# Notes on Inequalities.

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[The following is a digest, consisting mainly of extracts, of Mr Ramaswami's paper. The author mentions that the Notes are intended for readers of Chrystal's "Algebra."]

*On a General Inequality Theorem.*

1. The important inequality of which Prof. Chrystal has given so many examples may be called the "Power Inequality."

There is a simple theorem of the Differential Calculus which is to a general function $f(x)$, what the Power Inequality is to the function $x^m$.

In what follows it will be supposed that the functions and the differential coefficients considered are finite, single-valued, and continuous between the limits of the variable considered, though they may be infinite, *at* either limit.

2. *Theorem*: If $f''(x)$ be always positive, or always negative, as $x$ increases from a value B to a value A, and if $a$ and $b$ be any two quantities lying between the limits A and B, $a$ being greater than $b$,

then $$f'(a) \gtrless \frac{f(a)-f(b)}{a-b} \gtrless f'(b)$$

according as $f''(x) \gtrless 0$, between the limits A and B.

[Then follow several proofs of the theorem (which is practically an aspect of the Mean Value Theorem); the simplest is that obtained from consideration of the fact that the curve $y=f(x)$ is, under the specified conditions, either convex or concave to the axis of $x$ throughout the range of values considered.]

Applying the theorem to the elementary functions, we have

(i) If $x$ and $y$ be positive, and $x>y$, then

$$mx^{m-1} \gtrless \frac{x^m-y^m}{x-y} \gtrless my^{m-1}$$

according as $m(m-1) \gtrless 0$. (The Power-Inequality.)

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(ii) If $a$ is any positive quantity $\neq 1$, and $x>y$,

$$a^x \log a > \frac{a^x - a^y}{x-y} > a^y \log a.$$

(iii) If $x$ and $y$ be positive, and $x>y$,

$$\frac{1}{x} < \frac{\log x - \log y}{x-y} < \frac{1}{y}.$$

(iv) If $\frac{\pi}{2} > x > y > 0$,

$$\cos x < \frac{\sin x - \sin y}{x-y} < \cos y;$$

etc.

We proceed to deduce some consequences from the general theorem.

3. *Theorem*: If $f''(x)$ be constantly positive, or constantly negative, as $x$ increases from B to A, and if $x$, $y$, $z$ be any three quantities in descending order of magnitude, lying between the limits A and B, then

$$f(x).(y-z)+f(y).(z-x).)+f(z).(x-y) \gtreqless 0,$$

according as $f''(x) \gtreqless 0$, between the limits A and B.

*Demonstration*: Suppose $f''(x)$ to be positive. Then, by the general theorem,

$$\frac{f(x)-f(y)}{x-y} > f'(y) > \frac{f(y)-f(z)}{y-z};$$

$$\therefore \quad \frac{f(x)-f(y)}{x-y} > \frac{f(y)-f(z)}{y-z}.$$

The denominators being positive, we have multiplying out, etc., the result

$$f(x).(y-z)+f(y).(z-x)+f(z).(x-y)>0.$$

If $f''(x)$ be negative, the inequality sign is reversed throughout.

*Examples*: (i) $f(x)=a^x$, (ii) $f(x)=x^m$, (iii) $f(x)=\log x$.

4. *Theorem*: If $f''(x)$ be constantly positive, or constantly negative, as $x$ increases from B to A, and $a$ be any fixed quantity lying between A and B, then the expression $\frac{f(x)-f(a)}{x-a}$ constantly increases, or constantly decreases, as $x$ increases from B to A (passing through the value $f'(a)$ as $x$ passes through $a$).

*Demonstration*: Suppose $f''(x)$ to be positive. Let $x$ and $y$ be any two quantities lying between the limits A and B, $x$ being greater than $y$. We have to show that

$$\frac{f(x)-f(a)}{x-a} > \frac{f(y)-f(a)}{y-a}.$$

First, if $x > y > a$, we have

$$\frac{f(x)-f(y)}{x-y} > \frac{f(y)-f(a)}{y-a};$$

Secondly, if $x > a > y$, we have

$$\frac{f(x)-f(a)}{x-a} > \frac{f(a)-f(y)}{a-y};$$

Thirdly, if $a > x > y$, we have

$$\frac{f(a)-f(x)}{a-x} > \frac{f(x)-f(y)}{x-y}.$$

And in each case the result reduces to

$$\frac{f(x)-f(a)}{x-a} > \frac{f(y)-f(a)}{y-a}.$$

If $f''(x)$ be negative, the inequality sign is reversed throughout.

*Examples*: $\quad \dfrac{x^m - a^m}{x-a}, \quad \dfrac{a^x - 1}{x}, \quad \dfrac{\tan x}{x}.$

5. ***Theorem***: If $f''(x)$ be constantly positive or constantly negative, as $x$ increases from B to A; and if $a, b, \ldots k$ be any $n$ quantities, not all equal, lying between the limits A and B; and $p, q, \ldots t$ be any system of positive multiples corresponding to $a, b, \ldots k$, respectively, then

$$\frac{pf(a)+qf(b)+\ldots+tf(k)}{p+q+\ldots+t} \gtrless f\left(\frac{pa+qb+\ldots+tk}{p+q+\ldots+t}\right)$$

according as $f''(x) \gtrless 0$, between the limits A and B.

*Demonstration*: Suppose $f''(x)$ to be positive. We shall first prove the theorem in the case of two quantities $a$ and $b$. Let $a$ be $> b$. Then $x$ being any quantity between $a$ and $b$, we have

$$\frac{f(a)-f(x)}{a-x} > \frac{f(x)-f(b)}{x-b}.$$

Now, for $x$ write $\dfrac{pa+qb}{p+q}$. This is permissible as the value of this fraction lies between $a$ and $b$.

Substituting and reducing, we get

$$\frac{pf(a)+qf(b)}{p+q} > f\left(\frac{pa+qb}{p+q}\right).$$

The result is thus proved for two unequal quantities $a$ and $b$. If $a$ and $b$ be equal, the inequality becomes an equality; so that, *in any case*, we can write

$$\frac{pf(a)+qf(b)}{p+q} \nless f\left(\frac{pa+qb}{p+q}\right).$$

Hence, by induction, we obtain

$$\frac{pf(a)+qf(b)+\ldots+tf(k)}{p+q+\ldots+t} > f\left(\frac{pa+qb+\ldots+tk}{p+q+\ldots+t}\right).$$

If $f''(x)$ be negative, the inequality signs are reversed throughout.

*Examples* :

(i) $f(x)=x^m$; (ii) $f(x)=y^x$; (iii) $f(x)=\sin^x$; (iv) $f(x)=\tan x$.

[The author then points out that inequalities of a different form can be obtained by writing for $f(x)$, say, $\log f(x)$; so that constancy of sign in $f''(x)$ is replaced by that in $u \equiv f(x) . f''(x) - \{f'(x)\}^2$.

The results are given for this particular case, and a great many interesting results arise out of it.

*E.g.*, 1. (i) $e^{\frac{1}{x}} < \left(\dfrac{x}{y}\right)^{\frac{1}{x-y}} < e^{\frac{1}{y}}$ if $x>y>0$.

(ii) $x^{y-z} . y^{z-x} . z^{x-y} < 1$ if $x>y>z>0$.

(iii) $x^{\frac{1}{x-1}}$ constantly decreases as $x$ increases from 0 to $\infty$, passing through the value $e$ as $x$ passes through the value 1.

(iv) $(a^p . b^q \ldots k^t)^{\frac{1}{p+q+\ldots+t}} < \dfrac{pa+qb+\ldots+tk}{p+q+\ldots+t}$

where $a, b, \ldots k$ are not all equal, and the symbols all denote positive numbers.

2. From $S_x \equiv a^x + b^x + \dots + k^x$,

(iii) $\left(\frac{a^x + b^x + \dots + k^x}{n}\right)^{\frac{1}{x}}$ constantly increases as $x$ increases from $-\infty$ to $+\infty$, and has the limiting value $(a.b.\dots k)^{\frac{1}{n}}$ when $x = 0$.

3. From $\cos x$,

(ii) $(\cos x)^{y-z} . (\cos y)^{z-x} . (\cos z)^{x-y} < 1$, if $\frac{\pi}{2} > x > y > z > 0$.

(iii) $(\cos x)^{\frac{1}{x}}$ constantly decreases as $x$ increases from 0 to $\frac{\pi}{2}$ and has the limiting value 1, when $x = 0$.

(iv) $(\cos x)^p . (\cos y)^q . < \left(\cos\frac{px + qy}{p + q}\right)^{p+q}$, $\frac{\pi}{2} > x > y > 0$ and $p$ and $q$ positive.]

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## On Mathematical Instruments and the accuracy to be obtained with them in some elementary practical problems.

By J. H. A. M'INTYRE.

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