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A CHARACTER-THEORETIC CRITERION FOR THE SOLVABILITY OF FINITE GROUPS

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Abstract

Let p be an odd prime. In this note, we show that a finite group G is solvable if all degrees of irreducible complex characters of G not divisible by p are either 1 or a prime.

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1. Introduction

Let p be a prime and G a finite group. Define $\mathrm{Irr}(G)$ to be the set of all irreducible complex characters of G and $\mathrm{Irr}_{p'}(G)$ the subset of those irreducible complex characters of G of p'-degrees, that is, characters whose degrees are not divisible by p. Irreducible characters of a finite group of p'-degrees have attracted considerable attention, partly due to the famous McKay conjecture asserting that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$, where $N_G(P)$ is the normaliser of a Sylow p-subgroup P of G. It was known that they (or some of them) have an influence on the structure of G. For instance, the Ito–Michler theorem states that a finite group G has a normal abelian Sylow p-subgroup if and only if all of the irreducible complex characters of G have p'-degrees [14, Theorem 2.3], and a special case of the recently proved Gluck–Wolf theorem for arbitrary finite groups states that if $\lambda \in \mathrm{Irr}(Z)$ is a linear complex character of a normal subgroup Z of G such that $\chi(1)$ is not divisible by p for all $\chi \in \mathrm{Irr}(G)$ lying over Z, then G/Z has abelian Sylow p-subgroups [16, Theorem A].

As usual, let $\mathrm{cd}(G)$ and $\mathrm{cd}_{p'}(G)$ be the degree sets of $\mathrm{Irr}(G)$ and $\mathrm{Irr}_{p'}(G)$, respectively. The main purpose of this note is to investigate finite groups G under some assumption on $\mathrm{cd}_{p'}(G)$. This is motivated by the classification of finite groups with only one nonlinear irreducible character of p'-degree and the recent work of the authors on finite groups almost all of whose irreducible character degrees are primes

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(see [9, Theorem A] and [11], respectively). Our main result gives a character-theoretic criterion for a finite group to be solvable.

Theorem 1.1. *Let p be an odd prime. If G is a finite group such that each member of $\mathrm{cd}_{p'}(G)$ is either 1 or a prime, then G is solvable.*

We remark here that Theorem 1.1 does not hold if $p = 2$. A counterexample is $G = S_5$ with $\mathrm{cd}(G) = \{1, 4, 5, 6\}$. As another illustration, note that $G = \mathrm{Aut}(L_2(27))$ is not solvable and $\mathrm{cd}_{3'}(G) = \{1, 26\}$. Finally, we mention that it is not in general possible to determine the solvability of a group from its character degrees [15].

2. Preliminaries

Here we list some results for later use. We begin with a result that plays an important role in the proof of Theorem 1.1.

Lemma 2.1 [4, Lemma 5]. *Let N be a minimal normal subgroup of G so that $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$ is a nonabelian simple group. Let A be the automorphism group of S. If $\sigma \in \mathrm{Irr}(S)$ extends to A, then $\sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$ extends to G.*

The Steinberg character of a finite simple group of Lie type is significant in our investigation.

Lemma 2.2 [17, 18]. *Let S be a finite simple group of Lie type of characteristic r and St the Steinberg character of S. Then $\mathrm{St}(1) = |S|_r$ and St extends to the automorphism group of S.*

The following lemma gives the classification of faithful irreducible characters of prime degree of quasi-simple groups.

Lemma 2.3. *Let G be a quasi-simple group such that $S := G/Z(G)$ is nonabelian and simple and let χ be a faithful irreducible complex character of G. Suppose that $\chi(1) = r$ is a prime. Then one of the following holds:*

(1) *$G = S$ is a simple group of Lie type of characteristic r and χ is the Steinberg character of G (so $\chi(1) = |G|_r$);*
(2) *$S = \mathrm{L}_2(q)$ and $\chi(1) \in \{q \pm 1\}$, or q is an odd prime and $\chi(1) \in \{(q \pm 1)/2\}$;*
(3) *$S = \mathrm{L}_n(q)$, $q > 2$, n is an odd prime, $(n, q-1) = 1$, $\chi(1) = (q^n - 1)/(q - 1)$;*
(4) *$S = \mathrm{U}_n(q)$, n is an odd prime, $(n, q+1) = 1$, $\chi(1) = (q^n + 1)/(q + 1)$;*
(5) *$S = \mathrm{PSp}_{2n}(q)$, $n > 1$, $q = p^k$ with p an odd prime, kn is a 2-power, $\chi(1) = (q^n + 1)/2$;*
(6) *$S = \mathrm{PSp}_{2n}(3)$, $n > 1$ is a prime, $\chi(1) = (3^n - 1)/2$;*
(7) *$r = 7$, $G = \mathrm{Sp}_6(2)$.*

Proof. This is a special case of [13, Theorem 1.1] and [13, Conjecture], which has been proved in [2, 3]. □

To prove Theorem 1.1, we also need the following result, which is a slightly stronger version of [4, Theorems 3 and 4].

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Table 1. Aut(S)-extendible irreducible characters of coprime composite degrees.

Group	Chars.	Degrees	Group	Chars.	Degrees
M_{11}	χ_8	$44 = 2^2 \cdot 11$	Co_3	χ_5	$275 = 5^2 \cdot 11$
	χ_9	$45 = 3^2 \cdot 5$		χ_{21}	$26\,082 = 2 \cdot 3^4 \cdot 7 \cdot 23$
M_{24}	χ_7	$252 = 2^2 \cdot 3^2 \cdot 7$	Co_2	χ_4	$275 = 5^2 \cdot 11$
	χ_8	$253 = 11 \cdot 23$		χ_{36}	$312\,984 = 2^3 \cdot 3^5 \cdot 7 \cdot 23$

Lemma 2.4. *Let S be a sporadic simple group, the Tits group or the alternating group A_n for $n \geq 7$. Then S has two nonlinear* Aut(S)*-extendible irreducible characters χ_1 and χ_2 such that $(\chi_1(1), \chi_2(1)) = 1$ and neither $\chi_1(1)$ nor $\chi_2(1)$ is a prime.*

Proof. The result follows directly from [4, Theorem 3] if $S \cong A_n$. For S a sporadic simple group or the Tits group, the result follows from the Atlas [5] (or by [4, Table 1] for most of the cases and from Table 1 for the remaining four cases). □

Finally, we mention a result of Isaacs and Knutson [8], which is a strengthened version of a theorem of Berkovich [1]. For a group G, G' denotes the derived group of G and, for $N \lhd G$, we write $\mathrm{Irr}(G \mid N) = \{\chi \in \mathrm{Irr}(G) \mid N \nsubseteq \ker(\chi)\}$ and $\mathrm{cd}(G \mid N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G \mid N)\}$.

Lemma 2.5 [8, Theorem D]. *Let $N \lhd G$ and suppose that every member of* $\mathrm{cd}(G \mid N')$ *is divisible by some fixed prime p. Then N is solvable and has a normal p-complement.*

3. Proof of Theorem 1.1

Lemma 3.1. *Let G be a finite group such that each member of* $\mathrm{cd}_{p'}(G)$ *is either 1 or a prime. Suppose that G has a unique minimal normal subgroup N, which is nonabelian and has order divisible by p. Then N is a simple group of Lie type of characteristic p and has an irreducible character θ such that $\theta(1)$ is a prime different from p.*

Proof. Let $N = S \times \cdots \times S$ be the direct product of t copies of a nonabelian simple group S. By Lemma 2.5, we may choose $\chi \in \mathrm{Irr}(G)$ such that $N \nleq \ker(\chi)$ and $\chi(1) = r$, for a prime r different from p. Observe that the trivial character 1_N of N is the unique irreducible character of N of degree 1. We have $\theta = \chi_N \in \mathrm{Irr}(N)$. In particular, $I_G(\theta) = G$ and $\theta(1) = r$ is a prime. Since $\theta = \theta_1 \times \cdots \times \theta_t$ for some $\theta_i \in \mathrm{Irr}(S)$, we have $\theta_1 = \cdots = \theta_t$, so that $t = 1$ and hence N is simple.

By Lemma 2.4, N is a simple group of Lie type. Let ℓ be the defining characteristic of N. If $\ell \neq p$, then, by Lemma 2.2, $|S|_\ell \in \mathrm{cd}(G)$, whence $N \cong L_2(\ell)$ with $\ell \geq 5$. In particular, ℓ is odd. Notice that $\mathrm{Aut}(N) = \mathrm{PGL}_2(\ell)$ and $\{\ell - 1, \ell, \ell + 1\} \subset \mathrm{cd}(G)$. Since p is odd, it follows that p divides at most one of $\ell - 1$ and $\ell + 1$. Therefore, G has an irreducible character of p'-degree that is not a prime. This contradiction shows that $\ell = p$. □

Proof of Theorem 1.1. We first suppose that $p \nmid |G|$, so that all degrees of irreducible characters of G are 1 or a prime. By [10, Theorem 4.1], $|\mathrm{cd}(G)| \leq 3$. Hence, by [7, Theorem 12.15], G is solvable. So, we now suppose that p is a prime divisor of $|G|$.

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Let N be a minimal normal subgroup of G. If G has another minimal normal subgroup M, then, by induction on $|G|$, both G/N and G/M are solvable, so that $\Gamma = G/N \times G/M$ is solvable. Since G can be viewed as a subgroup of Γ, we conclude that G is solvable. Therefore, we may assume that N is the unique minimal normal subgroup of G.

If N is a p-group or an abelian p'-group, then G/N and so G is solvable. Assume that N is a nonabelian p'-group, so that $N \cong S_1 \times \cdots \times S_t$, where $S_i \cong S$, a nonabelian simple group. If $N \not\cong L_2(r)$ for some prime $r \geq 7$, then, by Lemmas 2.1, 2.2 and 2.4, N and so G has an irreducible character whose degree is composite and not divisible by p, which is a contradiction. So, we have $N \cong L_2(r)$ for some prime $r \geq 7$. Let $\overline{G} = G/C_G(N)$, so that $\overline{G}$ has socle isomorphic to N. Note that both $r-1$ and $r+1$ are composite and $p \mid |\overline{G}/\overline{N}|$. Checking the degrees of irreducible characters of $\overline{G}$ from [19, Theorem A], we get a contradiction.

From now on, we assume that N is nonabelian with $p \mid |N|$. Suppose that N has a P-invariant irreducible character θ with $\theta(1)$ composite and coprime to p. By [7, Theorem 8.15], θ extends to P. Let $\widehat{\theta} \in \mathrm{Irr}(P)$ be an extension of θ. Then $\widehat{\theta}^G$ has p'-degree, whence it has an irreducible constituent of p'-degree divisible by $\theta(1)$, which is a contradiction.

So, it remains to show that N has a P-invariant irreducible character θ with $\theta(1)$ composite and coprime to p. By Lemma 3.1, N is a simple group of Lie type of characteristic p and has an irreducible character θ such that $\theta(1)$ is some prime r different from p. Since N is one of the groups in Lemma 2.3, we can take a case-by-case analysis to the possibilities for N. Clearly, $N \not\cong \mathrm{PSp}_6(2)$ since p is odd and, if $N \cong L_2(q)$, then the result follows from [19, Theorem A]. So, we may assume that N is one of the groups listed in (3)–(6) of Lemma 2.3.

Let $N = \mathbf{G}^F/Z(\mathbf{G}^F)$, where $\mathbf{G}$ is a simple simply-connected algebraic group defined over an algebraically closed field of characteristic p, and F is a standard Frobenius map of $\mathbf{G}$ with finite group of fixed points $G := \mathbf{G}^F$. Let $(\mathbf{G}^*, F^*)$ be the dual pair of $(\mathbf{G}, F)$ and $G^* = \mathbf{G}^{*F^*}$. Let s be a semisimple element of G^*. Recall that a Lusztig series $\mathcal{E}(G, s)$ associated to the geometric conjugacy class (s) is the set of irreducible characters of G which occur in some Deligne–Lusztig character [6, Definition 13.16]. By [6, Theorem 13.23 and Remark 13.24], there is a bijection between $\mathcal{E}(G, s)$ and the set of unipotent characters of $C_{G^*}(s)$. Moreover, if ψ denotes this bijection, then $\chi(1) = (|G|/|C_{G^*}(s)|)_{p'}\psi(\chi)(1)$ for any $\chi \in \mathcal{E}(G, s)$. In particular, the semisimple character $\chi_s \in \mathcal{E}(G, s)$, which corresponds to the trivial character of $C_{G^*}(s)$, has degree $(|G|/|C_{G^*}(s)|)_{p'}$. If we choose $s \neq 1$ to be a semisimple element in the derived subgroup of G^*, then, by [16, Lemma 4.4], $Z(G) \subseteq \ker\chi_s$, so that χ_s can be viewed as a character of N. In addition, if (s) is $\mathrm{Aut}(G^*)$-invariant, then, by [12], χ_s is $\mathrm{Aut}(G)$-invariant. In the following, we choose a suitable s such that $\theta = \chi_s$ is the desired character.

If $N \cong \mathrm{PSp}_{2n}(3)$, then $p = 3$ and G^* has a maximal torus T of order $|T| = 3^n + 1$. Moreover, $T \cap (G^*)'$ has a regular element s of order r_0, where r_0 is a primitive prime divisor of $|T|$. Now $\chi_s(1) = (|G|/|T|)_{3'}$, which can be easily seen to be composite, and the result follows from the fact that $|P/N| = 1$ in this case.

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To check the remaining cases, we write $q = p^f$, where $f = p^a k$ with $(p, k) = 1$. Let $N \cong L_n(q)$, $U_n(q)$ for an odd prime n or $\mathrm{PSp}_{2n}(q)$ for a prime power n of 2, as listed in Lemma 2.3(3)–(5). We first assume that $a = 0$. Since G^* has a maximal torus T of order $(q^n - 1)/(q - 1)$, $(q^n + 1)/(q + 1)$ or $q^n + 1$, we may choose a regular element s in $T \cap (G^*)'$ of order r_0, where r_0 is a primitive prime divisor of $q^n - 1, q^n + 1$ or $q^n + 1$, respectively. Since $|P/N| = 1$, it is easy to see that $\theta = \chi_s$ is the desired character.

We now assume that $a > 0$ and $|P/N| > 1$. Let α be a field automorphism of G^* of order p^a and $q_0 = p^k$. Then the centraliser C of α in G^* is $\mathrm{PGL}_n(q_0)$, $\mathrm{PGU}_n(q_0)$ or $\mathrm{PCSp}_{2n}(q_0)$, respectively. As we did in the previous paragraph, we may choose a semisimple element s in the derived subgroup C' of C. Also, $\theta = \chi_s$ is the desired character, finishing the proof. □

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