hat{\otimes}} N').$$

Since $\boldsymbol{k}$ is the identity with respect to $\hat{\otimes}$, this will give us a ring structure on $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$, as well as an action of $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ on $\mathrm{Ext}^*_{k\mathcal{C}}(M, N)$ for arbitrary $M, N \in k\mathcal{C}$-mod. We shall compare our construction with [**2**, § 3.2] for co-commutative Hopf algebras.

We first provide some elementary results and then describe the cup products in detail. Suppose $\mathbb{C}$ is a complex of $k\mathcal{C}$-modules. The homology group $H_n(\mathbb{C})$, $n \in \mathbb{Z}$, is a $k\mathcal{C}$-module such that $H_n(\mathbb{C})(x) = H_n(\mathbb{C}(x))$ for each $x \in \mathrm{Ob}\,\mathcal{C}$. Thus, the complex $\mathbb{C}$ is exact if and only if its evaluation at each object $x \in \mathrm{Ob}\,\mathcal{C}$, $\mathbb{C}(x)$, is exact. Let $\mathbb{C}$ and $\mathbb{D}$ be two complexes of $k\mathcal{C}$-modules. We can define the product of them as

$$(\mathbb{C} \mathbin{\hat{\otimes}} \mathbb{D})_n = \bigoplus_{i+j=n} \mathbb{C}_i \mathbin{\hat{\otimes}} \mathbb{D}_j,$$

with the differential (a natural transformation) given by

$$\partial_x(a \otimes b) = \partial^{\mathbb{C}}_x a \otimes b + (-1)^i a \otimes \partial^{\mathbb{D}}_x b,$$

where $a \in \mathbb{C}_i(x)$ and $b \in \mathbb{D}_j(x)$ for each $x \in \mathrm{Ob}\,\mathcal{C}$. We need a result deduced from the Künneth Formula.

Lemma 3.1. *Given two complexes of $k\mathcal{C}$-modules $\mathbb{C}$ and $\mathbb{D}$, for each integer n we have*

$$H_n(\mathbb{C} \mathbin{\hat{\otimes}} \mathbb{D}) \cong \bigoplus_{i+j=n} H_i(\mathbb{C}) \mathbin{\hat{\otimes}} H_j(\mathbb{D}).$$

Proof. For each $x \in \operatorname{Ob}\mathcal{C}$, we apply the Künneth Formula on $(\mathbb{C} \mathbin{\hat{\otimes}} \mathbb{D})(x) = \mathbb{C}(x) \otimes \mathbb{D}(x)$, and then the above result follows. □

Now we are ready to give a precise definition to the cup product. Suppose that $\zeta \in \operatorname{Ext}^m_{k\mathcal{C}}(M, N)$ is represented by

$$0 \to N \to L_{m-1} \to \cdots \to L_0 \to M \to 0,$$

and $\zeta' \in \operatorname{Ext}^n_{k\mathcal{C}}(M', N')$ is represented by

$$0 \to N' \to L'_{n-1} \to \cdots \to L'_0 \to M' \to 0.$$

Then, applying Lemma 3.1 to $0 \to N \to L_{m-1} \to \cdots \to L_0$ and $0 \to N' \to L'_{n-1} \to \cdots \to L'_0$, we get an exact sequence

$$0 \to N \mathbin{\hat{\otimes}} N' \to (L_{m-1} \mathbin{\hat{\otimes}} N) \oplus (N \mathbin{\hat{\otimes}} L'_{n-1}) \to \cdots \to L_0 \mathbin{\hat{\otimes}} L'_0 \to M \mathbin{\hat{\otimes}} M' \to 0,$$

which is defined to be the cup product of ζ and ζ', $\zeta \cup \zeta' \in \operatorname{Ext}^{m+n}_{k\mathcal{C}}(M \mathbin{\hat{\otimes}} M', N \mathbin{\hat{\otimes}} N')$.

Lemma 3.2. *Let ζ, ζ' be as above. The cup product $\zeta \cup \zeta'$ is the Yoneda splice of*

$$\zeta \mathbin{\hat{\otimes}} \mathrm{Id}_{N'} \in \operatorname{Ext}^i_{k\mathcal{C}}(M \mathbin{\hat{\otimes}} N', N \mathbin{\hat{\otimes}} N')$$

with

$$\mathrm{Id}_M \mathbin{\hat{\otimes}} \zeta' \in \operatorname{Ext}^j_{k\mathcal{C}}(M \mathbin{\hat{\otimes}} M', M \mathbin{\hat{\otimes}} N').$$

The image of $\operatorname{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \to \operatorname{Ext}^*_{k\mathcal{C}}(M, M)$ lies in the graded centre for any $M \in k\mathcal{C}$-mod. In particular, $\operatorname{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ is graded commutative.

Proof. This is entirely analogous to [**2**, Proposition 3.2.1], but here we use the tensor structure on $k\mathcal{C}$-mod. The key fact is that, given the cocommutativity $\tau\Delta = \Delta$, we can establish an isomorphism of complexes of $k\mathcal{C}$-modules

$$\mathbb{C} \mathbin{\hat{\otimes}} \mathbb{D} \to \mathbb{D} \mathbin{\hat{\otimes}} \mathbb{C},$$

by $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a$. □

In terms of projective resolutions, we can describe the cup product as follows. Let M, M', N, N' be $k\mathcal{C}$-modules. Take two projective resolutions $\mathcal{P}_* \to M \to 0$ and $\mathcal{Q}_* \to M' \to 0$. Then, by our previous observation, $\mathcal{P}_* \mathbin{\hat{\otimes}} \mathcal{Q}_* \to M \mathbin{\hat{\otimes}} M' \to 0$ is an exact sequence. This is usually not a projective resolution as the tensor product of two projective is not projective, in contrast to the case of a co-commutative Hopf algebra. However, we can

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build a projective resolution, unique up to chain homotopy, $\mathcal{R}_* \to M \hat{\otimes} M' \to 0$ such that there exists a chain map $\delta\colon \mathcal{R}_* \to \mathcal{P}_* \hat{\otimes} \mathcal{Q}_*$ and a commutative diagram

$$\begin{tikzcd} \mathcal{R}_* \arrow[r] \arrow[d, "\delta"'] & M \hat{\otimes} M' \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{P}_* \hat{\otimes} \mathcal{Q}_* \arrow[r] & M \hat{\otimes} M' \arrow[r] & 0 \end{tikzcd}$$

If, for two integers m, n, $\zeta \in \mathrm{Ext}^m_{k\mathcal{C}}(M, N)$ and $\zeta' \in \mathrm{Ext}^n_{k\mathcal{C}}(M', N')$ are represented by two cocycles $f\colon P_m \to N$ and $g\colon Q_n \to N'$, then the product $\zeta \cup \zeta'$ is represented by $(f \hat{\otimes} g) \circ \delta\colon \mathcal{R}_{m+n} \to N \hat{\otimes} N'$.

As an example we show how to establish the algebra isomorphism $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \cong H^*(B\mathcal{C}, k)$. It is well known that $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ and $H^*(B\mathcal{C}, k)$ are isomorphic as graded vector spaces (see, for instance, [**15**]), and our proof is based on that.

Theorem 3.3. *With the above cup product,* $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \cong H^*(B\mathcal{C}, k)$ *as algebras.*

Proof. We begin with a quick description of $H^*(B\mathcal{C}, k)$. This ring can be computed using a simplicial complex from the nerve $N_*\mathcal{C}$ of $\mathcal{C}$. For each $n \geqslant 0$, $N_n\mathcal{C}$ is the set of n-chains of morphisms in $\mathcal{C}$. In particular, we have $N_0\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$. On the simplicial complex $kN_*\mathcal{C} \to 0$, for each $x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n \in kN_n\mathcal{C}$, we have

$$\delta(x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n) = \sum_{i=0}^{n} (-1)^i x_0 \xrightarrow{\alpha_1} \cdots \to \hat{x}_i \to \cdots \xrightarrow{\alpha_n} x_n,$$

in which the hat denotes the removal of an object. The cohomology ring $H^*(B\mathcal{C}, k)$ is computed as a graded k-vector space as the homology of the cochain complex $0 \to \mathrm{Hom}_k(kN_*\mathcal{C}, k)$, and the cup product is obtained using the Alexander–Whitney map on a simplicial complex.

In order to compare $H^*(B\mathcal{C}, k)$ with $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$, we note that each $kN_n\mathcal{C}$ has a $k\mathcal{C}$-module structure and, in particular, $kN_0\mathcal{C} \cong \boldsymbol{k}$. This enables us to modify the above chain complex to get the bar resolution $\mathcal{B}^{\mathcal{C}}_* = \mathcal{B}_*$ of $\boldsymbol{k}$ as follows. For each $x \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{B}_n(x)$ is the k-vector space with base elements of the form

$$x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x$$

(see [**8**] or [**15**] for more details) with $x_i, x \in \mathrm{Ob}\,\mathcal{C}$, $\alpha_i, \alpha \in \mathrm{Mor}(\mathcal{C})$, and a non-negative $n \in \mathbb{Z}$. The differential, as a natural transformation, is defined subsequently as

$$\delta_x(x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x) = \sum_{i=0}^{n} (-1)^i x_0 \xrightarrow{\alpha_1} \cdots \to \hat{x}_i \to \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x.$$

The complex of $k\mathcal{C}$-modules $\mathcal{B}_* \to \boldsymbol{k} \to 0$ is exact (see, for instance, [**15**]) and, furthermore, since, for any $M \in k\mathcal{C}$-mod,

$$\mathrm{Hom}_{k\mathcal{C}}(\mathcal{B}_n, M) \cong \prod_{x_0 \to x_1 \to \cdots \to x_n \in N_n\mathcal{C}} M(x_n),$$

we see $\mathcal{B}_n$ is projective and hence we get a projective resolution of $\boldsymbol{k}$. When $\mathcal{C} = G$ is a group, this is exactly the bar resolution for the trivial module k.

Using the bar resolution $\mathcal{B}_* \to \boldsymbol{k} \to 0$, we can describe the cup product on $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$. In fact, we can explicitly write out a diagonal approximation map (unique up to chain homotopy) $D\colon \mathcal{B}_* \to \mathcal{B}_* \hat{\otimes} \mathcal{B}_*$, as a natural transformation, given by

$$\begin{aligned} &D_x(x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x) \\ &\qquad = \sum_{i=0}^{n} (x_0 \xrightarrow{\alpha_1} \cdots \to x_i \xrightarrow{\alpha\cdots\alpha_{i+1}} x) \otimes (x_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x) \end{aligned}$$

for any $x \in \mathrm{Ob}\,\mathcal{C}$ and integer n. We note that $\mathrm{Hom}_{k\mathcal{C}}(\mathcal{B}_*, \boldsymbol{k}) \cong \mathrm{Hom}_k(kN_*\mathcal{C}, k)$ as complexes and that the diagonal approximation map D corresponds to the Alexander–Whitney map, which is used to calculate the cup product in the cohomology ring. These imply $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \cong H^*(B\mathcal{C}, k)$ as algebras. $\square$

The bar resolution can also be constructed via the nerve of overcategories associated with the identity functor $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, as $\mathcal{B}_* \cong \mathbb{C}_*(\mathrm{Id}_{\mathcal{C}}/?) := kN_*(\mathrm{Id}_{\mathcal{C}}/?)$ (see, for example, [**23**, **25**]). In this form, $\mathcal{B}_n(x) \cong \mathbb{C}_n(\mathrm{Id}_{\mathcal{C}}/x)$, for each $x \in \mathrm{Ob}\,\mathcal{C}$ and integer $n \geqslant 0$, consists of chains of the following form as base elements:

$$(x_0, \beta_0) \xrightarrow{\gamma_1} (x_1, \beta_1) \to \cdots \to (x_{n-1}, \beta_{n-1}) \xrightarrow{\gamma_n} (x_n, \beta_n),$$

in which β_i is a morphism in $\mathrm{Hom}_{\mathcal{C}}(x_i, x)$, and $\gamma_i \in \mathrm{Hom}_{\mathcal{C}}(x_{i-1}, x_i)$ such that $\beta_{i-1} = \beta_i\gamma_{i-1}$. The previously defined diagonal map D is given by

$$\begin{aligned} &D_x((x_0, \beta_0) \xrightarrow{\gamma_1} (x_1, \beta_1) \to \cdots \to (x_{n-1}, \beta_{n-1}) \xrightarrow{\gamma_n} (x_n, \beta_n)) \\ &\qquad = \sum_{i=0}^{n} [(x_0, \beta_0) \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_i} (x_i, \beta_i)] \otimes [(x_i, \beta_i) \xrightarrow{\gamma_{i+1}} \cdots \xrightarrow{\gamma_n} (x_n, \beta_n)]. \end{aligned} \tag{3.1}$$

This is exactly the Alexander–Whitney map for the simplicial chain complex from the nerve of the overcategory $\mathrm{Id}_{\mathcal{C}}/x$.

Using the above description of cup products, one may continue to describe the split surjective algebra homomorphism from the Hochschild cohomology ring of $k\mathcal{C}$ to the ordinary cohomology ring [**25**]

$$\phi_{\mathcal{C}}\colon \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \to \mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}).$$

This algebra homomorphism is given by $-\otimes_{k\mathcal{C}} \boldsymbol{k}$, and we have proved it is split surjective. The proof of it relies on the use of $F(\mathcal{C})$ of [**17**], which is the category of factorizations in $\mathcal{C}$ and plays the role of the diagonal subgroup $\Delta G \subset G \times G \cong G \times G^{\mathrm{op}}$ when $\mathcal{C} = G$ is a group. Indeed, there exists a commutative diagram of categories and functors:

$$\begin{tikzcd} F(\mathcal{C}) \arrow[rr, "\tau"] \arrow[dr, "t"'] & & \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \arrow[dl, "pr"] \\ & \mathcal{C} & \end{tikzcd}$$

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The category of factorizations in $\mathcal{C}$, $F(\mathcal{C})$, has all the morphisms in $\mathcal{C}$ as its objects. For any two objects in $F(\mathcal{C})$, there exists a morphism from one to the other if the first is a factor of the second in $\mathrm{Mor}(\mathcal{C})$. The category $F(\mathcal{C})$ is used by Quillen [**17**] to show that there exist natural functors $\mathcal{C} \xleftarrow{t} F(\mathcal{C}) \xrightarrow{s} \mathcal{C}^{\mathrm{op}}$ which induce homotopy equivalences. One can find more information about $F(\mathcal{C})$ in [**25**, § 2.1]. Here we just describe the key steps in comparing the cup products in the Hochschild cohomology ring and the ordinary cohomology ring. The preceding diagram naturally induces new commutative diagrams of module categories and functors, among them:

$$\begin{tikzcd} kF(\mathcal{C})\text{-mod} & & k\mathcal{C}^e\text{-mod} \arrow[ll, "\mathrm{Res}_\tau"'] \\ & k\mathcal{C}\text{-mod} \arrow[ul, "\mathrm{Res}_t"] \arrow[ur, "\mathrm{Res}_{\mathrm{pr}}"'] & \end{tikzcd} \qquad \begin{tikzcd} kF(\mathcal{C})\text{-mod} \arrow[rr, "LK_\tau"] \arrow[dr, "LK_t"'] & & k\mathcal{C}^e\text{-mod} \arrow[dl, "LK_{\mathrm{pr}}"] \\ & k\mathcal{C}\text{-mod} & \end{tikzcd}$$

Here Res_τ is the functor induced by τ by precomposition and is called the restriction along τ. The functor LK_τ is the well-known left adjoint of it, called the left Kan extension of τ. The other two pairs of functors, Res_t, LK_t and $\mathrm{Res}_{\mathrm{pr}}, LK_{\mathrm{pr}}$, are constructed in the same way over t and pr, respectively. We shall focus on the functors LK_τ, LK_t and LK_{pr}. It is known that LK_{pr} acts as $- \otimes_{k\mathcal{C}} \boldsymbol{k}$ and sends each projective resolution of $k\mathcal{C}$ to a projective resolution of $\boldsymbol{k}$. We showed in [**25**] that LK_t maps the bar resolution of $kF(\mathcal{C})$-modules $\mathcal{B}_*^{F(\mathcal{C})} \to \boldsymbol{k} \to 0$ to a projective resolution of $k\mathcal{C}$-modules

$$LK_t\mathcal{B}_*^{F(\mathcal{C})} \to LK_t\boldsymbol{k} \cong \boldsymbol{k} \to 0.$$

Meanwhile, LK_τ takes the bar resolution to a projective resolution of $k\mathcal{C}^e$-modules $LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \to LK_\tau\boldsymbol{k} \cong k\mathcal{C} \to 0$. These are the main facts that we need to proceed. We have to warn the reader that since normally a functor, such as Res_τ and LK_τ, between the module categories of two category algebras does *not* come from an algebra homomorphism, module-theoretic methods are not sufficient for our work. This is the main reason why simplicial constructions, such as the bar resolution, are necessary. Along with the adjunctions with corresponding restrictions, the previous observations on the three left Kan extensions lead to a commutative diagram of cohomology rings:

$$\begin{tikzcd} \mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k},\boldsymbol{k}) \arrow[rr, "\tau^*"] \arrow[dr, "\cong", "t^*"'] & & \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C},k\mathcal{C}) \arrow[dl, "pr^*=\phi_{\mathcal{C}}"] \\ & \mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k}) & \end{tikzcd}$$

The map pr^* is the same as the one induced by $- \otimes_{k\mathcal{C}} \boldsymbol{k}$ and is often written as $\phi_{\boldsymbol{k}}$ or $\phi_{\mathcal{C}}$. The map t^* is an isomorphism and can be thought as the inverse to

$$Bt^* \colon H^*(B\mathcal{C},k) \to H^*(BF(\mathcal{C}),k),$$

induced by the topological map $Bt\colon BF(\mathcal{C}) \to B\mathcal{C}$ (a homotopy equivalence). In fact, the composite of the chain maps,

$$\mathrm{Hom}_{k\mathcal{C}}(\mathcal{B}_*^{\mathcal{C}},\boldsymbol{k}) \xrightarrow{\mathrm{Res}_t} \mathrm{Hom}_{kF(\mathcal{C})}(\mathcal{B}_*^{F(\mathcal{C})},\boldsymbol{k}) \xrightarrow[\cong]{LK_t} \mathrm{Hom}_{k\mathcal{C}}(LK_t\mathcal{B}_*^{F(\mathcal{C})},\boldsymbol{k}),$$

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comes from a chain homotopy $LK_t\mathcal{B}_*^{F(\mathcal{C})} \to \mathcal{B}_*^{\mathcal{C}}$ lifting the identity map $\boldsymbol{k} \to \boldsymbol{k}$. It gives rise to the isomorphisms

$$\operatorname{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k}) \xrightarrow{Bt^*} \operatorname{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k},\boldsymbol{k}) \xrightarrow{t^*} \operatorname{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k}).$$

Then $\tau^*(t^*)^{-1}$ is a right inverse of $pr^* = \phi_{\mathcal{C}}$.

We shall take some time to explain how to show that τ^* is an algebra homomorphism. Based on our new description on the cup products in the source ring, we illustrate how this map τ^* preserves cup products. Take the bar resolution $\mathcal{B}_*^{F(\mathcal{C})} \to \boldsymbol{k} \to 0$. We have seen that $LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \to LK_\tau\boldsymbol{k} \cong k\mathcal{C} \to 0$ is a projective resolution of the $k\mathcal{C}^e$-module $k\mathcal{C}$. Let f, g be two cocycles representing two cohomology classes. Then we construct the following diagram:

$$\begin{tikzcd}
\mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "D^{F(\mathcal{C})}"] & \mathcal{B}_*^{F(\mathcal{C})} \hat{\otimes} \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "f\otimes g"] \arrow[dr, phantom, "\Downarrow LK_\tau"] & \boldsymbol{k} \hat{\otimes} \boldsymbol{k} \arrow[r, "\cong"] & \boldsymbol{k} \\
LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "LK_\tau(D^{F(\mathcal{C})})"] \arrow[d, equal] & LK_\tau(\mathcal{B}_*^{F(\mathcal{C})} \hat{\otimes} \mathcal{B}_*^{F(\mathcal{C})}) \arrow[r, "LK_\tau(f\otimes g)"] & LK_\tau(\boldsymbol{k} \hat{\otimes} \boldsymbol{k}) \arrow[r, "\cong"] & k\mathcal{C} \arrow[d, equal] \\
LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "D^{\mathcal{C}^e}"'] & LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \otimes_{k\mathcal{C}} LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "LK_\tau(f)\otimes LK_\tau(g)"'] \arrow[u, dotted, "\Theta_\tau"] & LK_\tau(\boldsymbol{k}) \otimes_{k\mathcal{C}} LK_\tau(\boldsymbol{k}) \arrow[r, "\cong"'] \arrow[u, "\Theta_0", "\cong"'] & k\mathcal{C}
\end{tikzcd}$$

The first row represents $f \cup g$, the cup product of f and g. The left Kan extension LK_τ maps it to the second row of $k\mathcal{C}^e$-modules, which represents the image of the cup product, and we want to show that it gives rise to the cup product of $LK_\tau(f)$ and $LK_\tau(g)$ as Hochschild cohomology classes. Since we have

$$LK_\tau(D^{F(\mathcal{C})})\colon LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \to LK_\tau(\mathcal{B}_*^{F(\mathcal{C})} \hat{\otimes} \mathcal{B}_*^{F(\mathcal{C})}),$$

and $LK_\tau(\mathcal{B}_*^{F(\mathcal{C})})$ and $LK_\tau(\mathcal{B}_*^{F(\mathcal{C})}) \otimes_{k\mathcal{C}} LK_\tau(\mathcal{B}_*^{F(\mathcal{C})})$ are chain homotopy equivalent as both of them are projective resolutions of $k\mathcal{C}$, we can construct chain maps $D^{\mathcal{C}^e}$ and Θ_τ, unique up to chain homotopy, such that the above diagram is commutative. Because the Hochschild diagonal approximation map always exists and is unique up to chain homotopy, independent of the choice of a projective resolution of $k\mathcal{C}$ [**18**], $D^{\mathcal{C}^e}$ will serve as the Hochschild diagonal approximation map. Then since the lower two rows form a commutative diagram, we know they represent the same cohomology class, i.e. the cup product $LK_\tau(f) \cup LK_\tau(g)$, in $\operatorname{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$.

Remark 3.4. Slightly modifying the previous argument, we can also demonstrate the action of $\operatorname{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$ on $\operatorname{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M)$ alternatively via $\hat{\otimes}$ on $kF(\mathcal{C})$-mod. As we proved in [**25**], for any $M \in k\mathcal{C}^e$-mod, one gets

$$\operatorname{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M) \cong \operatorname{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \operatorname{Res}_\tau M).$$

It was also shown that the $kF(\mathcal{C})$-module $\operatorname{Res}_\tau k\mathcal{C}$ naturally splits as $\boldsymbol{k} \oplus N_{\mathcal{C}}$ for some $N_{\mathcal{C}} \in kF(\mathcal{C})$-mod. This provides a surjective homomorphism $\rho\colon \operatorname{Res}_\tau k\mathcal{C} \hat{\otimes} \operatorname{Res}_\tau M \to$

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$\mathrm{Res}_\tau M$, and hence a map

$$\rho^*\colon \mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau k\mathcal{C} \hat{\otimes} \mathrm{Res}_\tau M) \to \mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau M).$$

The latter fits into the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau k\mathcal{C}) \otimes \mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau M) \arrow[r, equal] \arrow[d, "\cup"'] & \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \otimes \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M) \arrow[d, "\cup"] \\
\mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau k\mathcal{C} \hat{\otimes} \mathrm{Res}_\tau M) \arrow[d, "\cong"'] & \mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau M) \arrow[d, "\rho^*"] \\
\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C} \otimes_{k\mathcal{C}} M) \arrow[r, equal] & \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M),
\end{tikzcd}$$

which reduces to [**18**, Proposition 3.1] when $\mathcal{C} = G$ is a group. The left-hand column is the so-called cup product with respect to the pairing ρ. Since $\mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \boldsymbol{k})$ is a direct summand of $\mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \mathrm{Res}_\tau k\mathcal{C})$, it also exhibits the action of $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ on $\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M)$, via its identification with $\mathrm{Ext}^*_{kF(\mathcal{C})}(\boldsymbol{k}, \boldsymbol{k})$.

Remark 3.5. It is interesting to point out a conjecture by Etingof and Ostrik which asserts that the cohomology ring of the tensor identity in a finite tensor category is finitely generated [**6**, Conjecture 2.18]. Their cohomology ring is comparable to our ordinary cohomology ring of a category algebra, which is not finitely generated modulo nilpotents (see [**25**]). The only difference between our settings seems to be the condition that they require the tensor identity to be simple, while in our situation $\boldsymbol{k}$ is not simple unless the category is equivalent to a group.

4. Finite generation of certain Hochschild cohomology rings

In [**25**] we saw that the ordinary cohomology ring of a finite category can be infinitely generated even after we quotient out nilpotents. Based on the main theorem therein, the Hochschild cohomology ring of such a finite category algebra is not finitely generated either. However, in § 4, we show there are some interesting finite category algebras whose Hochschild cohomology rings are finitely generated. The motivating question is to find out whether or not $\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$ modulo nilpotents is finitely generated over $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k})$ if the latter is Noetherian (see Remark 3.5 and the discussions below). On the first attempt to solve that question, one may want to check whether the Evens–Venkov Theorem on the finite generation of group cohomology could be generalized to category cohomology. Unfortunately, the answer is negative.

Example 4.1. Let k be a field of characteristic 2 and let $\mathcal{C}$ be the following category:

$$\begin{tikzcd}
x \arrow[loop left, "\{1_x\}"] \arrow[r, "\alpha"] & y \arrow[loop right, "\{1_y, g\}"]
\end{tikzcd}$$

with $g\alpha = \alpha$. There are two one-dimensional simple modules $S_{x,k}$ and $S_{y,k}$, together with their projective covers $P_{x,k} = k\{1_x, \alpha\}$ and $P_{y,k} = \{1_y, g\}$. We see that $\boldsymbol{k} \cong P_{x,k}$ is projective; hence, $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, M) \cong M(x)$ for any $M \in k\mathcal{C}$-mod. In particular, $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, \boldsymbol{k}) \cong k$. However, both modules $\mathrm{Ext}^*_{k\mathcal{C}}(S_{x,k}, S_{y,k})$ and $\mathrm{Ext}^*_{k\mathcal{C}}(S_{y,k}, S_{y,k})$ are infinite dimensional.

If we look at the opposite category $\mathcal{C}^{\mathrm{op}}$, again we have $\mathrm{Ext}^*_{k\mathcal{C}^{\mathrm{op}}}(\boldsymbol{k},\boldsymbol{k}) \cong \mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k}) \cong k$, and $\mathrm{Ext}^*_{k\mathcal{C}^{\mathrm{op}}}(\boldsymbol{k}, S_{y,k}) \cong \mathrm{Ext}^*_{k\mathcal{C}^{\mathrm{op}}}(S_{y,k}, S_{y,k})$ is infinite dimensional. This means we do not have the finite generation of $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, M)$ over $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k})$, even if they can be calculated by using the same projective resolution.

The computation of Hochschild cohomology can be found in [**25**, § 3.2]. Indeed, the Hochschild cohomology ring of $k\mathcal{C}$ is infinite dimensional but its quotient ring by nilpotents is just k.

The above example implies that we cannot expect a finite generation property of $\mathrm{Ext}^*_{k\mathcal{C}}(M,N)$ over $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k},\boldsymbol{k})$. Thus, we need to develop other means to examine the finite generation of Hochschild cohomology ring modulo nilpotents as a module over the ordinary cohomology ring. In what follows, we show certain categories constructed over a finite group have their Hochschild cohomology closely related to the group cohomology. This is inspired by the well-known fact that one may approximate group cohomology via the ordinary cohomology of certain finite categories (see, for example, [**5**]). Based on this observation, we prove the finite generation of the Hochschild cohomology rings of a certain kind of finite category algebras. Before we state any results, we construct several categories, and along the way one can see why we are interested in them and the finite generation of their cohomology rings.

Let G be a finite group and let k be an algebraically closed field with positive characteristic $p \mid |G|$. Given a p-block b, according to Alperin and Broué (see [**22**]), one has a poset of b-Brauer pairs (Brauer pairs associated to b) $\mathcal{S}_b = \mathcal{S}_b(G)$, in which the objects are of the form (Q, e_Q) for Q a subgroup of some defect of b, and e_Q a Brauer correspondent of b in $kC_G(Q)$. When $b = b_0$ is the principal block, this poset is isomorphic to $\mathcal{S}^1_p$, that is, the poset of all p-subgroups of G. Just like the poset $\mathcal{S}^1_p$, every $\mathcal{S}_b$ has a natural G-action and thus is a so-called G-poset. By adding all possible morphisms among b-Brauer pairs, induced by conjugations by elements in G, one obtains a larger category which has the same objects as $\mathcal{S}_b$. Such a category may be called a 'b-transporter category', and is denoted by $\mathrm{Tr}_b(G)$. In fact, $\mathcal{S}_b$ can be identified with a subcategory of $\mathrm{Tr}_b(G)$. (We note that our transporter categories are usually not transporter systems in the sense of Oliver and Ventura [**16**].) When $b = b_0$, $\mathrm{Tr}_{b_0}(G)$ is exactly the p-transporter category of G, usually written as $\mathrm{Tr}_p(G)$. A concise but conceptual way to introduce $\mathrm{Tr}_b(G)$ is to assert that $\mathrm{Tr}_b(G)$ is the Grothendieck construction (recalled below) for the G-poset $\mathcal{S}_b$. In practice, one often fixes a defect P of b, and only considers the full subcategory $\mathrm{Tr}_b(G)_{\leqslant(P,e_P)}$, which is equivalent to $\mathrm{Tr}_b(G)$ as categories. Moreover, since the second entry in each b-Brauer pair (Q, e_Q) is uniquely determined by b and the first entry Q, for convenience one often writes Q instead of (Q, e_Q) as an object when a certain block b is chosen. Under this convention, one replaces the notation $\mathrm{Tr}_b(G)_{\leqslant(P,e_P)}$ by $\mathrm{Tr}_b(G)_{\leqslant P}$. The category $\mathrm{Tr}_b(G)_{\leqslant P}$ has an important quotient category $\mathcal{F}_b = \mathcal{F}_b(G)$, called the fusion system of b on the defect group P. The fusion system is a key concept in the modular representation theory of finite groups [**22**] and the homotopy theory of classifying spaces [**4**]. There are some objects (certain subgroups of P) in $\mathcal{F}_b$ which are of particular interest. These subgroups are called $\mathcal{F}$-centric subgroups, and P is always one of them. One normally denotes the full subcategory of $\mathcal{F}_b$, consisting of all

$\mathcal{F}$-centric subgroups, by $\mathcal{F}_b^{\mathrm{c}}$. Similarly, we have a full subcategory $\mathrm{Tr}_b^{\mathrm{c}}(G) \subset \mathrm{Tr}_b(G)$ with $\mathrm{Ob}\,\mathrm{Tr}_b^{\mathrm{c}}(G) = \mathrm{Ob}\,\mathcal{F}_b^{\mathrm{c}}$. Note that, when $b = b_0$, the $\mathcal{F}_{b_0}$-centric subgroups are exactly the p-centric subgroups.

A p-local finite group on a defect group P of b is defined to be a triple $(P, \mathcal{F}_b, \mathcal{L}_b)$, where $\mathcal{L}_b$ is a certain categorical extension of $\mathcal{F}_b^{\mathrm{c}}$ satisfying some axioms (see [**4**] for general definition). If $b = b_0$ is the principal block, one can explicitly construct $\mathcal{L}_{b_0}$ (usually written as $\mathcal{L}_p^c$) as a quotient category of $\mathrm{Tr}_p^{\mathrm{c}}(G)_{\leqslant P}$, although in general the existence and uniqueness of $\mathcal{L}_b$ for a given $\mathcal{F}_b$ are still unknown. If $\mathcal{L}_b$ exists, there will be important implications in both representation theory [**11**] and homotopy theory [**4**]. For example, the ordinary cohomology ring $H^*(\mathcal{L}_b; \boldsymbol{k}) \cong H^*(B\mathcal{L}_b, k)$ is isomorphic to the cohomology ring of b, usually written as $H^*(b)$, which is defined as the stable elements in $H^*(P, k)$ [**10**]. It is known that $H^*(b)$ is Noetherian [**4**, **10**]. Now one has the following algebra homomorphisms:

$$HH^*(k\mathcal{L}_b) \cong \mathrm{Ext}^*_{k\mathcal{L}_b^e}(k\mathcal{L}_b, k\mathcal{L}_b) \to \mathrm{Ext}^*_{k\mathcal{L}_b}(\boldsymbol{k}, \boldsymbol{k}) \cong H^*(B\mathcal{L}_b, k) \cong H^*(b) \to HH^*(b),$$

in which the left-hand '$\to$' is split surjective by [**25**], while the right-hand '$\to$' is injective [**10**]. Note that the algebras $k\mathcal{L}_b$ and b are usually not derived to be equivalent, and the rightmost map induces an isomorphism upon passing to the quotient rings by nilpotents [**12**]. Naturally, we want to compare $HH^*(k\mathcal{L}_b)$ with $HH^*(b)$, and we would like to see whether $HH^*(k\mathcal{L}_b)$ is finitely generated or, even better, if the two rings are isomorphic after modulo nilpotents. At this stage we do not know the answer to our questions. However, we are able to prove the finite generation of the Hochschild cohomology rings of certain transporter categories.

Let G be a finite group and let $\mathcal{P}$ be a finite G-poset. Then we have a *Grothendieck construction* (see, for instance, [**5**]) which is a finite category, written as $\mathrm{Tr}_{\mathcal{P}}(G)$. The objects are just objects of $\mathcal{P}$, but a morphism $x \to y$ is a pair $(g, gx \leqslant y)$ for some $g \in G$. Such a category admits a natural functor to G, regarded as a category with one object $*$. In fact, there exists a sequence of functors:

$$\mathcal{P} \xrightarrow{\gamma} \mathrm{Tr}_{\mathcal{P}}(G) \xrightarrow{\alpha_G} G,$$

where γ is the natural embedding, and whose topological realization is a fibration:

$$\begin{tikzcd} {|\mathcal{P}|} \arrow[r, "|\gamma|"] \arrow[d, "="'] & {|\mathrm{Tr}_{\mathcal{P}}(G)|} \arrow[r, "|\alpha_G|"] \arrow[d, "\simeq"'] & {|G|} \arrow[d, "\simeq"'] \\ {|\mathcal{P}|} \arrow[r] & EG \times_G |\mathcal{P}| \arrow[r] & EG \times_G *. \end{tikzcd}$$

In order to study the finite generation of cohomology rings, we need to recall the Grothendieck cohomology spectral sequence for a functor $\theta \colon \mathcal{D} \to \mathcal{C}$ (see Appendix A)

$$H^i(\mathcal{C}; H^j(?\backslash\theta; N)) \Rightarrow H^{i+j}(\mathcal{D}; N),$$

where $x\backslash\theta$ is the undercategory for each $x \in \mathrm{Ob}\,\mathcal{C}$ (i.e. the comma category in [**13**, X.3]) and N is a $k\mathcal{D}$-module that can be regarded as an $x\backslash\theta$-module through the forgetful functor $x\backslash\theta \to \mathcal{D}$. As a reminder, the cohomology of a small category $\mathcal{C}$ with coefficients in some $M \in k\mathcal{C}$-mod is written as $H^*(\mathcal{C}; M)$, and is isomorphic to $\mathrm{Ext}^*_{k\mathcal{C}}(\boldsymbol{k}, M)$, as a well-known result. Since G only has one object, the Grothendieck spectral sequence for $\alpha_G\colon \mathrm{Tr}_{\mathcal{P}}(G) \to G$ reads as follows:

$$H^i(G; H^j(\tilde{\mathcal{P}}; N)) \Rightarrow H^{i+j}(\mathrm{Tr}_{\mathcal{P}}(G); N),$$

or, to be consistent with our Ext notation,

$$\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, N)) \Rightarrow \mathrm{Ext}^{i+j}_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, N)$$

for any $N \in k\,\mathrm{Tr}_{\mathcal{P}}(G)$-mod, in which $\tilde{\mathcal{P}}$ is the (finite) undercategory (the 'fibre') $*\backslash\alpha_G$ whose skeleton is isomorphic to the poset $\mathcal{P}$ (thus $\tilde{\mathcal{P}} \simeq \mathcal{P}$ as categories). By Appendix A, the above spectral sequence is a module over

$$\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, \boldsymbol{k})) \Rightarrow \mathrm{Ext}^{i+j}_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, \boldsymbol{k}).$$

Meanwhile, we have a morphism between the following Grothendieck spectral sequences, induced by

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\gamma"] \arrow[d] & \mathrm{Tr}_{\mathcal{P}}(G) \arrow[r, "\alpha_G"] \arrow[d, "\alpha_G"'] & G \arrow[d, equal] \\
\mathrm{pt} \arrow[r] & \mathrm{Tr}_{\mathrm{pt}}(G) \arrow[r, "="'] & G
\end{tikzcd}$$

where 'pt' is a point fixed by G,

$$\begin{tikzcd}
\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{k\mathrm{pt}}(\boldsymbol{k}, \boldsymbol{k})) \arrow[r, Rightarrow] \arrow[d] & \mathrm{Ext}^{i+j}_{k\,\mathrm{Tr}_{\mathrm{pt}}(G)}(\boldsymbol{k}, \boldsymbol{k}) \arrow[d] \\
\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, \boldsymbol{k})) \arrow[r, Rightarrow] & \mathrm{Ext}^{i+j}_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, \boldsymbol{k})
\end{tikzcd}$$

This makes the lower spectral sequence, and hence

$$\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, N)) \Rightarrow \mathrm{Ext}^{i+j}_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, N)$$

modules over $\mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathrm{pt}}(G)}(\boldsymbol{k}, \boldsymbol{k}) = \mathrm{Ext}^*_{kG}(k, k)$. We point out that the group cohomology ring $H^*(G, k) \cong \mathrm{Ext}^*_{kG}(k, k)$ acts on $H^*(\mathrm{Tr}_{\mathcal{P}}(G); N) \cong \mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, N)$ via the algebra homomorphism induced by α_G (or $|\alpha_G|$),

$$\mathrm{Ext}^*_{kG}(k, k) \cong H^*(G, k) \to H^*(|\mathrm{Tr}_{\mathcal{P}}(G)|, k) \cong H^*(\mathrm{Tr}_{\mathcal{P}}(G); \boldsymbol{k}) \cong \mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, \boldsymbol{k}).$$

Since $\mathcal{P}$ has the property that $\boldsymbol{k}$ is of finite projective dimension, $H^j(\tilde{\mathcal{P}}; -) \cong \mathrm{Ext}^*_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, -)$ vanishes for large j. Furthermore, the well-known theorem of Evens and Venkov says that, for each j, $\mathrm{Ext}^*_{kG}(k, \mathrm{Ext}^j_{k\tilde{\mathcal{P}}}(\boldsymbol{k}, N))$ is a finitely generated $\mathrm{Ext}^*_{kG}(k, k)$-module. Since E_∞ is a subquotient of E_2 of a cohomology spectral sequence, we have the following statement. We comment that our argument works for any small G-categories $\mathcal{P}$ with the property that $\boldsymbol{k} \in k\mathcal{P}$-mod is of finite projective dimension.

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Lemma 4.2. *For any $N \in k\,\mathrm{Tr}_{\mathcal{P}}(G)$-mod, $\mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k}, N)$ is a finitely generated $\mathrm{Ext}^*_{kG}(k,k)$- and $\mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathcal{P}}(G)}(\boldsymbol{k},\boldsymbol{k})$-module.*

To examine the Hochschild cohomology ring

$$\mathrm{Ext}^*_{k\mathrm{Tr}_{\mathcal{P}}(G)^e}(k\mathrm{Tr}_{\mathcal{P}}(G), k\mathrm{Tr}_{\mathcal{P}}(G)) \cong \mathrm{Ext}^*_{kF(\mathrm{Tr}_{\mathcal{P}}(G))}(\boldsymbol{k}, \mathrm{Res}_{\tau}(k\mathrm{Tr}_{\mathcal{P}}(G))),$$

we have to consider the category of factorizations $F(\mathrm{Tr}_{\mathcal{P}}(G))$. In fact, we can always define a functor $\tilde{\alpha}_G = \alpha_G \circ t$:

$$\begin{tikzcd} F(\mathrm{Tr}_{\mathcal{P}}(G)) \arrow[rr, "t"] \arrow[dr, "\tilde{\alpha}_G"'] & & \mathrm{Tr}_{\mathcal{P}}(G) \arrow[dl, "\alpha_G"] \\ & G & \end{tikzcd}$$

When we look at the Grothendieck spectral sequence for $\tilde{\alpha}_G$, the undercategory $*\backslash\tilde{\alpha}_G$ similarly amounts to a finite poset, and hence we have the next result.

Lemma 4.3. *For any $U \in kF(\mathrm{Tr}_{\mathcal{P}}(G))$-mod, $\mathrm{Ext}^*_{kF(\mathrm{Tr}_{\mathcal{P}}(G))}(\boldsymbol{k}, U)$ becomes a finitely generated $\mathrm{Ext}^*_{kG}(k,k)$- and $\mathrm{Ext}^*_{kF(\mathrm{Tr}_{\mathcal{P}}(G))}(\boldsymbol{k},\boldsymbol{k})$-module.*

Now we can state our main result in this section.

Theorem 4.4. *Let G be a finite group and let $\mathcal{P}$ be a finite G-poset. The Hochschild cohomology ring $\mathrm{Ext}^*_{k\,\mathrm{Tr}_{\mathcal{P}}(G)^e}(k\,\mathrm{Tr}_{\mathcal{P}}(G), k\,\mathrm{Tr}_{\mathcal{P}}(G))$ is a finitely generated algebra.*

If $\mathcal{P} = \mathrm{pt}$, we get the usual assertion that $\mathrm{Ext}^*_{kG^e}(kG,kG)$ is finitely generated over $\mathrm{Ext}^*_{kG}(k,k)$. If $\mathcal{P} = \mathcal{S}_b$, we have $\mathrm{Ext}^*_{kG}(k,k)$ acting on $\mathrm{Ext}^*_{k\,\mathrm{Tr}_b(G)^e}(k\,\mathrm{Tr}_b(G), k\,\mathrm{Tr}_b(G))$ via $\mathrm{Ext}^*_{k\,\mathrm{Tr}_b(G)}(\boldsymbol{k},\boldsymbol{k})$. In particular, when $b = b_0$, $\mathrm{Ext}^*_{k\,\mathrm{Tr}_p(G)}(\boldsymbol{k},\boldsymbol{k}) \cong H^*(b_0) \cong \mathrm{Ext}^*_{kG}(k,k)$.

Corollary 4.5. *If k is a field with positive characteristic $p \mid |G|$ and b is a p-block, then, for any full subcategory $\mathrm{Tr} \subset \mathrm{Tr}_b(G)$ whose objects are closed under G-conjugation, $\mathrm{Ext}^*_{k\,\mathrm{Tr}^e}(k\,\mathrm{Tr}, k\,\mathrm{Tr})$ is a finitely generated algebra.*

Given the principal block b_0 of a group algebra kG, we have a fusion system $\mathcal{F}_{b_0} = \mathcal{F}_p$ over a fixed Sylow p-subgroup S. As we mentioned earlier, there exists a centric linking system $\mathcal{L}_{b_0} = \mathcal{L}^c_p$, which is determined by the full subcategory $\mathrm{Tr}^{\mathrm{c}}_p(G)_{\leqslant S}$ of the transporter category $\mathrm{Tr}_p(G)$, consisting of all p-centric subgroups contained in S [**3**]. In fact, $\mathcal{L}^c_p$ is a quotient category of $\mathrm{Tr}^{\mathrm{c}}_p(G)_{\leqslant S}$ by some p'-groups. In other words, if one looks at the canonical functor $\pi\colon \mathrm{Tr}^{\mathrm{c}}_p(G)_{\leqslant S} \to \mathcal{L}^c_p$, each undercategory has the property such that it has a minimal object whose automorphism group is p' and, if one regards this p'-automorphism group as a subcategory, the left Kan extension along the inclusion is exact. Furthermore, the left Kan extension of the trivial group module is the trivial module of the undercategory. This implies, by an Eckmann–Shapiro-type result, that the cohomology of each undercategory can be reduced to the cohomology of the automorphism group of the above-specified minimal object in it. Consequently, the mod-p cohomology of each undercategory of π with arbitrary coefficients vanishes in positive degrees. To summarize, since $\mathrm{Tr}_p^c(G)_{\leqslant S}$ is equivalent to $\mathrm{Tr}_p^c(G)$ and the Grothendieck spectral sequence for π collapses, we have an isomorphism

$$\mathrm{Ext}^*_{k\mathrm{Tr}_p^c(G)}(\boldsymbol{k}, V) \cong \mathrm{Ext}^*_{k\mathrm{Tr}_p^c(G)_{\leqslant S}}(\boldsymbol{k}, V) \cong \mathrm{Ext}^*_{k\mathcal{L}_p^c}(\boldsymbol{k}, RK_\pi V),$$

where

$$RK_\pi V = H^0(?\backslash\pi; V) \cong \varprojlim_{?\backslash\pi} V$$

is the right Kan extension along π of V. It is similar to [**3**, Lemma 1.3 (iii)], in which right modules are considered and thus the left Kan extension is applied. In particular, the functors

$$G \leftarrow \mathrm{Tr}_p^c(G) \hookleftarrow \mathrm{Tr}_p^c(G)_{\leqslant S} \to \mathcal{L}_p^c$$

induce isomorphisms of mod-p ordinary cohomology rings.

Proposition 4.6. *Let $\mathcal{L}_p^c$ be the centric linking system associated to the principal block of a finite group algebra kG. Then $\mathrm{Ext}^*_{k\mathcal{L}_p^c}(\boldsymbol{k}, RK_\pi V)$ is finitely generated as an $\mathrm{Ext}^*_{k\mathcal{L}_p^c}(\boldsymbol{k}, \boldsymbol{k})$-module.*

However this still leaves us some way from understanding the finite generation of the Hochschild cohomology ring $\mathrm{Ext}^*_{k(\mathcal{L}_p^c)^e}(k\mathcal{L}_p^c, k\mathcal{L}_p^c) \cong \mathrm{Ext}^*_{kF(\mathcal{L}_p^c)}(\boldsymbol{k}, \mathrm{Res}_\tau(k\mathcal{L}_p^c))$.

Appendix A. Grothendieck spectral sequence

Let $\theta\colon \mathcal{D} \to \mathcal{C}$ be a functor between two small categories. For each module $M \in k\mathcal{D}$-mod, it is known to the experts that there exists a Grothendieck spectral sequence $E_2^{*,*}(M) \Rightarrow E_\infty^{*,*}(M)$ or, more explicitly,

$$H^*(\mathcal{C}; H^*(?\backslash\theta); M) \Rightarrow H^*(\mathcal{D}; M),$$

which comes from a double complex $E_0^{*,*}(M)$. Here we mainly want to assure the reader that there exists a natural pairing of such double complexes $E_0^{*,*}(M) \otimes E_0^{*,*}(N) \to E_0^{*,*}(M \hat{\otimes} N)$. With the pairing, we have a product on each page of the Grothendieck spectral sequences

$$E_n^{i,j}(M) \otimes E_n^{s,t}(N) \to E_n^{i+s,j+t}(M \hat{\otimes} N),$$

and

$$E_\infty^{i,j}(M) \otimes E_\infty^{s,t}(N) \to E_\infty^{i+s,j+t}(M \hat{\otimes} N),$$

Thus, we have a ring structure on

$$H^*(\mathcal{C}; H^*(?\backslash\theta); \boldsymbol{k}) \Rightarrow H^*(\mathcal{D}; \boldsymbol{k}),$$

over which the following is a module

$$H^*(\mathcal{C}; H^*(?\backslash\theta); M) \Rightarrow H^*(\mathcal{D}; M).$$

Now we recall how one constructs the Grothendieck spectral sequence. Consider the composite of the following two functors:

$$k\mathcal{D}\text{-mod} \xrightarrow{RK_\theta} k\mathcal{C}\text{-mod} \xrightarrow{\varprojlim_{\mathcal{C}}} \mathrm{Vect}_k,$$

in which $RK_\theta = \varprojlim_{?\backslash\theta}$ is the right Kan extension along θ. Since RK_θ sends injectives to injectives and $\varprojlim_{\mathcal{C}} RK_\theta \cong \varprojlim_{\mathcal{D}}$, we have a Grothendieck spectral sequence (see, for instance, [**9**, § VIII.9]) for each $M \in k\mathcal{D}$-mod. Note that, for a given small category $\mathcal{E}$, one has

$$\varprojlim_{\mathcal{E}}{}^{i} \cong H^i(\mathcal{E}; -) \cong \mathrm{Ext}^i_{k\mathcal{E}}(\boldsymbol{k}, -).$$

Fix a $k\mathcal{D}$-module M; we start with a double complex $E_0^{*,*}(M)$. First take an injective resolution of M

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots .$$

Then we apply RK_θ to get a complex of injective $k\mathcal{C}$-modules

$$RK_\theta I_0 \to RK_\theta I_1 \to RK_\theta I_2 \to \cdots ,$$

and consequently a commutative diagram

$$\begin{tikzcd}
RK_\theta I_0 \arrow[r] \arrow[d] & RK_\theta I_1 \arrow[r] \arrow[d] & RK_\theta I_2 \arrow[r] \arrow[d] & \cdots \\
J_{0,0} \arrow[r] \arrow[d] & J_{1,0} \arrow[r] \arrow[d] & J_{2,0} \arrow[r] \arrow[d] & \cdots \\
J_{0,1} \arrow[r] \arrow[d] & J_{1,1} \arrow[r] \arrow[d] & J_{2,1} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

in which every column is an injective resolution of the top module. Now we apply $\varprojlim_{\mathcal{C}}$ and obtain a double cochain complex, denoted by $E_0^{*,*}(M)$,

$$\begin{tikzcd}
\varprojlim_{\mathcal{C}} J_{0,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{1,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{2,0} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J_{0,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{1,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{2,1} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J_{0,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{1,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J_{2,2} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

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and it gives rise to the Grothendieck spectral sequence recorded above. We omit the details, as the construction is standard and we are more interested in finding a pairing. Suppose we also have a double complex $E_0^{*,*}(N)$ for another $k\mathcal{D}$-module N:

$$\begin{tikzcd}
\varprojlim_{\mathcal{C}} J'_{0,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{1,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{2,0} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J'_{0,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{1,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{2,1} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J'_{0,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{1,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J'_{2,2} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

and, furthermore, a double complex $E_0^{*,*}(M \hat{\otimes} N)$ for the $k\mathcal{D}$-module $M \hat{\otimes} N$:

$$\begin{tikzcd}
\varprojlim_{\mathcal{C}} J''_{0,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{1,0} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{2,0} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J''_{0,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{1,1} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{2,1} \arrow[r] \arrow[d] & \cdots \\
\varprojlim_{\mathcal{C}} J''_{0,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{1,2} \arrow[r] \arrow[d] & \varprojlim_{\mathcal{C}} J''_{2,2} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

We want to establish a natural map

$$\varprojlim_{\mathcal{C}} J_{i,j} \otimes \varprojlim_{\mathcal{C}} J'_{s,t} \to \varprojlim_{\mathcal{C}} J''_{i+s,j+t}$$

that is compatible with the differentials. In fact, since there is a unique map, given by the universal property of $\varprojlim$,

$$\varprojlim_{\mathcal{C}} J_{i,j} \otimes \varprojlim_{\mathcal{C}} J'_{s,t} \to \varprojlim_{\mathcal{C}} J_{i,j} \hat{\otimes} J'_{s,t},$$

we only need to construct a map

$$\varprojlim_{\mathcal{C}} J_{i,j} \hat{\otimes} J'_{s,t} \to \varprojlim_{\mathcal{C}} J''_{i+s,j+t}.$$

Our definition is again based on the universal property of $\varprojlim$, along with the tensor product of complexes of functors in § 3. We emphasize that

$$\varprojlim_{\mathcal{C}} J_{i,j} \otimes \varprojlim_{\mathcal{C}} J'_{s,t} \to \varprojlim_{\mathcal{C}} J_{i,j} \hat{\otimes} J'_{s,t}$$

respects the differentials in $E_0^{*,*}$ due to its construction via the universal property. This is the case when we define

$$\varprojlim_{\mathcal{C}} J_{i,j} \hat{\otimes} J'_{s,t} \to \varprojlim_{\mathcal{C}} J''_{i+s,j+t}$$

and thus we will not verify that the map we are about to construct does respect differentials.

From the two injective resolutions $0 \to M \to I_*$ and $0 \to N \to I'_*$, we can build a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \hat{\otimes} N \arrow[r] \arrow[d, equal] & I_* \hat{\otimes} I'_* \arrow[d] \\ 0 \arrow[r] & M \hat{\otimes} N \arrow[r] & I''_* \end{tikzcd}$$

in which the upper row is an exact sequence and the lower one is the injective resolution used to define $E_0^{*,*}(M \hat{\otimes} N)$. Applying RK_θ, we obtain a chain map $RK_\theta(I_* \hat{\otimes} I'_*) \to RK_\theta I''_*$. In particular, we have for any non-negative integers i and s a map $RK_\theta(I_i \hat{\otimes} I'_s) \to RK_\theta I''_{i+s}$. The universal property of $\varprojlim$ provides a morphism $RK_\theta I_i \hat{\otimes} RK_\theta I'_s \to RK_\theta(I_i \hat{\otimes} I'_s)$. Thus, we have a natural map

$$RK_\theta I_i \hat{\otimes} RK_\theta I'_s \to RK_\theta I''_{i+s}.$$

Next we repeat the above tensor construction for the two injective resolutions $0 \to RK_\theta I_i \to J_{i,*}$ and $0 \to RK_\theta I'_s \to J'_{s,*}$. It follows from our discussions that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & RK_\theta I_i \hat{\otimes} RK_\theta I'_s \arrow[r] \arrow[d] & J_{i,*} \hat{\otimes} J'_{s,*} \arrow[d] \\ 0 \arrow[r] & RK_\theta I''_{i+s} \arrow[r] & J''_{i+s,*+*}. \end{tikzcd}$$

In particular there exists $J_{i,j} \hat{\otimes} J'_{s,t} \to J''_{i+s,j+t}$, and consequently the desired map

$$\varprojlim_{\mathcal{C}} J_{i,j} \hat{\otimes} J'_{s,t} \to \varprojlim_{\mathcal{C}} J''_{i+s,j+t}.$$

Hence, we do obtain a pairing $E_0^{*,*}(M) \otimes E_0^{*,*}(N) \to E_0^{*,*}(M \hat{\otimes} N)$.

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