KHINCHIN'S INEQUALITY FOR OPERATORS

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(Received 13 April, 1995)

1. Introduction. Let $\mathscr{A}$ be either a C^*-algebra (with norm $\| \ \|$) or a symmetric ideal of operators on a Hilbert space (with norm denoted by σ). Let $a_1, \ldots, a_n$ be self-adjoint elements, and let $a_0 = \left(\sum_{j=1}^{n} a_j^2\right)^{1/2}$.

Let $D_n = \{-1, 1\}^n$, with elements denoted by $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Write $A_\varepsilon = \sum_j \varepsilon_j a_j$. We shall consider inequalities involving

$$\mathrm{av}_\varepsilon |A_\varepsilon| =: \frac{1}{2^n} \sum_{\varepsilon \in D_n} |A_\varepsilon|.$$

In the same way as for scalars, it is elementary that $\mathrm{av}_\varepsilon A_\varepsilon^2 = a_0^2$, and hence

$$\mathrm{av}_\varepsilon |A_\varepsilon| \le a_0. \tag{1}$$

The most important case of the inequality of Khinchin (alias Khintchine, etc.) for scalars is the following converse of (1):

$$a_0 \le \sqrt{2}\, \mathrm{av}_\varepsilon |A_\varepsilon|. \tag{2}$$

Many proofs are known. Until recently, methods giving the best constant $\sqrt{2}$ were substantially harder, but a short and elegant proof has now been given in [**6**]. (This method actually proves the vector-valued version, i.e. Kahane's inequality.)

Three possible generalizations of (2) for operators, in descending order of strength, are:

(C1) $a_0 \le C \mathrm{av}_\varepsilon |A_\varepsilon|$ (an operator inequality),
(C2) $\sigma(a_0) \le C\sigma(\mathrm{av}_\varepsilon |A_\varepsilon|)$ for the norm considered,
(C3) $\sigma(a_0) \le C \mathrm{av}_\varepsilon \sigma(A_\varepsilon)$.

When the a_j are general (not self-adjoint) elements, both $|a|$ and "a_0" appear in three different versions, so that these statements can be reformulated in various ways.

A very simple example shows that (C1) is false, seemingly beyond hope of rescue by any reasonable reformulation. Even (C3) fails for the trace-class norm. (We give a direct example to show this, although it is implicit in the results of [**9**].) In the light of these facts, it is interesting that a statement midway between (C1) and (C2) is correct. We shall prove the following operator inequality:

$$a_0^2 \le \sqrt{3}\, \|a_0\|\, \mathrm{av}_\varepsilon |A_\varepsilon|,$$

from which it follows that (C2) holds for C^*-algebra norms. We actually prove versions for the left, right and symmetric modulus. These have to be formulated with some care: the simple-minded generalization obtained by writing (for example) $a_j^* a_j$ throughout is easily seen to be false.

The method imitates a version of the classic one using fourth powers. For scalars, this is based on the inequality $\mathrm{av}_\varepsilon A_\varepsilon^4 \le 3a_0^4$. The generalization of this to self-adjoint elements of C^*-algebras was given by Pisier [**10**]. (This was a vital step in the proofs of the

https://doi.org/10.1017/S001708950003175X Published online by Cambridge University Press

non-commutative Grothendieck inequality in **[10]** and **[3]**.) Our proof requires an extension of Pisier's theorem to non-self-adjoint elements, which we give in two variants.

2. Preliminaries. Before proceeding further, we show that two 2×2 matrices are enough to provide a counter-example to (C1).

EXAMPLE 1. Let

$$a_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\lambda > 0$, $\mu > 0$ and $\lambda\mu \geq 1$. Then $a_1 + a_2 \geq 0$ and $a_1 - a_2 \geq 0$; hence $\mathrm{av}_\varepsilon |A_\varepsilon| = a_1$. Also, $a_2^2 = I$ and hence (with the above notation)

$$a_0 = \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix},$$

where $\lambda' = (1 + \lambda^2)^{1/2}$, $\mu' = (1 + \mu^2)^{1/2}$. Given any $K > 0$, let $\lambda = K^{-1}$. Since $\lambda' > 1$, the inequality $a_0 \leq K\, \mathrm{av}_\varepsilon |A_\varepsilon|$ is false.

Now let $(E, \|\ \|_E)$ be a symmetric Banach sequence space (cf. **[2, 11]**. Define $\mathcal{A}$ to be the space of compact operators a on l_2 such that

$$\sigma_E(a) =: \|[s_n(a)]\|_E$$

is finite, where $[s_n(a)]$ is the sequence of singular numbers of a. By a "symmetric ideal" of operators, and a "symmetric" norm, we mean an algebra $\mathcal{A}$ and a norm σ_E defined in this way. The properties of such norms that matter for our purposes are:

$$\sigma(a) \geq \|a\|,$$
$$\sigma(a^*) = \sigma(a),$$
$$\sigma(xay) \leq \|x\|\, \sigma(a)\, \|y\|,$$
$$\text{if } -a \leq b \leq a, \text{ then } \sigma(b) \leq \sigma(a).$$

The last property is an immediate consequence of the characterization in terms of orthonormal bases (e.g. **[11**, Theorem 2.6**]**). We write σ_p for σ_{l_p}, so that (for example) σ_2 is the Hilbert-Schmidt norm.

We use the following notation for the three moduli of an element:

$$|a|_R = (a^*a)^{1/2},$$
$$|a|_L = (aa^*)^{1/2},$$
$$|a|_S = \left(\frac{1}{2}(a^*a + aa^*)\right)^{1/2}.$$

($|a|_S$ only appears as an optional extra in the results below.) Since $|a|_R$ and $|a|_L$ have the same singular numbers as a, we have $\sigma(|a|_R) = \sigma(|a|_L) = \sigma(a)$ for any symmetric norm σ. Note that $|a|_S^2 = \frac{1}{2}|a|_R^2 + \frac{1}{2}|a|_L^2 = b^2 + c^2$, where $a = b + ic$. Clearly, $|a|_R^2 \leq 2|a|_S^2$; hence $|a|_R \leq \sqrt{2}\,|a|_S$ (and similarly for $|a|_L$), by the well-known fact that if a, b are positive and $0 \leq a^2 \leq b^2$, then $a \leq b$.

Given elements $a_1, \ldots, a_n$, we define

$$a_R = \Big(\sum_j a_j^* a_j\Big)^{1/2} = \Big(\sum_j |a_j|_R^2\Big)^{1/2}$$

and a_L, a_S correspondingly. For self-adjoint a_j, each of these clearly coincides with the a_0 above. When $n = 1$, a_R coincides with $|a|_R$. Again, $2a_S^2 = a_R^2 + a_L^2$ and hence $a_R \le \sqrt{2} a_S$.

It is a well-known property of symmetric norms (proved by considering suitable operators on the product H^n) that if each a_j is in $\mathscr{A}$, then so are a_R and a_L; also $\sigma(a_R)$ and $\sigma(a_L)$ are not greater than $\sum_j \sigma(a_j)$. The norm σ is said to be (strictly) *2-convex* if for self-adjoint elements a_j with $a_0 = a_R = a_L$, we have

$$\sigma(a_0)^2 \le \sum_j \sigma(a_j)^2,$$

and (strictly) *2-concave* if the opposite inequality holds. (The word "strictly" implies that this occurs with constant 1, but we shall leave it to be understood.) For non-self-adjoint a_j, we see by applying the definition to the elements $|a_j|_R$ that the same inequality then holds with a_0 replaced by a_R (or a_L). It is elementary that a C^*-algebra norm is 2-convex and σ_p is 2-convex for $p \ge 2$, 2-concave for $p \le 2$. More generally, σ_E is 2-convex or 2-concave if $\| \ \|_E$ is. See **[9]**.

The next lemma clarifies the relationship with a_S.

LEMMA 1. *We have* $\max[\sigma(a_R), \sigma(a_L)] \le \sqrt{2}\,\sigma(a_S) \le \sigma(a_R) + \sigma(A_L)$.

If σ is 2-convex, then $\sigma(a_S) \le \max[\sigma(a_R), \sigma(a_L)]$.

If σ is 2-concave, then $\sigma(a_S) \ge \min[\sigma(a_R), \sigma(a_L)]$.

In particular, for one element, $(1/\sqrt{2})\sigma(a) \le \sigma(|a|_S) \le \sqrt{2}\,\sigma(a)$. *The right-hand constant becomes* 1 *when σ is 2-convex and the left-hand constant becomes* 1 *when σ is 2-concave.*

Proof. The left-hand inequality follows from the fact that $a_R \le \sqrt{2}\,a_S$ (and similarly for a_L), and the right-hand inequality from $2a_S^2 = a_R^2 + a_L^2$, together with the remark above. The statements for 2-convex and 2-convave norms also follow easily from this identity. The statements for a single element a follow at once when we recall that $\sigma(|a|_R) = \sigma(|a|_L) = \sigma(a)$.

By the last statement in Lemma 1, it is clear that a_0 can be replaced by a_S in the definition of 2-convex and 2-concave.

We shall require the following Cauchy-Schwarz inequality. It is actually a special case of the Cauchy-Schwarz inequality for "inner-product $\mathscr{A}$-modules" given in [**5**, Proposition 1.1], but the proof is short and so we include it for completeness.

PROPOSITION 1. *For elements x_j, y_j $(1 \le j \le n)$ of a C^*-algebra, we have*

$$\Big(\sum_j x_j y_j^*\Big)\Big(\sum_j y_j x_j^*\Big) \le \Big\|\sum_j y_j y_j^*\Big\| \Big(\sum_j x_j x_j^*\Big).$$

In particular, for scalars λ_j (with the above notation),

$$\Big|\sum_j \lambda_j x_j\Big|_R \le \Big(\sum_j |\lambda_j|^2\Big)^{1/2} x_R,$$

(and similarly for x_L, x_S).

Proof. Consider $\mathscr{A}^n$, with elements $x = (x_1, \dots, x_n)$. For $a \in \mathscr{A}$ and $x,y \in \mathscr{A}^n$, define:

$$ax = (ax_1, \dots, ax_n),$$

$$\langle x, y\rangle = \sum_j x_j y_j^*.$$

Our statement is $\langle x, y\rangle\langle y, x\rangle \le \|\langle y, y\rangle\| \langle x, x\rangle$. Note that $\langle x, x\rangle \ge 0$ and $\langle x, ay\rangle = \langle x, y\rangle a^*$. It is enough to consider the case where $\|\langle y, y\rangle\| = 1$. For any $a \in \mathscr{A}$, we then have

$$\begin{aligned} 0 &\le \langle x - ay, x - ay\rangle \\ &= \langle x, x\rangle - a\langle y, x\rangle - \langle x, y\rangle a^* + a\langle y, y\rangle a^*. \end{aligned}$$

Since $\|\langle y, y\rangle\| = 1$, we have $a\langle y, y\rangle a^* \le aa^*$. Take $a = \langle x, y\rangle$ to obtain

$$\begin{aligned} 0 &\le \langle x, x\rangle - aa^* - aa^* + aa^* \\ &= \langle x, x\rangle - \langle x, y\rangle\langle y, x\rangle, \end{aligned}$$

as required. The second statement is obtained by putting $y_j = \lambda_j e$, where e is the identity.

(Note that x_j and y_j cannot be interchanged on the right-hand side, even when $n = 1$. Also, Example 2 below shows that $\sum_j y_i y_j^*$ cannot be replaced by $\sum_j y_j^* y_j$.)

In particular, for positive elements $a_1, \dots, a_n$, we have

$$\frac{1}{n}\sum_j a_j \le \left(\frac{1}{n}\sum_j a_j^2\right)^{1/2}.$$

Returning to our basic problem, let $a_1, \dots, a_n$ be elements of $\mathscr{A}$ (not necessarily self-adjoint), and let $A_\varepsilon = \sum_j \varepsilon_j a_j$. Then

$$A_\varepsilon^* A_\varepsilon = \sum_j a_j^* a_j + \sum_{i \ne j} \varepsilon_i \varepsilon_j a_i^* a_j.$$

Since $\sum_{\varepsilon \in D_n} \varepsilon_i \varepsilon_j = 0$ for each fixed i, j, we have (exactly as in the scalar case)

$$\mathrm{av}_\varepsilon(A_\varepsilon^* A_\varepsilon) = \sum_j a_j^* a_j;$$

that is,

$$\mathrm{av}_\varepsilon\, |A_\varepsilon|_R^2 = a_R^2 \tag{3}$$

and hence, by Proposition 1,

$$\mathrm{av}_\varepsilon\, |A_\varepsilon|_R \le a_R. \tag{4}$$

Similar statements apply to a_L and a_S. (This is the full version of our original inequality (1).)

We mention at this point an elementary fact about $\mathrm{av}_\varepsilon\, |A_\varepsilon|$ in the self-adjoint case.

LEMMA 2. *Let $a_1, \dots, a_n$ be self-adjoint and $A = \mathrm{av}_\varepsilon\, |A_\varepsilon|$. Then $\pm a_j \le A$, for each j. Hence if σ is any symmetric norm, then $\sigma(a_j) \le \sigma(A)$. If σ is 2-convex, then $\sigma(a_0) \le n^{1/2}\sigma(A)$.*

Proof. For fixed j, we have $\operatorname{av}_\varepsilon \varepsilon_j A_\varepsilon = a_j$. Now $\pm A_\varepsilon \le |A_\varepsilon|$; hence $\pm a_j \le A$ and $\sigma(a_j) \le \sigma(A)$. If σ is 2-convex, then $\sigma(a_0)^2 \le \sum_j \sigma(a_j)^2$.

Note. It is not true in general that $\sigma(a_j) \le \sigma(\operatorname{av}_\varepsilon |A_\varepsilon|_R)$. This is shown by the elements b_j in Example 2 below.

3. Extensions of Pisier's theorem for fourth powers. In our notation, Pisier's Khinchin-type inequality for fourth powers **[10]** states that for self-adjoint elements $a_1, \dots, a_n$,

$$\operatorname{av}_\varepsilon A_\varepsilon^4 \le a_0^4 + 2\,\|a_0\|^2\, a_0^2.$$

This is a C^*-algebra inequality, implying (for norms) $\|\operatorname{av}_\varepsilon A_\varepsilon^4\| \le 3\,\|a_0\|^4$. We remark that in Example 1, the statement $\operatorname{av}_\varepsilon A_\varepsilon^4 \le K a_0^4$ is false for all K. Pisier's result is sufficient for the self-adjoint case in our theorem, but for the general case, we need the following adaptation: the proof is essentially the same, but with careful attention to the positions in which elements a_j^* occur.

PROPOSITION 2. *Let $a_1, \dots, a_n$ be elements of $\mathcal{A}$. Then we have (with the above notation)*

$$\operatorname{av}_\varepsilon |A_\varepsilon|_R^4 \le a_R^4 + 2\,\|a_L\|^2 a_R^2.$$

In particular, if $M = \max(\|a_R\|, \|a_L\|)$, then

$$\operatorname{av}_\varepsilon |A_\varepsilon|_R^4 \le (\|a_R\|^2 + 2\,\|a_L\|^2) a_R^2 \le 3M^2 a_R^2.$$

Proof. We have

$$|A_\varepsilon|_R^2 = A_\varepsilon^* A_\varepsilon = \sum_i a_i^* a_i + \sum_{i<j} \varepsilon_i \varepsilon_j (a_i^* a_j + a_j^* a_i).$$

After squaring again and removing terms that average to 0, we have

$$\operatorname{av}_\varepsilon |A_\varepsilon|_R^4 = a_R^4 + \sum_{i<j} (a_i^* a_j + a_j^* a_i)^2.$$

By the elementary inequality $(x + x^*)^2 \le 2(xx^* + x^*x)$, we have

$$(a_i^* a_j + a_j^* a_i)^2 \le 2(a_i^* a_j a_j^* a_i + a_j^* a_i a_i^* a_j).$$

Hence

$$\begin{aligned}
\sum_{i<j} (a_i^* a_j + a_j^* a_i)^2 &\le 2 \sum_i a_i^* \Big(\sum_{j>i} a_j a_j^*\Big) a_i + 2 \sum_j a_j^* \Big(\sum_{i<j} a_i a_i^*\Big) a_j \\
&= 2 \sum_i a_i^* \Big(\sum_{j \ne i} a_j a_j^*\Big) a_i \\
&\le 2 \sum_i a_i^* \Big(\sum_j a_j a_j^*\Big) a_i \\
&\le 2 \Big\|\sum_j a_j a_j^*\Big\| \Big(\sum_i a_i^* a_i\Big) \\
&= 2\,\|a_L\|^2 a_R^2.
\end{aligned}$$

The statement follows.

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We shall see later (Example 2) that the a_L appearing in Proposition 1 cannot be replaced by a_R.

Proposition 2 is adequate for our main theorem, but it is clearly highly unsymmetrical. Before going on to the main theorem, we show how to derive a symmetrical version.

PROPOSITION 3. *With the same notation, we have*

$$\text{av}_\varepsilon\, |A_\varepsilon|_S^4 \le a_S^4 + \|a_L\|^2 a_R^2 + \|a_R\|^2 a_L^2 \le a_S^4 + 4\,\|a_S\|^2\, a_S^2.$$

Proof. We have

$$|A_\varepsilon|_S^2 = \frac{1}{2} A_\varepsilon^* A_\varepsilon + \frac{1}{2} A_\varepsilon A_\varepsilon^* = a_S^2 + \frac{1}{2} \sum_{i<j} \varepsilon_i \varepsilon_j (u_{ij} + v_{ij}),$$

where $u_{ij} = a_i^* a_j + a_j^* a_i$, $v_{ij} = a_i a_j^* + a_j a_i^*$. Squaring again and cancelling as before, we have

$$\text{av}_\varepsilon\, |A_\varepsilon|_S^4 = a_S^4 + \frac{1}{4} \sum_{i<j} (u_{ij} + v_{ij})^2.$$

For self-adjoint u, v, we have $(u+v)^2 = 2(u^2+v^2) - (u-v)^2 \le 2(u^2+v^2)$. Hence

$$\text{av}_\varepsilon\, |A_\varepsilon|_S^4 \le a_S^4 + \frac{1}{2} \sum_{i<j} (u_{ij}^2 + v_{ij}^2).$$

As shown in proposition 2, $\sum_{i<j} u_{ij}^2 \le 2\,\|a_L\|^2 a_R^2$. Substituting a_i^* for a_i, we obtain also $\sum_{i<j} v_{ij}^2 \le 2\,\|a_R\|^2\, a_L^2$. The left-hand inequality follows. For the right-hand inequality, note that

$$\|a_L\|^2 a_R^2 + \|a_R\|^2 a_L^2 \le 2\,\|a_S\|^2 (a_R^2 + a_L^2) = 4\,\|a_S\|^2\, a_S^2.$$

4. The main theorem: statement (C2). The following imitates the proof for the scalar case found in [**4**]. Other known methods (including the elegant new one of [**6**]) do not appear to adapt readily to a form relevant to (C2). The key step is the following lemma.

LEMMA 3. *For any self-adjoint element a and any $t > 0$, we have*

$$|a| \ge \frac{3}{2} t a^2 - \frac{1}{2} t^3 a^4.$$

Proof. For real $x > 0$ it is elementary that $3x - x^3 \le 2$. Hence $|x| \ge \frac{3}{2}x^2 - \frac{1}{2}x^4$ for all real x. By the Gelfand representation, it follows that the same inequality holds for any self-adjoint element a of $\mathcal{A}$. Hence for any $t > 0$, we have $t\,|a| \ge \frac{3}{2}t^2 a^2 - \frac{1}{2}t^4 a^4$. The statement follows.

Our Theorem now follows quite easily. We remark that the *statement* becomes much simpler when the elements are self-adjoint, though the *proof* is the same.

THEOREM 1. *Let $a_1, \ldots, a_n$ be elements of $\mathcal{A}$. Write $A^R = \text{av}_\varepsilon\, |A_\varepsilon|_R$, and define A^L, A^S similarly. Let a_R, a_L be as above, and let $M = \max(\|a_R\|, \|a_L\|)$. Then*
(i) $a_R^2 \le \sqrt{3}\, M A^R$ *(and similarly for A_L)*;
(ii) $M \le \sqrt{3} \max(\|A^R\|, \|A^L\|)$.

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In particular, if the a_j are self-adjoint and $A = \mathrm{av}_\varepsilon |A_\varepsilon|$, then

(i) $a_0^2 \leq \sqrt{3}\, \|a_0\|\, A$,

(ii) $\|a_0\| \leq \sqrt{3}\, \|A\|$.

Further, we have $a_S^2 \leq \sqrt{5}\, \|a_S\|\, A^S$ and $\|a_S\| \leq \sqrt{5}\, \|A^S\|$.

Proof. By Lemma 3, for any $t > 0$, we have

$$A^R \geq \frac{3}{2} t\, \mathrm{av}_\varepsilon\, |A_\varepsilon|_R^2 - \frac{1}{2} t^3\, \mathrm{av}_\varepsilon\, |A_\varepsilon|_R^4.$$

Hence by (3) and Proposition 2,

$$A^R \geq \frac{3}{2} t a_R^2 - \frac{3}{2} t^3 M^2 a_R^2.$$

The maximum value of $\frac{3}{2}(t - M^2 t^3)$ is $1/(\sqrt{3}\, M)$, occurring when $t = 1/(\sqrt{3}\, M)$. This proves (i).

It follows that

$$\|a_R\|^2 = \|a_R^2\| \leq \sqrt{3}\, M\, \|A^R\|.$$

Together with the similar inequality for a_L, this gives

$$M^2 \leq \sqrt{3}\, M \max(\|A^R\|, \|A^L\|),$$

and so (ii) is proved.

By Proposition 3, $\mathrm{av}_\varepsilon\, |A_\varepsilon|_S^4 \leq 5\, \|a_S\|^2\, a_S^2$. Hence

$$A^S \geq \frac{3}{2} t a_S^2 - \frac{5}{2} t^3\, \|a_S\|^2\, a_S^2.$$

Choosing $t = 1/(\sqrt{5}\, \|a_S\|)$, we obtain $A^S \geq (1/\sqrt{5}) a_S^2 / \|a_S\|$, and hence $\|a_S\| \leq \sqrt{5}\, \|A^S\|$. This inequality, with $\sqrt{6}$ instead of $\sqrt{5}$, can also be deduced from (ii) without Proposition 3, using the elementary relations $\|a_S\| \leq M$ (from Lemma 1 noting that $\|\ \|$ is 2-convex) and $A^R \leq \sqrt{2}\, A^S$.

Note. Let σ be any symmetric norm, and let $\sigma^{(2)}$ be its 2-convexification, defined (for self-adjoint elements) by $\sigma^{(2)}(a) = (\sigma(a^2))^{1/2}$. In particular, if $\sigma = \sigma_p$, then $\sigma^{(2)} = \sigma_{2p}$. For self-adjoint a_j, we have $a_0^2 \leq \sqrt{3}\, \|a_0\|\, A$; hence $\sigma^{(2)}(a_0)^2 \leqq \sqrt{3}\, \|a_0\|\, \sigma(A)$. Since $\|a_0\| \leq \sigma^{(2)}(a_0)$, this gives $\sigma^{(2)}(a_0) \leq \sqrt{3}\, \sigma(A)$. The corresponding statement for non-self-adjoint elements is

$$\max(\sigma^{(2)}(a_R), \sigma^{(2)}(a_L)) \leq \sqrt{3} \max(\sigma(A^R), \sigma(A^L)).$$

The following example shows how comprehensively the one-sided version $\|a_R\| \leq C\, \|A^R\|$ fails.

EXAMPLE 2. Consider operators on n-dimensional Hilbert space. Let $e_1, \ldots, e_n$ denote the usual basis. For $1 \leq j \leq n$, let $b_j = e_1 \otimes e_j$, so that $b_j(x) = \langle x, e_j \rangle e_1$, and let $p_j = e_j \otimes e_j$. From the relation $(x \otimes y)(u \otimes v) = \langle u, y \rangle (x \otimes v)$, one has $b_j^* b_j = p_j$ and $b_j b_j^* = p_1$, so that

$$\sum_j b_j^* b_j = I, \qquad \sum_j b_j b_j^* = n p_1;$$

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hence $b_R = I$, $b_L = n^{1/2}p_1$. Also, $\mathrm{av}_\varepsilon(B_\varepsilon^* B_\varepsilon) = \sum_j b_j^* b_j = I$. Now

$$B_\varepsilon(x) = \sum_j \varepsilon_j \langle x, e_j \rangle e_1 = \langle x, \varepsilon \rangle e_1,$$

in which we now regard $\varepsilon = \sum_j \varepsilon_j e_j$ as an element of l_2^n. Also, $b_j^*(e_1) = e_j$, so that $B_\varepsilon^*(e_1) = \varepsilon$ and hence $B_\varepsilon^* B_\varepsilon = \varepsilon \otimes \varepsilon$. Therefore

$$|B_\varepsilon|_R = (B_\varepsilon^* B_\varepsilon)^{1/2} = n^{-1/2} \varepsilon \otimes \varepsilon = n^{-1/2} B_\varepsilon^* B_\varepsilon.$$

and so $B^R = \mathrm{av}_\varepsilon |R_\varepsilon|_R = n^{-1/2} I = n^{-1/2} b_R$.

We remark that $B_\varepsilon B_\varepsilon^* = np_1$ and hence $B^L = b_L = n^{1/2}p_1$. Also, to give a direct counter-example to the one-sided version of Proposition 2, note that

$$\sum_{i<j} (b_i^* b_j + b_j^* b_i)^2 = \sum_{i<j} (p_i + p_j) = (n-1)I.$$

5. Problems, remarks and special cases. An obvious problem is to find the best constant in our statements. Can the $\sqrt{3}$ be replaced by $\sqrt{2}$, as in the scalar case? As mentioned above, the author does not see any way of adapting the method of [6] to operators: it appears to depend fundamentally on $|\sum a_j| \le \sum |a_j|$. However, it is not surprising that $\sqrt{2}$ is correct in the case $n = 2$. In fact, two different stronger variants of our basic result apply in this case. One of these is the statement for 2-convex norms given in Lemma 2. The other is the following result.

LEMMA 4. *For the elements a_1, a_2, we have $a_R^2 \le \sqrt{2}\, \|a_R\|\, A^R$; hence $\|a_R\| \le \sqrt{2}\, \|A^R\|$.*

Proof. We have

$$a_R^2 = \frac{1}{2}|a_1 + a_2|_R^2 + \frac{1}{2}|a_1 - a_2|_R^2;$$

hence $\|a_1 \pm a_2\| \le \sqrt{2}\, \|a_R\|$ and

$$\begin{aligned} a_R^2 &\le \frac{1}{2}\|a_1 + a_2\|\, |a_1 + a_2|_R + \frac{1}{2}\|a_1 - a_2\|\, |a_1 - a_2|_R \\ &\le \sqrt{2}\, \|a_R\| \frac{1}{2}(|a_1 + a_2|_R + |a_1 - a_2|_R) \\ &= \sqrt{2}\, \|a_R\| A^R. \end{aligned}$$

Note that this differs from Theorem 1 in being one-sided throughout. For n elements, the $\sqrt{2}$ becomes $2^{(n-1)/2}$.

Failing an answer to the general question, does the constant $\sqrt{2}$ apply for 2×2 matrices?

REMARKS ON (C3). If σ is 2-convex, then we have from (3)

$$\sigma(a_R) \le [\mathrm{av}_\varepsilon\, \sigma(A_\varepsilon)^2]^{1/2}.$$

By Kahane's inequality (with the constant $\sqrt{2}$ obtained in [**6**]).

$$[\mathrm{av}_\varepsilon \sigma(A_\varepsilon)^2]^{1/2} \le \sqrt{2}\,\mathrm{av}_\varepsilon \sigma(A_\varepsilon).$$

Hence (C3) holds trivially for any 2-convex norm σ (also for variants such as $\sigma(a_S)$ and $\max[\sigma(a_R), \sigma(a_L)]$).

Clearly, the opposite inequalities hold for 2-concave norms. However, the true comparison in this case is with the following quantity, introduced in [**9**]. Define

$$\hat{\sigma}(a_1, \dots, a_n) = \inf\{\sigma(b_R) + \sigma(c_L) : a_j = b_j + c_j \text{ for each } j\}.$$

The significance of this norm is that $(\mathcal{A}^n, \hat{\sigma})$ is predual to $\mathcal{A}'^n$ with norm $\max[\sigma'(x_R), \sigma' x_L)]$, where σ' is the dual norm to σ. In the easy direction, we have the following result.

LEMMA 5. *If σ is 2-concave, then* $[\mathrm{av}_\varepsilon \sigma(A_\varepsilon)^2]^{1/2} \le \hat{\sigma}(a_1, \dots, a_n)$.

Proof. For any decomposition $a_j = b_j + c_j$, we have $\sigma(A_\varepsilon) \le \sigma(B_\varepsilon) + \sigma(C_\varepsilon)$ and

$$[\mathrm{av}_\varepsilon\, \sigma(B_\varepsilon)^2]^{1/2} \le \sigma(b_R),$$
$$[\mathrm{av}_\varepsilon\, \sigma(C_\varepsilon)^2]^{1/2} \le \sigma(c_L);$$

hence by Minkowski's inequality

$$[\mathrm{av}_\varepsilon \sigma(A_\varepsilon)^2]^{1/2} \le \sigma(b_R) + \sigma(c_L).$$

Similarly, we have $\left[\sum_j \sigma(a_j)^2\right]^{1/2} \le \hat{\sigma}(a_1, \dots, a_n)$.

The Hilbert-Schmidt norm σ_2 satisfies $\sigma_2(a_R)^2 = \sigma_2(a_L)^2 = \sum_j \sigma_2(a_j)^2$ and so if $a_j = b_j + c_j$, then $\sigma_2(a_R) \le \sigma_2(b_R) + \sigma_2(c_R) = \sigma_2(b_R) + \sigma_2(c_L)$; hence

$$\hat{\sigma}_2(a_1, \dots, a_n) = \sigma_2(a_R) = \sigma_2(a_L).$$

(This also follows from (3) and Lemma 5.) This property is quite special to σ_2. The next example shows that for other norms (in particular, $\|\ \|$ and σ_1), $\hat{\sigma}(a_1, \dots, a_n)$ is not equivalent to $\min[\sigma(a_R), \sigma(a_L)]$, even for self-adjoint elements. For σ_1, this constitutes a counter-example to (C2) and (C3) (which of course coincide in the case of σ_1).

EXAMPLE 3. Consider again the operators b_j in Example 2. Let $c_j = b_j^*$ and $a_j = b_j + c_j$. Clearly,

$$b_R = c_L = I, \qquad b_L = c_R = n^{1/2} p_1.$$

For $j \ge 2$, we have $b_j^2 = 0$ and hence $a_j^2 = p_1 + p_j$. Therefore $\sum_j a_j^2 = \mathrm{diag}(n+3, 1, \dots, 1)$ and

$$a_0 = \left(\sum_j a_j^2\right)^{1/2} = \mathrm{diag}((n+3)^{1/2}, 1, \dots, 1).$$

It is now clear that $\|a_0\| = (n+3)^{1/2}$, while if $\sigma = \|\ \ \|$, then

$$\hat{\sigma}(a_1, \ldots, a_n) \le \|b_R\| + \|c_L\| = 2.$$

Also, $\sigma_1(a_0) = (n-1) + (n+3)^{1/2}$, while

$$\hat{\sigma}_1(a_1, \ldots, a_n) \le \sigma_1(b_L) + \sigma_1(c_R) = 2n^{1/2}.$$

It is shown in [**9**] that for 2-concave norms subject to certain conditions, the reverse inequality to Lemma 5 holds (with an intervening constant), so that $\text{av}_\varepsilon\, \sigma(A_\varepsilon)$ is equivalent to $\hat{\sigma}$. This is the proper "Khinchin" inequality for this case. Two proofs are given in [**9**], both quite deep. One of the methods shows that the statement is essentially equivalent to the Grothendieck-type factorization theorem of [**8**] (given the results of [**1**]) and so a simple direct proof, though desirable, seems unlikely.

Problem. Is (C2) true for Hilbert-Schmidt norm σ_2, or even for 2-convex norms generally? By Theorem 1, if the a_j are such that $\sigma_2(a_0)\, \|a_0\| \le C\sigma_2(a_0^2)$, then $\sigma_2(a_0) \le \sqrt{3}\, C\sigma_2(A)$, and so a counter-example must avoid this condition. The a_j in Example 3 fail to provide a counter-example for the even simpler reason that they satisfy $\sigma_2(a_0) \le 2\, \|a_0\|$. The general difficulty in constructing counter-examples is of course that it is very laborious to calculate $\text{av}_\varepsilon\, |A_\varepsilon|$.

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