On the Fourier coefficients of a discontinuous function

By S. P. BHATNAGAR.

(*Received* 29*th May*, 1940. *Read* 1*st June*, 1940.)

§ 1.

Introduction.

We suppose throughout that $f(t)$ is periodic with period 2π, and Lebesgue-integrable in $(-\pi, \pi)$.

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

and suppose that the Fourier series of $\phi(t)$ and $\psi(t)$ are respectively $\sum_{n=0}^{\infty} A_n \cos nt$ and $\sum_{n=1}^{\infty} B_n \sin nt$. Then the Fourier series and allied series of $f(t)$ at the point $t = x$ are respectively $\sum_{n=0}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$, where $A_0 = \frac{1}{2}a_0$, $A_n = a_n \cos nx + b_n \sin nx$, $B_n = b_n \cos nx - a_n \sin nx$ and a_n, b_n are the Fourier coefficients of $f(t)$.

We write, for $t > 0$,

$$\theta(t) = \frac{2}{\pi}\int_t^{\infty} \frac{\psi(u)}{u}\, du,$$

$$\Phi_a(t) = \frac{1}{\Gamma(a)}\int_0^t (t-u)^{a-1}\,\phi(u)\, du, \quad (a > 0),$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_a(t) = \Gamma(a+1)\, t^{-a}\, \Phi_a(t), \quad (a \geqq 0),$$

and we define $\Psi_a(t)$, $\psi_a(t)$, $\Theta_a(t)$, $\theta_a(t)$, etc., in a similar way.

We also write s_n^a, $\bar{s}_n^a$, τ_n^a, $\bar{\tau}_n^a$ for the n-th Cesàro means of order a of the sequences $s_n = \sum_{\nu=0}^{n} A_\nu$, $\bar{s}_n = \sum_{\nu=1}^{n} B_\nu$, $\tau_n = nA_n = n\Delta s_n$ and $\bar{\tau}_n = nB_n = n\Delta \bar{s}_n$ respectively[1], and $s_{-1} = 0$, $\bar{s}_0 = \bar{s}_{-1} = 0$.

---

[1] Here $\Delta p_n = p_n - p_{n-1}$, and $\bar{s}_n^a = \bar{S}_n^a / A_n^a$, where $\bar{S}_n^a$ and A_n^a are defined formally by

$$\sum_{n=1}^{\infty} \bar{S}_n^a x^n = (1-x)^{-a-1} \sum_{n=1}^{\infty} B_n x^n \text{ and } \sum_{n=0}^{\infty} A_n^a x^n = (1-x)^{-a-1}.$$

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

Finally we write[1], for $\alpha \geqq 0$,

$$r_\alpha(\omega) = \omega^{-\alpha} \sum_{n<\omega} (\omega - n)^\alpha A_n,$$

$$\bar{r}_\alpha(\omega) = \omega^{-\alpha} \sum_{n<\omega} (\omega - n)^\alpha B_n.$$

Concerning the Cesàro summability of a Fourier series and its allied series at the point $t = x$, the following two theorems of Hardy and Littlewood are well known[2].

Theorem A. A necessary and sufficient condition that s_n should tend to a limit s (C), is that $\phi_\lambda(t)$ should tend to s as $t \to +0$, for some positive λ.

Theorem B. A necessary and sufficient condition that $\bar{s}_n$ should tend to a limit s (C), is that $\theta_\lambda(t)$ should tend to s as $t \to +0$, for some positive λ.

Concerning the existence of the Cesàro limits of the sequences nA_n and nB_n, we have the following known results.

Theorem C. A necessary and sufficient condition[3] that nA_n should tend to a limit s (C), is that $t\dfrac{d}{dt}\phi_\lambda(t)$ should tend to $-s$ as $t \to +0$, for some positive λ.

Theorem D. A necessary and sufficient condition[4] that nB_n should tend to a limit s (C), is that $\psi_\lambda(t)$ should tend to $\frac{1}{2}\pi s$ as $t \to +0$, for some positive λ, or what is the same thing that $t\dfrac{d}{dt}\theta_\lambda(t)$ should tend to $-s$ as $t \to +0$.

We next observe that the condition that "$nA_n = n\Delta s_n = \tau_n$ tends to a limit $s(C, \lambda)$," or what is the same thing, that "$\tau_n^\lambda = n\Delta s_n^\lambda$ tends to the limit s" is equivalent[5] to the condition that

---

[1] Here $r_\alpha(\omega)$, $\bar{r}_\alpha(\omega)$ denote the Rieszian arithmetic means of order α of the Fourier series and allied series respectively.

[2] Hardy and Littlewood, **14** and **14** (a). More precise results have been given by other writers. For references see Bosanquet and Hyslop, **9**, 491-2.

[3] See Bosanquet, **7**, where more precise results are given.

[4] See Paley, **22**. Though results of this type are not explicitly stated there, much more precise results are implied by his analysis.

[5] See Bosanquet, **4**. This is stated there.

$$\frac{\lambda}{\omega} \sum_{n<\omega} \left(1 - \frac{n}{\omega}\right)^{\lambda - 1} n A_n \to s,$$

or

$$\omega \frac{d}{d\omega} \sum_{n<\omega} \left(1 - \frac{n}{\omega}\right)^{\lambda} A_n \to s,$$

or

$$\omega \frac{d}{d\omega} r_\lambda (\omega) \to s,$$

as $\omega \to \infty$.

Regarding the sequences $\left(\omega \dfrac{d}{d\omega}\right)^\lambda r_\beta(\omega)$ and $\left(\omega \dfrac{d}{d\omega}\right)^\lambda \bar{r}_\beta(\omega), \beta \geqq \lambda$, we have the following two theorems of Bosanquet[1].

Theorem E. If λ is a non-negative integer, a necessary and sufficient condition that

$$\left(\omega \frac{d}{d\omega}\right)^\lambda r_\beta (\omega) = O(1)$$

as $\omega \to \infty$ for some $\beta \geqq \lambda$, is that

$$\left(t \frac{d}{dt}\right)^\lambda \phi_\kappa (t) = O(1)$$

in the interval $(0, \pi)$ for some $\kappa \geqq \lambda$, where $\phi_\kappa(t)$ is a λ-th integral except at $t = 0$.

Theorem F. If λ is a non-negative integer, a necessary and sufficient condition that

$$\left(\omega \frac{d}{d\omega}\right)^\lambda \bar{r}_\beta (\omega) = O(1)$$

as $\omega \to \infty$ for some $\beta \geqq \lambda$, is that

$$\left(t \frac{d}{dt}\right)^\lambda \theta_\kappa (t) = O(1)$$

in the interval $(0, \pi)$ for some $\kappa \geqq \lambda$, where $\theta_\kappa(t)$ is a λ-th integral except at $t = 0$, or what is the same thing, if $\lambda \geqq 1$, that

$$\left(t \frac{d}{dt}\right)^{\lambda - 1} \psi_\kappa (t) = O(1)$$

in the interval $(0, \pi)$ for some $\kappa \geqq \lambda$.

Theorem D is illustrated by the following examples.

(i) If
$$\psi(t) = \begin{cases} 1 & (0, \pi) \\ 0 & t = 0 \\ -1 & (-\pi, 0) \end{cases}$$

---

[1] See Bosanquet, 7.

then $nB_n = \frac{2}{\pi}(1 - \cos n\pi)$, which tends to $\frac{2}{\pi}(C, \delta), \delta > 0$, as $n \to \infty$.

(ii) If $nB_n = 1, (n \geqq 1)$ then

$$\psi(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n} = \begin{cases} \frac{1}{2}(\pi - t) & (0, \pi) \\ 0 & t = 0 \\ -\frac{1}{2}(\pi - t) & (-\pi, 0), \end{cases}$$

which tends to $\frac{1}{2}\pi$ as $t \to +0$.

To illustrate Theorem C, we may either take $nA_n = 1, (n \geqq 1)$, in which case we have

$$\phi(t) = \sum_{n=1}^{\infty} \frac{\cos nt}{n} = \log\left|\frac{1}{2 \sin \frac{1}{2} t}\right|,$$

or we may consider the function $\phi(t) = \log\left|\frac{1}{t}\right|$ in $(-\pi, \pi)$.

Finally to illustrate Theorems E and F, we may consider respectively the following functions.

$$\phi(t) = \log^{\lambda}\left|\frac{1}{t}\right| \text{ in } (-\pi, \pi)$$

and[1]

$$\psi(t) = \operatorname{sign} t \log^{\lambda - 1}\left|\frac{1}{t}\right| \text{ in } (-\pi, \pi).$$

It is reasonable to suppose that the means $\left(\omega \frac{d}{d\omega}\right)^{\lambda} r_{\beta}(\omega)$ and $\left(\omega \frac{d}{d\omega}\right)^{\lambda} \bar{r}_{\beta}(\omega)$ in Theorems E and F can be replaced by $(n\Delta)^{\lambda} s_n^{\beta}$ and $(n\Delta)^{\lambda} \bar{s}_n^{\beta}$ respectively, and also the O by o, or appropriate limits[2]. The latter means have the advantage that they can be used when $\lambda - 1 < \beta < \lambda$, whereas the former become infinite for integral values of ω.

For example if we consider the particular case of Theorem F when $\lambda = 2$ and suppose that in $(-\pi, \pi)$

$$\psi(t) = \operatorname{sign} t \log\left|\frac{\pi}{t}\right|,$$

then $(n\Delta)^2 \bar{s}_n$ tends to the limit $\frac{2}{\pi}$ as $n \to \infty$.

In fact, for $n \geqq 1$,

---

[1] Sign $z = \frac{z}{|z|}$ if $z \neq 0$, and sign $0 = 0$.

[2] $(n\Delta)^{\lambda} a_n = n\Delta (n\Delta)^{\lambda - 1} a_n$, λ being a positive integer, $(n\Delta)^0 a_n = a_n$.

$$B_n = \frac{2}{\pi}\int_0^{\pi} \log\frac{\pi}{t} \sin nt \, dt$$

$$= \left[\frac{2}{\pi}\,\frac{1-\cos nt}{n} \log \frac{\pi}{t}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos nt}{nt}\, dt,$$

that is,

$$nB_n = \frac{2}{\pi}\int_0^{n\pi} \frac{1-\cos t}{t}\, dt,$$

and hence, for $n \geqq 2$,

$$(n\Delta)^2\, \bar{s}_n = \frac{2}{\pi}\, n \int_{(n-1)\pi}^{n\pi} \frac{1-\cos t}{t}\, dt,$$

which tends to $\dfrac{2}{\pi}$ as $n \to \infty$.

Again, if we take $nB_n = \dfrac{2}{\pi} \sum\limits_{\nu=1}^{n} \dfrac{1}{\nu}$, then $(n\Delta)^2 \bar{s}_n = \dfrac{2}{\pi}$ $(n \geqq 1)$, and it can be proved that $t \dfrac{d}{dt}\psi(t)$ tends to the limit -1 as $t \to +0$.

To show this we consider the two functions

$$\xi(t) = \sum_{n=1}^{\infty} a_n \sin nt, \quad \eta(t) = \sum_{n=1}^{\infty} \beta_n \sin nt$$

such that in $(0, \pi)$ $\qquad \xi(t) = \log \dfrac{\pi}{t}$,

and $$n\beta_n = \frac{2}{\pi} \sum_{\nu=1}^{n} \frac{1}{\nu}, \quad (n \geqq 1).$$

Then to prove that $t \dfrac{d}{dt}\eta(t)$ tends to -1 as $t \to +0$, it is enough to show that $t \dfrac{d}{dt}\{\eta(t) - \xi(t)\}$ tends to zero as $t \to +0$.

We write

$$\eta(t) - \xi(t) = \sum_{n=1}^{\infty} (\beta_n - a_n) \sin nt$$

$$= \sum_{n=1}^{\infty} \frac{d_n}{n} \sin nt.$$

If we now show that $d_n = C + C_n$, where C_n is steadily decreasing and tends to zero, it will follow that

$$t\frac{d}{dt}\{\eta(t) - \xi(t)\} = -\tfrac{1}{2}Ct + \sum_{n=1}^{\infty} C_n t \cos nt = o(1)$$

as $t \to +0$, since $\sum_{n=1}^{\infty} C_n \cos nt$ converges uniformly in any interval $\delta \leqq t \leqq 2\pi - \delta$, $0 < \delta < \pi$, while $\sum_{n=1}^{\infty} C_n t \cos nt$ converges uniformly in $0 \leqq t \leqq \pi$.

We next write

$$d_n = \frac{2}{\pi} \sum_{\nu=1}^{n} \frac{1}{\nu} - \frac{2}{\pi} \int_0^{n\pi} \frac{1-\cos t}{t}\, dt,$$

so that

$$\Delta d_n = \frac{2}{\pi} \int_{(n-1)\pi}^{n\pi} \left\{ \frac{1}{n\pi} - \frac{1-\cos t}{t} \right\} dt$$

$$< \frac{2}{n\pi^2} \int_{(n-1)\pi}^{n\pi} \cos t\, dt$$

$$= 0,$$

which shows that d_n is monotonic and steadily decreasing.

Also we have

$$\Delta d_n > \frac{2}{n(n-1)\pi^2} \int_{(n-1)\pi}^{n\pi} (n \cos t - 1)\, dt = -\frac{2}{n(n-1)\pi},$$

and hence $d_n - d_1 > -\frac{2}{\pi}\left(1 - \frac{1}{n}\right) > -\frac{2}{\pi}$, which shows that d_n is bounded below and hence, being monotonic and decreasing, tends to a finite limit. Thus d_n is of the form required.

In § 2 of this paper we give some general lemmas, which are required in particular cases in the subsequent work. In § 3 we obtain some results related to Theorem D, which complete some of the known results about the connection between the jump of a function and its Fourier coefficients. Finally in § 4 we consider analogous problems related to Theorem F in the case $\lambda = 2$, using Cesàro means instead of Rieszian means.

§ 2.

We write $\kappa^a(n, t)$ and $\bar{\kappa}^a(n, t)$ for the n-th Cesàro means of order a (> -1) of the series

$$\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \cos nt, \quad \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nt$$

respectively and suppose that, for $a > -1$,

$$K^{a}(n, t) + i\,\overline{K}^{a}(n, t) = \frac{2}{\pi}\frac{1}{A_n^a}\cdot\frac{e^{i\{(n+\frac{1}{2}+\frac{1}{2}a)t - \frac{1}{2}(a+1)\pi\}}}{(2\sin\frac{1}{2}t)^{1+a}} \tag{1}$$

where

$$A_n^\sigma = \frac{\Gamma(n+1+\sigma)}{\Gamma(n+1)\,\Gamma(\sigma+1)}.$$

Lemma 1. *For* $0 < t < \pi$, *and* $a > \sigma - 1$, *we have*

$$\left|\left(\frac{d}{dt}\right)^{\rho}\Delta^{\sigma}\,\kappa^{a}(n, t)\right| < A n^{1+\rho-\sigma}, \tag{2}$$

$$\left|\left(\frac{d}{dt}\right)^{\rho}\Delta^{\sigma}\left\{\bar{\kappa}^{a}(n, t) - \overline{K}^{a}(n, t) - \frac{1}{\pi}\cot\tfrac{1}{2}t\right\}\right| < A n^{-1-\sigma}\,t^{-2-\rho}, \tag{3}$$

where ρ *and* σ *are non-negative integers and the* A'*s are independent of* n *and* t.

The inequalities (2) and (3) are well known for the case[1] $\sigma = 0$, and by the method of induction we now obtain the results for $\sigma = \lambda$, where λ is a positive integer. Assuming the result of (2) for $\sigma = 0, 1, 2, \ldots\ldots \lambda - 1$, we have[2] for $a > \lambda - 1$ and $n \geqq \lambda$,

$$\left(\frac{d}{dt}\right)^{\rho}\Delta^{\lambda}\,\bar{\kappa}^{a}(n, t) = \left(\frac{d}{dt}\right)^{\rho}\Delta^{\lambda-1}\left[\frac{a}{n}\{\bar{\kappa}^{a-1}(n, t) - \bar{\kappa}^{a}(n, t)\}\right]$$

which in turn[3] may be written in the form

$$\sum_{p=0}^{\lambda-1}\binom{\lambda-1}{p}\Delta^{p}\frac{a}{n}\,\Delta^{\lambda-p-1}\left(\frac{d}{dt}\right)^{\rho}\{\bar{\kappa}^{a-1}(n-p, t) - \bar{\kappa}^{a}(n-p, t)\}$$

$$= \sum_{p=0}^{\lambda-1} O(n^{-p-1})\,O(n^{p+2+\rho-\lambda}) = \sum_{p=0}^{\lambda-1} O(n^{1+\rho-\lambda}) = O(n^{1+\rho-\lambda}).$$

This proves (2).

---

[1] See Zygmund, 27. Obrechkoff, 21, 86-93. Gergen, 13, 264-7.

[2] It is easy to verify that for $a > 0$, we have the following identities.

$$\frac{a}{n+a}\frac{d}{dt}\bar{\kappa}^{a-1}(n, t) = n\Delta\,\kappa^{a}(n, t) = a\{\kappa^{a-1}(n, t) - \kappa^{a}(n, t)\}$$

$$-\frac{a}{n+a}\frac{d}{dt}\kappa^{a-1}(n, t) = n\Delta\,\bar{\kappa}^{a}(n, t) = a\{\bar{\kappa}^{a-1}(n, t) - \bar{\kappa}^{a}(n, t)\}$$

$$\frac{a}{n+a}\frac{d}{dt}\overline{K}^{a-1}(n, t) = n\Delta K^{a}(n, t) = a\{K^{a-1}(n, t) - K^{a}(n, t)\}$$

$$-\frac{a}{n+a}\frac{d}{dt}K^{a-1}(n, t) = n\Delta\overline{K}^{a}(n, t) = a\{\overline{K}^{a-1}(n, t) - \overline{K}^{a}(n, t)\}$$

The first two of these sets of identities follow from Lemma 5, and the last two were pointed out to me by Dr Bosanquet.

[3] Here we use the following result

$$\Delta^{m}\,p_n\,q_n = \sum_{l=0}^{m}\binom{m}{l}\Delta^{l}p_n\,\Delta^{m-l}q^{n-l}.$$

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

Following the same argument we can write

$$\left(\frac{d}{dt}\right)^{\rho} \Delta^{\lambda} \left\{ \bar{\kappa}^{\alpha}(n, t) - \bar{K}^{\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t \right\} = \left(\frac{d}{dt}\right)^{\rho} \Delta^{\lambda} T^{\alpha}(n, t)$$

$$= \left(\frac{d}{dt}\right)^{\rho} \Delta^{\lambda-1} \left[\frac{\alpha}{n} \{T^{\alpha-1}(n, t) - T^{\alpha}(n, t)\} \right]$$

$$= \sum_{p=0}^{\lambda-1} \binom{\lambda-1}{p} \Delta^{p} \frac{\alpha}{n} \Delta^{\lambda-p-1} \left(\frac{d}{dt}\right)^{\rho} \{T^{\alpha-1}(n-p, t) - T^{\alpha}(n-p, t)\}$$

$$= \sum_{p=0}^{\lambda-1} O(n^{-p-1})\, O(n^{p-\lambda} t^{-2-\rho}) = \sum_{p=0}^{\lambda-1} O(n^{-1-\lambda} t^{-2-\rho}) = O(n^{-1-\lambda} t^{-2-\rho}).$$

This proves (3).

Lemma 2. *For* $0 < t < \pi$, $\alpha > \sigma - 1$, $\rho \geqq 0$, $\sigma \geqq 0$, *we have*

$$\left| \left(\frac{d}{dt}\right)^{\rho} (n\Delta)^{\sigma} \bar{\kappa}^{\alpha}(n, t) \right| < A n^{1+\rho}, \tag{4}$$

$$\left| \left(\frac{d}{dt}\right)^{\rho} (n\Delta)^{\sigma} \left\{ \bar{\kappa}^{\alpha}(n, t) - \bar{K}^{\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t \right\} \right| < A n^{-1} t^{-2-\rho}. \tag{5}$$

This lemma can be obtained from Lemma 1 and the relation

$$(n\Delta)^{\lambda} g(n, t) = \sum_{p=1}^{\lambda} O(n^{p})\, \Delta^{p} g(n, t).$$

Lemma 3. *For* $0 < t < \pi$, $\alpha > \sigma - 1$, $\rho \geqq 0$, $\sigma \geqq 0$, *we have*

$$\left| \left(\frac{d}{dt}\right)^{\rho} \Delta^{\sigma} \bar{K}^{\alpha}(n, t) \right| \begin{array}{ll} < A n^{-\sigma-\alpha} t^{-\rho-\alpha-1} & (nt \leqq 1) \\ < A n^{\rho-\alpha} t^{\sigma-\alpha-1} & (nt > 1). \end{array} \tag{6}$$

The result is easily proved when $\sigma = 0$. Assuming it for $\sigma = 0, 1, 2, \ldots \lambda - 1$, and using the argument of induction as before, we can write

$$\left(\frac{d}{dt}\right)^{\rho} \Delta^{\lambda} \bar{K}^{\alpha}(n, t) = \left(\frac{d}{dt}\right)^{\rho} \Delta^{\lambda-1} \left[\frac{\alpha}{n} \{\bar{K}^{\alpha-1}(n, t) - \bar{K}^{\alpha}(n, t)\} \right]$$

$$= \sum_{p=0}^{\lambda-1} \binom{\lambda-1}{p} \Delta^{p} \frac{\alpha}{n} \Delta^{\lambda-p-1} \left(\frac{d}{dt}\right)^{\rho} \{\bar{K}^{\alpha-1}(n-p, t) - \bar{K}^{\alpha}(n-p, t)\}$$

$$= \sum_{p=0}^{\lambda-1} O(n^{-p-1}) \begin{cases} O(n^{p+1-\lambda-\alpha} t^{-\rho-\alpha-1}) & (nt \leqq 1) \\ O(n^{\rho+1-\alpha} t^{\lambda-p-\alpha-1}) & (nt > 1) \end{cases}$$

$$= \sum_{p=0}^{\lambda-1} \begin{cases} O(n^{-\lambda-\alpha} t^{-\rho-\alpha-1}) & (nt \leqq 1) \\ O(n^{\rho-\alpha} t^{\lambda-\alpha-1}) & (nt > 1). \end{cases}$$

This proves (6).

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

Lemma 4. *For* $0 < t < \pi$, $a > \sigma - 1$, $\rho \geqq 0$, $\sigma \geqq 0$, *we have*

$$\left| \left(\frac{d}{dt}\right)^\rho (n\Delta)^\sigma \bar{K}^a (n, t) \right| \begin{array}{ll} < An^{-a} t^{-\rho-a-1} & (nt \leqq 1) \\ < An^{\rho+\sigma-a} t^{\sigma-a-1} & (nt > 1). \end{array} \tag{7}$$

This lemma can be obtained[1] from Lemma 3 in the same way as Lemma 2 was obtained from Lemma 1.

Lemma 5. *If*[2] $\nu > 0$, *then*

$$\nu (\bar{s}_n^{\nu-1} - \bar{s}_n^\nu) = n\Delta \bar{s}_n^\nu = \bar{\tau}_n^\nu . \tag{8}$$

Lemma 6. *If* $\nu > 0$ *and* λ *is a positive integer, then*

$$\nu \{(n\Delta)^\lambda \bar{s}_n^{\nu-1} - (n\Delta)^\lambda \bar{s}_n^\nu\} = (n\Delta)^{\lambda+1} \bar{s}_n^\nu = \bar{\tau}_{n,\lambda}^\nu, \tag{9}$$

where $\bar{\tau}_{n,\lambda}^\nu$ *denotes the n-th Cesàro mean of order* ν *of* $(n\Delta)^{\lambda+1} \bar{s}_n$.

This follows from Lemma 5.

§ 3.

In this section we shall be concerned with a function $f(t)$ which possesses a *simple discontinuity*, or a discontinuity of a similar nature, at the point $t = x$. If $f(x+0) - f(x-0)$ exists, its value is called the *jump* of the function $f(t)$ at $t = x$. Here we shall be dealing with functions which possess a jump in a *generalised sense*[3].

It is known[4] that if $a \geqq 0$ and

---

[1] Alternatively it can be obtained by repeated applications of the identities given in the footnote on page 237.

[2] See Kogbetliantz, **20**, 23 and 30, and also **19**.

[3] For example when (10) is satisfied, the number $2s$ may be called the jump of the function $f(t)$ at the point $t = x$ in a *generalised sense*. The expression *generalised jump* has been used by Szász, **24**, 362.

[4] The relation between the limit of the sequence nB_n and the jump of the function $f(t)$ was first pointed out by Fejér for a function satisfying Dirichlet's conditions ; Fejér, **12**, and later Young in 1916 proved that for a function of bounded variation nB_n tends to $\frac{1}{\pi}\{f(x+0) - f(x-0)\}$; Young, **26**, 44. In 1918 this result was also given by Csillag, **10**. Later Szidon proved that nB_n tends to the limit $\frac{1}{\pi}\{f(x+0) - f(x-0)\}$ $(C, 2)$, whenever this limit exists ; Szidon, **23** ; and Paley showed that if $a \geqq 0$, and $\psi(t)$ tends to a limit $s\,(C, a)$, then $n B^n$ tends to the limit $\frac{2}{\pi} s\,(C, a+1+\delta)$, $\delta > 0$; Paley, **22**, 184-9. Also Jacob showed that if $a = 0$ and (10) holds, then nB_n tends to the limit $\frac{2}{\pi} s\,(C, 1+\delta)$; Jacob, **18** ; and the general result stated above was given by Bosanquet, **4**, 23-9.

$$\frac{1}{t}\int_0^t |\psi_\alpha(u) - s|\, du = o(1) \tag{10}$$

as $t \to +0$, or more generally, if

$$\lim_{\epsilon \to +0} \frac{1}{t}\int_\epsilon^t u^{-\alpha}\, |d\,\Psi_{\alpha+1}(u)| = O(1) \tag{11}$$

in an interval $(0, \eta)$, $\eta > 0$ and $\psi(t)$ tends to a limit s (C) as $t \to +0$, then nB_n tends to the limit $\dfrac{2}{\pi}\, s$ $(C, \alpha + 1 + \delta)$, $\delta > 0$ as $n \to \infty$.

Both these results break down[1] when $-1 < \alpha < 0$, even if the integral in (10) is replaced by a Stieltjes integral. The second result remains true[2], however, if (11) holds *throughout the whole interval* $(0, \pi)$. In this section we shall show more generally that if we make an additional hypothesis that $B_n = o(n^\alpha)$, $-1 < \alpha < 0$, the above result will remain true even if condition (11) holds only in an interval $(0, \eta)$, $0 < \eta < \pi$. We give this result in Theorem 3 and apply it to obtain the result stated in Theorem 4. In order to prove Theorem 3 we first obtain necessary and sufficient conditions that nB_n should tend to a limit (C, ν), $0 < \nu < 1$, depending only upon the properties of the function near the point $t = x$ and give the result in Theorem 2.

We first prove the following theorem.

Theorem 1. *If* $-1 < \alpha < 0$, $\beta > \alpha$ *and* (11) *holds in the interval* $(0, \pi)$, *and* $\psi(t)$ *tends to a limit* s (C) *as* $t \to +0$, *then* nB_n *tends to the limit* $\dfrac{2}{\pi}\, s$ $(C, \beta + 1)$ *as* $n \to \infty$.

Proof. It will be enough to show that $nB_n = O(1)$ $(C, \alpha + 1 + \delta)$, $\delta > 0$. For since, by Theorem D, nB_n tends to the limit $\dfrac{2}{\pi}\, s$ (C), it will follow by a well-known theorem that nB_n tends to the limit $\dfrac{2}{\pi}\, s$ $(C, \alpha + 1 + \delta + \delta')$, $\delta' > 0$.

---

[1] The reason for the failure is that the existence of the Cesàro limit of order ν, $0 < \nu < 1$, of nB_n depends upon the nature of the function *throughout the whole interval* $(0, \pi)$. This can be illustrated by the following example. We can construct a function $\psi(t)$ which is zero in $(0, \frac{1}{2}\pi)$ and such that $B_n \neq o(n^{\nu - 1})$, so that nB_n does not tend to a limit (C, ν). Thus the (C, ν) limit of nB_n may be destroyed by altering $\psi(t)$ in the range $(\frac{1}{2}\pi, \pi)$. See Titchmarsh, 25. We simply integrate his series.

[2] See Theorem 1.

We write[1]

$$\begin{aligned}
\bar{\tau}_n^{a+1+\delta} &= n\Delta\, \bar{s}_n^{\,a+1+\delta} \\
&= \int_0^\pi \psi(t)\, n\Delta\, \bar{\kappa}^{a+1+\delta}(n, t)\, dt \\
&= \frac{1}{\Gamma(-a)} \int_0^\pi n\Delta\, \bar{\kappa}^{a+1+\delta}(n, t)\, dt \int_0^t (t-u)^{-a-1}\, d\Psi_{a+1}(u) \\
&= \frac{1}{\Gamma(-a)} \int_0^\pi d\Psi_{a+1}(u) \int_u^\pi (t-u)^{-a-1}\, n\Delta\, \bar{\kappa}^{a+1+\delta}(n, t)\, dt \\
&= \int_0^\pi L(n, u)\, d\Psi_{a+1}(u),
\end{aligned}$$

where

$$L(n, u) = \frac{1}{\Gamma(-a)} \int_u^\pi (t-u)^{-a-1}\, n\Delta\, \bar{\kappa}^{a+1+\delta}(n, t)\, dt.$$

We next state the following inequalities[2].

For $0 < u < \pi$

$$\left| L(n, u) \right| \begin{array}{l} < A n^{1+a} \\ < A n^{-\delta} u^{-1-a-\delta}. \end{array}$$

We now have

$$\bar{\tau}_n^{a+1+\delta} = \left\{ \int_0^{1/n} + \int_{1/n}^\pi \right\} L(n, u)\, d\Psi_{a+1}(u) = L_1 + L_2,$$

where[3]

$$| L_1 | \leqq A n^{1+a} \int_0^{1/n} | d\Psi_{a+1}(u) | = O(1),$$

and

$$\begin{aligned}
| L_2 | &\leqq A n^{-\delta} \int_{1/n}^\pi u^{-2-\delta}\, u^{-a}\, | d\Psi_{a+1}(u) | \\
&= O(1),
\end{aligned}$$

on integration by parts.

This completes the proof.

---

[1] The various steps in this argument can easily be justified. See Bosanquet, 5, 114; 7, 195-7.

[2] To obtain these inequalities we use (4) and (5) and follow the method used by Bosanquet, 7, 197.

[3] Bosanquet, 5, 114.

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

Theorem 2. *Necessary and sufficient conditions that* nB_n *should tend to a limit* s (C, ν) *for* $t = x$, *where* $0 < \nu < 1$, *are that* (i) $B_n = o\,(n^{\nu-1})$ *as* $n \to \infty$, *and* (ii)

$$\int_0^\delta \{\psi(t) - \tfrac{1}{2}\pi s\}\; n\, \Delta\{\bar{\kappa}^\nu\,(n, t) - \frac{t^2}{\delta^2}\, \bar{K}^\nu\,(n, t)\}\, dt = o\,(1), \tag{12}$$

as $n \to \infty$, *where* $0 < \delta < \pi$.

Proof. A necessary and sufficient condition that nB_n should tend to the limit s (C, ν) is that

$$\int_0^\pi \psi\,(t)\; n\, \Delta\bar{\kappa}^\nu\,(n,\, t)\; dt \to s \tag{13}$$

as $n \to \infty$.

It can easily be seen that (13) can be replaced by[1]

$$\left\{\int_0^\delta + \int_\delta^\pi\right\} \{\psi\,(t) - \tfrac{1}{2}\pi s\}\; n\, \Delta\bar{\kappa}^\nu\,(n,\, t)\; dt = \mathrm{I}_1 + \mathrm{I}_2 = o\,(1) \tag{14}$$

as $n \to \infty$.

We now observe that I_2 can be replaced by

$$\int_\delta^\pi \{\psi\,(t) - \tfrac{1}{2}\pi s\}\; n\, \Delta\bar{K}^\nu\,(n,\, t)\; dt. \tag{15}$$

For, on integration by parts, we have by (5)

$$\int_\delta^\pi \{\psi\,(t) - \tfrac{1}{2}\pi s\}\; n\; \Delta\{\bar{\kappa}^\nu\,(n,\, t) - \bar{K}^\nu\,(n,\, t)\}\; dt = o\,(1).$$

Now, since (i) is a necessary condition[2] that nB_n should tend to a limit s (C, ν), it can be assumed to be fulfilled.

We next prove that

$$\int_0^\delta \{\psi\,(t) - \tfrac{1}{2}\pi s\}\, \frac{t^2}{\delta^2}\, n\Delta\, \bar{K}^\nu\,(n,\, t)\; dt + \int_\delta^\pi \{\psi\,(t) - \tfrac{1}{2}\pi s\}\; n\Delta\bar{K}^\nu\,(n,\, t)\; dt = o\,(1) \tag{16}$$

as $n \to \infty$.

---

[1] Here we use the fact that if $\psi\,(t) = 1$ in $(0,\ \pi)$, then nB_n tends to the limit $\frac{2}{\pi}$ $(C,\ \delta)$, $\delta > 0$, as $n \to \infty$.

[2] Dienes, 11, 427.

Next let

$$h(t) = \begin{cases} \dfrac{t^2}{\delta^2} \dfrac{1}{(2 \sin \frac{1}{2}t)^\nu} & (o \leqq t \leqq \delta) \\ \dfrac{1}{(2 \sin \frac{1}{2}t)^\nu} & (\delta \leqq t \leqq \pi). \end{cases}$$

Then we have to show that

$$\mathrm{I}' = \int_0^\pi \{\psi(t) - \tfrac{1}{2}\pi s\}\, h(t) \frac{1}{(2 \sin \frac{1}{2}t)}\, n\Delta \frac{\cos \{(n + \frac{1}{2} + \frac{1}{2}\nu)\, t - \frac{1}{2}\nu\pi\}}{A_n^\nu}\, dt = o(1).$$

We can now write

$$\mathrm{I}' = -\frac{\nu}{A_n^\nu} \int_0^\pi \{\psi(t) - \tfrac{1}{2}\pi s\}\, h(t) \frac{1}{2 \sin \frac{1}{2}t} \cos \{(n - \tfrac{1}{2} + \tfrac{1}{2}\nu)\, t - \tfrac{1}{2}\nu\pi\}\, dt$$

$$- \frac{n}{A_n^\nu} \int_0^\pi \{\psi(t) - \tfrac{1}{2}\pi s\}\, h(t) \sin \{(n + \tfrac{1}{2}\nu)\, t - \tfrac{1}{2}\nu\pi\}\, dt$$

$$= \mathrm{I}_1' - \frac{n}{A_n^\nu} \mathrm{I}_2'.$$

Then $\mathrm{I}_1' = o(1)$, by the Riemann Lebesgue theorem, and so it remains to prove that $\mathrm{I}_2' = o(n^{\nu-1})$, which can be done by following an argument given by Bosanquet and Offord[1].

From (14), (15) and (16), we obtain (12).

Theorem 3. *If* $-1 < a < 0$, $\beta > a$ *and* (i) $B_n = o(n^\beta)$ *as* $n \to \infty$ [2], *and* (ii) *condition* (11) *holds in an interval* $(0, \eta)$, $\eta > 0$, *then either* nB_n *tends to a limit* $(C, \beta + 1)$ *or does not tend to a limit* (A). *A necessary and sufficient condition that it should tend to the limit* s (C) *is that* $\psi_1(t)$ *should tend to the limit* $\frac{1}{2}\pi s$ *as* $t \to +0$.

Proof. Without any loss of generality we can assume[3] that $\Psi_{a+1}(+0) = 0$, so that now from condition (ii) we have $\Psi_{a+1}(t) = O(t^{a+1})$, *i.e.*, $\psi(t) = O(1)$ $(C, a + 1)$. Also if nB_n tends to the limit s (C), $\psi(t)$ tends[4] to the limit $\frac{1}{2}\pi s$ (C), and therefore[5] $\psi(t)$ tends to the limit $\frac{1}{2}\pi s$ $(C, a + 1 + \delta)$, and so in particular $\psi_1(t)$ tends to the limit $\frac{1}{2}\pi s$ as $t \to +0$. Thus the necessary condition is

---

[1] Bosanquet and Offord, 8, 276-7.

[2] It is interesting to observe that if (11) holds in the whole interval $(0, \pi)$, then $B_n = O(n^a)$. See Bosanquet, 5.

[3] See Bosanquet, 5, 114.

[4] See Theorem D.

[5] Bosanquet, 3, 103.

established, and for the rest of the proof there will be no loss of generality if we take it to be satisfied.

Let us now define two functions $p(t)$ and $q(t)$ such that

$$p(t) = \begin{cases} \psi(t) \text{ in } (0, \eta) \\ 0 \quad \text{in } (\eta, \pi) \end{cases}$$

and $q(t) = \psi(t) - p(t)$, and let their Fourier series be denoted respectively by

$$\sum_{n=1}^{\infty} c_n \sin nt \text{ and } \sum_{n=1}^{\infty} d_n \sin nt.$$

Then since[1]

$$\lim_{\epsilon \to +0} \frac{1}{t} \int_{\epsilon}^{t} u^{-a} \, | dP_{a+1}(u) | = O(1) \tag{17}$$

in the interval $(0, \pi)$, we see by Theorem 1 that nc_n tends to the limit $\frac{2}{\pi} s \; (C, \beta + 1)$ and hence[2] that $c_n = o(n^{\beta})$ as $n \to \infty$.

Also since $q(t) = 0$ in $(0, \eta)$ and $d_n = B_n - c_n = o(n^{\beta})$, it follows from Theorem 2 that nd_n tends to the limit $0 \; (C, \beta + 1)$, and hence that nB_n tends to the limit $\frac{2}{\pi} s \; (C, \beta + 1)$. This completes the proof.

---

[1] In the interval $(0, \eta)$ (17) is the same as (11). If $\eta \leq t \leq u$ we have,

$$\begin{aligned}
\int_{\eta}^{t} u^{-a} \, | dP_{a+1}(u) | &\leq t^{-a} \int_{\eta}^{t} \left| \frac{d}{du} P_{a+1}(u) \right| du \\
&= t^{-a} \int_{\eta}^{t} \left| \frac{1}{\Gamma(a)} \int_0^{\eta} (u-v)^{a-1} \, p(v) \, dv \right| \\
&= t^{-a} \int_{\eta}^{t} \left| \frac{1}{\Gamma(a)} \int_0^{\eta} (u-v)^{a-1} \frac{dv}{\Gamma(-a)} \int_0^{v} (v-w)^{-a-1} \, dP_{a+1}(w) \right| du \\
&\leq \frac{t^{-a}}{| \Gamma(a) | \, \Gamma(-a)} \int_{\eta}^{t} du \int_0^{\eta} (u-v)^{a-1} \, dv \int_0^{v} (v-w)^{-a-1} \, | dP_{a+1}(w) | \\
&= \frac{t^{-a}}{| \Gamma(a) | \, \Gamma(-a)} \int_0^{\eta} | dP_{a+1}(w) | \int_{w}^{\eta} (v-w)^{-a-1} \, dv \int_{\eta}^{t} (u-v)^{a-1} \, du \\
&\leq \frac{t^{-a}}{\Gamma(a+1) \, \Gamma(-a)} \int_0^{\eta} | dP_{a+1}(w) | \int_{w}^{\eta} (\eta - v)^{a} \, (v-w)^{-a-1} \, dv \\
&= t^{-a} \int_0^{\eta} | dP_{a+1}(w) | \\
&= O(1)
\end{aligned}$$

by (11). See Bosanquet, **5**, **114**. This proof was pointed out by Dr Bosanquet.

[2] This is a necessary condition that nc_n should tend to a limit $(C, \beta + 1)$ as $n \to \infty$.

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

In virtue of the well-known result[1] that a necessary and sufficient condition that a series $\Sigma\, a_n$ be summable (C, ν), $\nu > -1$, is that it should be summable (A) and the sequence $na_n = o(1)\ (C, \nu + 1)$, we obtain the following theorem.

Theorem 4. *If*[2] $-1 < \alpha < 0$, $\beta > \alpha$ *and* (i) $B_n = o(n^\beta)$ *as* $n \to \infty$, *and* (ii) *condition* (11) *holds in an interval* $(0, \eta)$, $\eta > 0$, *then the allied series of* $f(t)$ *at* $t = x$ *is summable* (C, β), *if it is summable* (A).

We here state the following lemma.

Lemma 7. *If*[3] nB_n *tends to a limit* s (C, α), $\alpha > 0$, *as* $n \to \infty$, *then* $\psi(t)$ *tends to the limit* $\frac{1}{2}\pi s$ $(C, \alpha + \delta)$, $\delta > 0$, *as* $t \to +0$.

Remark. It is of independent interest to show that in Theorem 2, condition (ii) can be replaced by one of much simpler form[4], if we assume an extra condition (iii) that $\psi_1(t)$ tends to the limit $\frac{1}{2}\pi s$ as $t \to +0$, which has been shown in Lemma 7 to be a necessary condition that nB_n should tend to the limit s (C, ν), $0 < \nu < 1$.

We first observe that (12) can be replaced by

$$\int_{1/n}^{\delta} \{\psi(t) - \tfrac{1}{2}\pi s\}\, n\, \Delta\{\bar{\kappa}^\nu(n, t) - \frac{t^2}{\delta^2} \bar{K}^\nu(n, t)\}\, dt = o(1) \tag{18}$$

as $n \to \infty$.

For, on integration by parts, we have by (iii), (4) and (7)

$$\int_0^{1/n} \{\psi(t) - \tfrac{1}{2}\pi s\}\, n\, \Delta\{\bar{\kappa}^\nu(n, t) - \frac{t^2}{\delta^2} \bar{K}^\nu(n, t)\}\, dt = o(1).$$

Now in virtue of the identities given in the footnote on page 237, condition (18) can be replaced by

$$\frac{1}{n}\int_{1/n}^{\delta} \{\psi(t) - \tfrac{1}{2}\pi s\} \left\{\frac{d}{dt}\kappa^{\nu-1}(n, t) - \frac{t^2}{\delta^2}\frac{d}{dt} K^{\nu-1}(n, t)\right\} dt = o(1). \tag{19}$$

Next (19) can be replaced by

$$\frac{1}{n}\int_{1/n}^{\delta} \{\psi(t) - \tfrac{1}{2}\pi s\} \left(1 - \frac{t^2}{\delta^2}\right) \frac{d}{dt} K^{\nu-1}(n, t)\, dt = o(1). \tag{20}$$

---

[1] Hardy and Littlewood, **15**, 283. See also Kogbetliantz, **19**, 238.

[2] This result can also be obtained by an application of Theorem 1 of my paper, **1**, 318. A corresponding result for Fourier series has been given by Bosanquet, **5**, 115.

[3] This result, though not explicitly stated, is implied by Paley's results; Paley, **22**, 195-9. See also Bosanquet, **2**, 162-3. The lemma can be proved from his analysis on pages 162-3.

[4] Condition (19) below.

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

For, on integrating by parts and using (iii), and the inequalities[1] for κ and K analogous to (5) and (7), we have

$$\frac{1}{n}\int_{1/n}^{\delta}\{\psi(t)-\tfrac{1}{2}\pi s\}\frac{d}{dt}\{\kappa^{\nu-1}(n,t)-K^{\nu-1}(n,t)\}\,dt=o(1).$$

Alternatively, (20) can be replaced by[2]

$$\lim_{\tau\to\infty}\ \overline{\lim_{n\to\infty}}\left|n^{-\nu}\int_{\tau/n}^{\delta}\{\psi(t)-\tfrac{1}{2}\pi s\}\left(1-\frac{t^2}{\delta^2}\right)\frac{d}{dt}\frac{\sin(n;t)}{(2\sin\frac{1}{2}t)^{\nu}}\,dt\right|=0, \quad (21)$$

where $(n;t)$ is written for $(n+\frac{1}{2}\nu)t+\frac{1}{2}(1-\nu)\pi$.

§ 4.

In this section we consider functions $\psi(t)$ such that $t\dfrac{d}{dt}\psi(t)$ tends to a limit, and more general functions of the same type. We prove results giving precise relations between the existence of Cesàro limits of $t\dfrac{d}{dt}\psi(t)$ and of the sequence $(n\Delta)^2\bar{s}_n$.

We first state the following inequalities which will be used in the proof of the next theorem.

For $0<t<\pi$, $\alpha>\sigma-1$, $\rho\geqq 0$, $\sigma\geqq 0$, *we have*

$$\left|\left(\frac{d}{dt}\right)^{\rho}(n\Delta)^{\sigma}\bar{\kappa}^{\alpha}(n,t)\right| \begin{cases} <An^{1+\rho} & \\ <An^{\rho+\sigma-\alpha}t^{\sigma-\alpha-1} & (\alpha\leqq\sigma+\rho+1) \\ <An^{-1}t^{-2-\rho} & (\alpha>\sigma+\rho+1). \end{cases} \quad (22)$$

These inequalities can easily be obtained from Lemmas 2 and 4.

Theorem 5. *If*[3] $\alpha>-1$, $\beta>\alpha$ *and*

$$\frac{1}{t}\int_0^t u\,|\,d\psi_{\alpha+1}(u)\,|=O(1) \quad (23)$$

in the interval $(0,\pi)$, *then* $(n\Delta)^2\bar{s}_n=O(1)\ (C,\beta+2)$ *as* $n\to\infty$.

---

[1] *For* $0<t<\pi$, $\alpha>\sigma-1$, $\rho\geq 0$, $\sigma\geq 0$, *we have*

$$\left|\left(\frac{d}{dt}\right)^{\rho}(n\Delta)^{\sigma}\kappa^{\alpha}(n,t)\right|<An^{1+\rho},$$

$$\left|\left(\frac{d}{dt}\right)^{\rho}(n\Delta)^{\sigma}\{\kappa^{\alpha}(n,t)-K^{\alpha}(n,t)\}\right|<An^{-1}t^{-2-\rho}, \quad (20a)$$

$$\left|\left(\frac{d}{dt}\right)^{\rho}(n\Delta)^{\sigma}K^{\alpha}(n,t)\right| \begin{cases} <An^{-\alpha}t^{-\rho-\alpha-1} & (nt\leq 1) \\ <An^{\rho+\sigma-\alpha}t^{\sigma-\alpha-1} & (nt>1). \end{cases}$$

[2] Here we use condition (iii) and follow an argument similar to one given by Bosanquet and Offord, **8**, 277.

[3] It can also be obtained from the same analysis that if $\alpha\geq 0$, $\beta>\alpha$ and (20) holds in an interval $(0,\eta)$, $\eta>0$, then $(n\Delta)^2\bar{s}_n=O(1)\ (C,\beta+2)$ as $n\to\infty$.

Proof. Let h be the greatest integer not greater than α, and suppose, as we may without any loss of generality, that $\beta < h + 1$. The n-th Cesàro mean of order $(\beta + 2)$ of $(n\Delta)^2 \bar{s}_n$ can be written in the form[1]

$$(n\Delta)^2 \bar{s}_n^{\beta+2} = \int_0^\pi \psi(t)\,(n\Delta)^3\,\bar{\kappa}^{\beta+2}(n, t)\,dt$$

$$= \left[\sum_{\rho=1}^{h+1} (-1)^{\rho-1}\,\Psi_\rho(t)\left(\frac{d}{dt}\right)^{\rho-1}(n\Delta)^2\,\bar{\kappa}^{\beta+2}(n, t)\right]_0^\pi$$

$$+ (-1)^{h+1}\int_0^\pi \Psi_{h+1}(t)\left(\frac{d}{dt}\right)^{h+1}(n\Delta)^2\,\bar{\kappa}^{\beta+2}(n, t)\,dt$$

$$= \sum_{\rho=1}^{h} O(n^{-1}) + O(n^{h-\beta}) + (-1)^{h+1}\frac{1}{\Gamma(1+h-\alpha)}\times$$

$$\int_0^\pi \left(\frac{d}{dt}\right)^{h+1}(n\Delta)^2\,\bar{\kappa}^{\beta+2}(n, t)\,dt \int_0^t (t-u)^{h-\alpha}\,d\Psi_{\alpha+1}(u)$$

$$= O(n^{-1}) + O(n^{h-\beta}) + (-1)^{h+1}\frac{1}{\Gamma(1+h-\alpha)}\times$$

$$\int_0^\pi d\Psi_{\alpha+1}(u)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}(n\Delta)^2\,\bar{\kappa}^{\beta+2}(n, t)\,dt$$

$$= O(n^{h-\beta}) + \int_0^\pi J(n, u)\,d\Psi_{\alpha+1}(u),$$

where

$$J(n, u) = (-1)^{h+1}\frac{1}{\Gamma(1+h-\alpha)}\int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}(n\Delta)^2\,\bar{\kappa}^{\beta+2}(n, t)\,dt.$$

We next state the following inequalities[2].

For $0 < u < \pi$,

$$\left|J(n, u)\right| \begin{array}{l} < An^{1+\alpha} \\ < An^{\alpha-\beta}\,u^{-\beta-1}. \end{array} \tag{24}$$

---

[1] See Lemma 6. The various steps below can easily be justified. See Bosanquet, 5, 114 ; 7, 195-7.

[2] To obtain these inequalities we use (22) and (20a) and follow the method used by Bosanquet, 7, 197.

Assuming these we now have

$$(n\Delta)^2\, \bar{s}_n^{\beta+2} = O\,(n^{h-\beta}) + \Big[\Psi_{a+1}\,(u)\; J\,(n,\, u)\Big]_0^{\pi}$$

$$- \frac{1}{\Gamma\,(a+2)} \int_0^{\pi} \psi_{a+1}\,(u)\; u^{a+1}\, \frac{d}{du}\, J\,(n,\, u)\, du$$

$$= O\,(n^{h-\beta}) + O\,(n^{a-\beta}) - \frac{1}{\Gamma\,(a+2)} \Big[\psi_{a+1}\,(u) \int_0^u v^{a+1}\, \frac{d}{dv}\, J\,(n,\, v)\, dv\Big]_0^{\pi}$$

$$+ \frac{1}{\Gamma\,(a+2)} \int_0^{\pi} d\psi_{a+1}\,(u) \int_0^u v^{a+1}\, \frac{d}{dv}\, J\,(n,\, v)\, dv$$

$$= O\,(n^{a-\beta}) - \Big[\psi_{a+1}\,(u)\;\, V\,(n,\, u)\Big]_0^{\pi} + \int_0^{\pi} V\,(n,\, u)\, d\psi_{a+1}\,(u),$$

where

$$V\,(n,\, u) = \frac{1}{\Gamma\,(a+2)} \int_0^u v^{a+1}\, \frac{d}{dv}\, J\,(n,\, v)\, dv.$$

We next obtain the following inequalities

For $0 < u < \pi$,

$$\Big|\, V\,(n,\, u)\, \Big| \begin{array}{l} < A n^{1+a}\, u^{1+a} \\ < A n^{a-\beta}\, u^{a-\beta}. \end{array} \tag{25}$$

The first inequality (25) follows, on integration by parts, from the first inequality (24).

To obtain the second inequality we first observe that if $\psi\,(t) \equiv 1$ in $(0,\, \pi)$, then $\psi_{a+1}\,(t) = 1$ in $(0,\, \pi)$ and it can be shown that, with this particular $\psi\,(t)$,

$$(n\Delta)^2\, \bar{s}_n^{\beta+2} = \begin{cases} O\,(n^{-1}) & (\beta \geqq o) \\ O\,(n^{-\beta-1}) & (\beta < o) \end{cases}$$

as $n \to \infty$.

To prove this we write

$$nB_n = n\Delta\, \bar{s}_n = \frac{2}{\pi} \int_0^{\pi} n \sin nt\, dt = \frac{2}{\pi}\,(1 - \cos n\pi).$$

Now following an argument given by Hyslop[1], we can write[2]

$$n\Delta\, \bar{s}_n^{\beta+2} = \bar{\tau}_n^{\beta+2} = \frac{2}{\pi}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\, A_\nu^{h+3} - \frac{1}{\pi}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\, A_\nu^{h+2}$$

$$-\frac{1}{\pi}\frac{1}{2}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\, A_\nu^{h+1} - \ldots\ldots - \frac{1}{\pi}\frac{1}{2^{h+1}}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\, A_\nu^{1}$$

$$-\frac{1}{\pi}\frac{1}{2^{h+2}}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\,(1+\cos\nu\pi)$$

$$= \frac{2}{\pi} - \frac{1}{\pi}\frac{A_n^{\beta+1}}{A_n^{\beta+2}} - \frac{1}{\pi}\frac{1}{2}\frac{A_n^{\beta}}{A_n^{\beta+2}} - \ldots\ldots - \frac{1}{\pi}\frac{1}{2^{h+1}}\frac{A_n^{\beta-h}}{A_n^{\beta+2}}$$

$$-\frac{1}{\pi}\frac{1}{2^{h+2}}\frac{1}{A_n^{\beta+2}}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-h-2}\,(1+\cos\nu\pi),$$

whence we have

$$(n\Delta)^2\, \bar{s}_n^{\beta+2} = O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right) + \ldots\ldots + O\left(\frac{1}{n^{h+2}}\right) + O\left(\frac{1}{n^{\beta+1}}\right) + O\left(\frac{1}{n^{\beta+2}}\right)$$

$$= \begin{cases} O\,(n^{-1}) & (\beta \geqq o) \\ O\,(n^{-\beta-1}) & (\beta < o). \end{cases}$$

Now repeating the first part of the argument with $\psi(t)$ replaced by the special function $\psi(t) \equiv 1$ in $(0, \pi)$, we find that

$$O\,(n^{-1}) + O\,(n^{-\beta-1}) = O\,(n^{\alpha-\beta}) - V\,(n, \pi),$$

i.e.,
$$\frac{1}{\Gamma(\alpha+2)}\int_0^{\pi} v^{\alpha+1}\frac{d}{dv}J\,(n, v)\,dv = O\,(n^{\alpha-\beta}).$$

Hence observing that

$$V\,(n, \pi) - V\,(n, u) = \frac{1}{\Gamma(\alpha+2)}\int_u^{\pi} v^{\alpha+1}\frac{d}{dv}J\,(n, v)\,dv,$$

and using the second inequality (24) after integrating by parts, we obtain[3] the second inequality (25).

Now returning to the original function $\psi(t)$, we have

$$(n\Delta)^2\, \bar{s}_n^{\beta+2} = O\,(n^{\alpha-\beta}) + \left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\} V\,(n, u)\, d\psi_{\alpha+1}(u)$$

$$= O\,(n^{\alpha-\beta}) + \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$|\,\mathrm{I}_1\,| \leqq A n^{1+\alpha}\int_0^{1/n} u^{\alpha}\,|\,u\,d\psi_{\alpha+1}(u)\,| = O\,(1),$$

and

$$|\,\mathrm{I}_2\,| \leqq A n^{\alpha-\beta}\int_{1/n}^{\pi} u^{\alpha-\beta-1}\,|\,u\,d\psi_{\alpha+1}(u)\,| = O\,(1).$$

---

[1] Hyslop, 17, 185.

[2] See Lemma 5.

[3] See Bosanquet, 7, 199.

This completes the proof of the theorem.

We next consider the converse problem.

We first give the following inequalities and a lemma which will be used in the proof of the next theorem.

For $\sigma > 0$, $t > 0$, $n > 0$, we have

$$\nabla^p \left(\frac{d}{dt}\right)^\lambda \bar{\gamma}_\sigma(nt) \left| \begin{array}{ll} < An^{\lambda+1}\, t^{p+1} & (\lambda \geqq 0,\, p \geqq 0) \\ < An^{-p-1}\, t^{-\lambda-1} & (p+\lambda \leqq \sigma - 1) \\ < An^{\lambda-\sigma}\, t^{p-\sigma} & (p+\lambda > \sigma - 1), \end{array} \right. \tag{26}$$

where, for $\sigma > 0$,

$$\gamma_\sigma(x) + i\bar{\gamma}_\sigma(x) = \int_0^1 (1-u)^{\sigma-1} e^{ixu}\, du,$$

and

$$\nabla f(n) = f(n) - f(n+1),\ \nabla^m f(n) = \nabla \cdot \nabla^{m-1} f(n),\ \nabla^0 f(n) = f(n).$$

These inequalities can be obtained in the same way as those obtained for $\Delta^p \left(\frac{d}{dt}\right)^\lambda \gamma_\sigma(nt)$ by Bosanquet[1].

Lemma 8. *If $(n\Delta)^2 \bar{s}_n$ tends to a limit s $(C, \nu+1)$, $\nu > 0$, as $n \to \infty$, then $B_n = o(n^{\nu-1})$ as $n \to \infty$.*

Since $(n\Delta)^2 \bar{s}_n^{\nu+1}$ tends to s as $n \to \infty$, we obtain from the consistency theorem for Cesàro limits and Lemma 6 that $(n\Delta)^3 \bar{s}_n^{\nu+2} = o(1)$ as $n \to \infty$. Whence by an argument given by Dienes[2], we have $(n\Delta)^3 \bar{s}_n^2 = o(n^\nu)$ and this gives $(n\Delta)^2 \bar{s}_n^2 = o(n^\nu)$. Now applying Lemma 6 we get $(n\Delta)^2 \bar{s}_n^1 = o(n^\nu)$. This in turn gives $n\Delta\, \bar{s}_n^1 = o(n^\nu)$ and then again applying Lemma 6 we obtain the required result that $n\Delta\, \bar{s}_n = o(n^\nu)$.

Theorem 6. *If $\alpha > 0$, $\beta > \alpha + 1$ and $(n\Delta)^2 \bar{s}_n$ tends to a limit s $(C, \alpha+1)$ as $n \to \infty$, then $t \frac{d}{dt} \psi_\beta(t)$ tends to the limit $-\frac{1}{2}\pi s$ as $t \to +0$.*

Proof. We again assume that h is the greatest integer not greater than α and also suppose, as we may without any loss of generality, that α is not an integer, that[3] $s = 0$ and that $\beta < h + 2$.

---

[1] See Bosanquet, **6**, 519-20. See also Bosanquet and Hyslop, **9**, 495.

[2] Dienes, **11**, 427.

[3] Here we use the fact that if $\psi(t) = \frac{1}{2}\pi s \log \frac{\pi}{t}$ in $(0, \pi)$, then $t \frac{d}{dt} \psi(t) = -\frac{1}{2}\pi s$ and $(n\Delta)^2 \bar{s}_n$ tends to the limit s as $n \to \infty$. See § 1.

If $\beta \geqq 2$, we can write[1]

$$t\frac{d}{dt}\psi_\beta(t) = \beta\{\psi_{\beta-1}(t) - \psi_\beta(t)\}$$

$$= \beta\left\{\frac{\beta-1}{t^{\beta-1}}\int_0^t (t-u)^{\beta-2}\,\psi(u)\,du - \frac{\beta}{t^\beta}\int_0^t (t-u)^{\beta-1}\,\psi(u)\,du\right\}$$

$$= \beta\left\{(\beta-1)\int_0^1 (1-u)^{\beta-2}\,\psi(ut)\,du - \beta\int_0^1 (1-u)^{\beta-1}\,\psi(ut)\,du\right\}$$

$$= \beta\sum_{n=1}^{\infty} B_n\{(\beta-1)\,\bar{\gamma}_{\beta-1}(nt) - \beta\bar{\gamma}_\beta(nt)\}$$

$$= \beta\sum_{n=1}^{\infty} nB_n t\{\gamma_\beta(nt) - \gamma_{\beta+1}(nt)\}$$

$$= \beta\sum_{n=1}^{\infty} B_n t\frac{d}{dt}\bar{\gamma}_\beta(nt).$$

If $1 < \beta < 2$, the same result follows from the formula

$$\psi_\beta(t) = \beta\sum_{n=1}^{\infty} B_n\bar{\gamma}_\beta(nt)$$

by term by term differentiation, since the resulting series is uniformly convergent[2] for $t \geqq \epsilon$, $\epsilon > 0$.

Hence for $\beta > 1$, we now have

$$t\frac{d}{dt}\psi_\beta(t) = \beta\sum_{n=1}^{\infty} n\Delta\,\bar{s}_n\cdot\frac{1}{n}\,t\frac{d}{dt}\bar{\gamma}_\beta(nt)$$

$$= \beta\sum_{n=1}^{\infty}(n\Delta)^2\,\bar{s}_n\cdot\frac{1}{n}\sum_{\mu=n}^{\infty}\frac{1}{\mu}\,t\frac{d}{dt}\bar{\gamma}_\beta(\mu t)$$

$$= \beta\sum_{n=1}^{\infty} X_n\,U_n(t),$$

where
$$X_n = (n\Delta)^2\,\bar{s}_n,$$

and
$$U_n(t) = \frac{1}{n}\sum_{\mu=n}^{\infty}\frac{1}{\mu}\,t\frac{d}{dt}\bar{\gamma}_\beta(\mu t),$$

provided the partial summation can be justified.

To show this we observe by (26) and Lemma 8 that for a fixed $t > 0$,

$$n\Delta\,\bar{s}_n\sum_{\mu=n}^{\infty}\frac{1}{\mu}\,t\frac{d}{dt}\bar{\gamma}_\beta(\mu t) = \begin{cases} o(n^\alpha)\,O(n^{1-\beta}) & (\alpha < 1)\\ o(n)\,O(n^{-1}) & (\alpha \geqq 1),\end{cases}$$

using the fact that $B_n = o(1)$ in the case $\alpha \geqq 1$.

---

[1] For amplification of this argument see Bosanquet and Hyslop, 9, 496.

[2] This can easily be shown, since by (26) and Lemma 8 we have

$$B_n t\frac{d}{dt}\bar{\gamma}_\beta(nt) = o(n^{\alpha-1})\,O(n^{1-\beta}).$$

Now again applying partial summations $(h+2)$ times, we have

$$t\frac{d}{dt}\psi_\beta(t)=\beta\sum_{n=1}^{\infty}X_n^{h+2}\nabla^{h+2}U_n(t),$$

where X_n^λ denotes the λ-th partial sum of the sequence X_n, provided that these steps can be justified.

To show this we first obtain the following inequalities.

For $0<t<\pi$, $q\geqq 0$, $\beta>1$, *we have*

$$\nabla^q U_n(t)=\left[\begin{array}{lll} O(n^{-q}t) & (q\geqq 0) & (nt\leqq 1) \\ O(n^{-\beta}t^{-\beta+1}) & (\beta<2,\ q=0) & \\ O(n^{-1-\beta}t^{-\beta+q}) & (\beta<q+1,\ q\geqq 1) & (nt>1) \\ O(n^{-q-2}t^{-1}) & (\beta\geqq q+1,\ q\geqq 1) & \\ O(n^{-2}t^{-1}) & (\beta\geqq 2,\ q=0) & \end{array}\right]. \qquad (27)$$

It is easy to verify that if $\psi(t)=\frac{1}{2}(\pi-t)$ in $(0,\pi)$, then $n\Delta\bar{s}_n=1$ and $t\frac{d}{dt}\psi_\beta(t)=-\frac{1}{2}\frac{t}{\beta+1}$. Now using these values for this special function $\psi(t)$ in the above reasoning, we obtain that

$$\sum_{\mu=1}^{\infty}\frac{1}{\mu}t\frac{d}{dt}\bar{\gamma}_\beta(\mu t)=-\frac{1}{2}\frac{t}{\beta(\beta+1)},$$

so that, for $nt\leqq 1$, we have

$$\sum_{\mu=n}^{\infty}\frac{1}{\mu}t\frac{d}{dt}\bar{\gamma}_\beta(\mu t)=\left\{\sum_{\mu=1}^{\infty}-\sum_{\mu=1}^{n-1}\right\}\frac{1}{\mu}t\frac{d}{dt}\bar{\gamma}_\beta(\mu t)$$
$$=O(t)+\sum_{\mu=1}^{n-1}O(t)=O(nt). \qquad (28)$$

We can now obtain (27) from (26), (28) and the identity

$$\nabla^q a_n b_n=\sum_{l=0}^{q}\binom{q}{l}\nabla^l a_n\nabla^{q-l}b_{n+l}.$$

The partial summations can now be justified.

For, since $\bar{\tau}_n=n\Delta\bar{s}_n=o(n)$, we have $\bar{\tau}_n^1=n\Delta\bar{s}_n^1=o(n)$ and hence using Lemma 6, we obtain $(n\Delta)^2\bar{s}_n^1=o(n)$. Thus we have $X_n^1=o(n^2)$ and this gives $X_n^p=o(n^{p+1})$, while, if $0<\alpha<1$, $X_n^1=o(n^{\alpha+1})$.

Now we observe from (27) that, if $\beta\geqq 2$,

$$X_n^p\nabla^{p-1}U_n(t)=o(n^{p+1})\,O(n^{-p-1}),$$

for $p=1,2,3,\ldots.(h+1)$, while, if $\alpha+1<\beta<2$,

$$X_n^1 U_n(t)=o(n^{\alpha+1})\,O(n^{-\beta}).$$

Also for $p=h+2$, we have

$$X_n^{h+2}\nabla^{h+1}U_n(t)=o(n^{h+2})\,O(n^{-\beta-1}).$$

For, since by hypothesis we have $(n\Delta)^2 \bar{s}_n^{1+\alpha} = o(1)$, we obtain by Lemma 6 and the consistency theorem for Cesàro limits that $(n\Delta)^2 \bar{s}_n^{h+2} = o(1)$ and hence $X_n^{h+2} = o(n^{h+2})$.

This justifies the partial summations.

Now we know that

$$X_n^{1+\alpha} = A_n^{1+\alpha} (n\Delta)^2 \bar{s}_n^{1+\alpha} = o(n^{1+\alpha}),$$

and
$$X_n^{h+2} = \sum_{\nu=1}^{n} A_{n-\nu}^{h-\alpha} X_\nu^{1+\alpha}.$$

Therefore,

$$\begin{aligned} t \frac{d}{dt} \psi_\beta(t) &= \beta \sum_{n=1}^{\infty} \nabla^{h+2} U_n(t) \sum_{\nu=1}^{n} A_{n-\nu}^{h-\alpha} X_\nu^{1+\alpha} \\ &= \beta \sum_{\nu=1}^{\infty} X_\nu^{1+\alpha} \sum_{n=\nu}^{\infty} A_{n-\nu}^{h-\alpha} \nabla^{h+2} U_n(t) \\ &= \beta \sum_{\nu=1}^{\infty} X_\nu^{1+\alpha} V_\nu(t), \end{aligned}$$

where
$$V_\nu(t) = \sum_{n=\nu}^{\infty} A_{n-\nu}^{h-\alpha} \nabla^{h+2} U_n(t),$$

provided the inversion of the order of summation can be justified.

To show this it is enough to prove that

$$\sum_{\nu=1}^{N} X_\nu^{1+\alpha} \sum_{n=N+1}^{\infty} A_{n-\nu}^{h-\alpha} \nabla^{h+2} U_n(t)$$

exists and tends to zero as $N \to \infty$.

We observe that for a fixed $t > 0$, we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} A_{n-\nu}^{h-\alpha} \nabla^{h+2} U_n(t) &= O\left\{ (N-\nu+1)^{h-\alpha} \max_{m>N} \left| \sum_{n=N+1}^{m} \nabla^{h+2} U_n(t) \right| \right\} \\ &= O\left\{ \frac{(N-\nu+1)^{h-\alpha}}{N^{\beta+1}} \right\}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{N^{\beta+1}} \sum_{\nu=1}^{N} X_\nu^{1+\alpha} O\{(N-\nu+1)^{h-\alpha}\} &= O\left\{ \frac{1}{N^{\beta+1}} \sum_{\nu=1}^{N} \nu^{1+\alpha} (N-\nu+1)^{h-\alpha} \right\} \\ &= O\left\{ \frac{1}{N^{\beta-h-1}} \right\} \\ &= o(1) \end{aligned}$$

as $N \to \infty$, since $\beta > h+1$.

Thus the inversion is justified.

https://doi.org/10.1017/S0013091500027206 Published online by Cambridge University Press

We shall next show that

$$V_\nu(t) = \begin{cases} O(\nu^{-h-2}\, t^{\alpha-h}) & (\nu t \leqq 1) \\ O(\nu^{-1-\beta}\, t^{1+\alpha-\beta}) & (\nu t > 1), \end{cases} \tag{29}$$

uniformly in ν and t.

We write, for $t > 0$,

$$V_\nu(t) = \sum_{n=\nu}^{\nu+\rho} + \sum_{n=\nu+\rho+1}^{\infty} = \Sigma_1 + \Sigma_2,$$

where ρ is the greatest integer not greater than $\dfrac{1}{t}$.

Now

$$\Sigma_2 = \sum_{n=\nu+\rho+1}^{\infty} A_{n-\nu}^{h-\alpha}\, \nabla^{h+2}\, U_n(t)$$

$$= A_{\rho+1}^{h-\alpha}\, \nabla^{h+1}\, U_{\nu+\rho+1}(t) + \sum_{n=\nu+\rho+1}^{\infty} \nabla^{h+1}\, U_{n+1}(t)\, A_{n-\nu+1}^{h-\alpha-1}$$

$$= O\left[\frac{\nu^{-h-1}\, t}{(1+\nu t)^{\beta-h}}\left\{\rho^{h-\alpha} + \sum_{n=\nu+\rho+1}^{\infty} (n-\nu+1)^{h-\alpha-1}\right\}\right]$$

$$= O\left[\frac{\nu^{-h-1}\, t^{1+\alpha-h}}{(1+\nu t)^{\beta-h}}\right],$$

which satisfies the inequality (29).

In Σ_1 we use the inequalities

$$\nabla^{h+2}\, U_n(t) = O\left\{\frac{n^{-h-2}\, t}{(1+nt)^{\beta-h-1}}\right\},$$

and we obtain

$$\Sigma_1 = O\left\{\frac{\nu^{-h-2}\, t}{(1+\nu t)^{\beta-h-1}} \sum_{n=\nu}^{\nu+\rho} (n-\nu+1)^{h-\alpha}\right\}$$

$$= O\left\{\frac{\nu^{-h-2}\, t^{\alpha-h}}{(1+\nu t)^{\beta-h-1}}\right\},$$

which also satisfies (29).

Thus (29) is proved.

Finally by using (29), we obtain

$$t\,\frac{d}{dt}\,\psi_\beta(t) = O(t^{\alpha-h}) \sum_{\nu t \leq 1} o(\nu^{\alpha-1-h}) + O(t^{1+\alpha-\beta}) \sum_{\nu t > 1} o(\nu^{\alpha-\beta}) = o(1),$$

since $\alpha > h$ and $\beta > \alpha + 1$, which completes the proof of the theorem.

If we observe that Theorem 5 remains true when O is replaced by o, and use one of the examples given in § 1, we obtain the following analogue of Theorem F.

Theorem 7. *A necessary and sufficient condition that* $(n\Delta)^2 \bar{s}_n$ *should tend to a limit* s (C) *as* $n \to \infty$ *is that* $t \dfrac{d}{dt} \psi_\kappa (t)$ *should tend to the limit* $-\frac{1}{2}\pi s$ *as* $t \to +0$ *for some* $\kappa \geqq 2$.

We also have the following result, analogous to Theorem 1.

Theorem 8. If $-1 < \alpha < 0$, $\beta > \alpha$, *and* (23) *holds in the interval* $(0, \pi)$, *and* $t \dfrac{d}{dt} \psi_\kappa (t)$ *tends to a limit* s *as* $t \to +0$, *for some* $\kappa \geqq 2$, *then* $(n\Delta)^2 \bar{s}_n$ *tends to the limit* $-\dfrac{2}{\pi} s$ $(C, \beta + 2)$ *as* $n \to \infty$.

Finally I should like to express my best thanks to Dr L. S. Bosanquet for his valuable suggestions and criticisms.

REFERENCES.

1. S. P. Bhatnagar, "A local property of the allied series of a Fourier series," *Proc. London Math. Soc.* (2), 44 (1938), 315-322.

2. L. S. Bosanquet, "On the summability of Fourier series," *Proc. London Math. Soc.* (2), 31 (1930), 144-164.

3. ———, "Note on the limit of a function at a point," *Journal London Math. Soc.*, 7 (1932), 100-105.

4. ———, "On the Cesàro summation of Fourier series and allied series," *Proc. London Math. Soc.* (2), 37 (1934), 17-32.

5. ———, "Some extensions of Young's criterion for the convergence of a Fourier series," *Quart. J. of Math.* (*Oxford series*), 6 (1935), 113-123.

6. ———, "The absolute Cesàro summability of Fourier series," *Proc. London Math. Soc.* (2), 41 (1936), 517-528.

7. ———, "Some arithmetic means connected with Fourier series," *Trans. American Math. Soc.*, 39 (1936), 189-204.

8. L. S. Bosanquet and A. C. Offord, "A local property of Fourier series," *Proc. London Math. Soc.* (2), 40 (1936), 273-280.

9. L. S. Bosanquet and J. M. Hyslop, "On the absolute summability of the allied series of a Fourier series," *Math. Zeitschrift*, 42 (1937), 489-512.

10. P. Csillag, "Korlátos ngadozásu függvénysorok Fourier-féle állandóiról," *Math. es Phys. Lapok*, 27 (1918), 301-308.

11. P. Dienes, *The Taylor series* (Oxford, 1931).

12. L. Fejér, "Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe," *Journal für Math.*, 142 (1913), 165-188.

13. J. J. Gergen, "Convergence and summability criteria for Fourier series," *Quart. J. of Math.* (*Oxford series*), 1 (1930), 252-275.

14. G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power series and Fourier series," *Math. Zeitschrift*, 19 (1924), 67-96.

14(a). ———, The allied series of a Fourier series. *Proc. London Math. Soc.* (2), 24 (1925), 211-246.

15. ———, "Notes on the theory of series (XVI): Two Tauberian theorems," *Journal London Math. Soc.*, 6 (1931), 281-286.

16. E. W. Hobson, *The theory of functions of a real variable*, 2 (Cambridge, 1926).

17. J. M. Hyslop, "On the approach of a series to its Cesàro limit," *Proc. Edinburgh Math. Soc.* (2), 5 (1938), 182-201.

18. M. Jacob, "Beitrag zur Darstellung der Ableitungen einer Funktion durch ihre Fourier'sche Reihe," *Bull. de l'Académie Polonaise* (*Cracovie*) (A) (1927), 287-294.

19. E. Kogbetliantz, "Sur les séries absolument sommables par la méthode des moyennes arithmétiques," *Bull. des Sci. Math.* (2), 49 (1925), 234-256.

20. ———, "Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques," *Mémorial des Sciences Math.*, 51 (1931), 1-84.

21. N. Obrechkoff, "Sur la sommation des séries trigonométriques de Fourier par les moyennes arithmétiques," *Bull. de la Soc. Math. de France*, 62 (1934), 84-109 and 167-184.

22. R. E. A. C. Paley, "On the Cesàro summability of Fourier series and allied series," *Proc. Cambridge Phil. Soc.*, 26 (1930), 173-203.

23. S. Szidon, "A függvény ugrásának meghatározása a fugvény Fourier-féle sorából," *Math. és Termés. Ért.*, 27 (1918), 309-311.

24. O. Szász, "Über die Arithmetischen Mittel Fourierscher Reihen," *Acta Math.*, 48 (1936), 353-362.

25. E. C. Titchmarsh, "The order of magnitude of the coefficients in a generalised Fourier series," *Proc. London Math. Soc.* (2), 22 (1923), xxv-xxvi.

26. W. H. Young, "On the order of magnitude of the coefficients of a Fourier series," *Proc. Royal Soc.* (A), 93 (1916), 42-55.

27. A. Zygmund, "Sur un théorème de M. Gronwall," *Bull. de l'Acad. Polonaise* (*Cracovie*) (A) (1925), 207-217.

UNIVERSITY COLLEGE,
LONDON.