ON ASYMMETRICAL DERIVATES OF NON-DIFFERENTIABLE FUNCTIONS

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1. Introduction. Let $f(x)$ be a non-differentiable function, i.e. a real-valued continuous function defined on a linear interval which has nowhere a finite or infinite derivative. We shall say that $f(x)$ has *symmetrical derivates* at a point x if the four Dini derivates of $f(x)$ at x satisfy the relations

$$D^{+}f(x) = D^{-}f(x), \qquad D_{+}f(x) = D_{-}f(x);$$

and otherwise we shall say that $f(x)$ has *asymmetrical derivates* at x. Let A denote the set of points where $f(x)$ has asymmetrical derivates and let K denote the set of *knot points* (**20**, p. 168) of $f(x)$, viz. the points where

$$D^{+}f = D^{-}f = +\infty, \qquad D_{+}f = D_{-}f = -\infty.$$

Clearly $K \subset C(A)$, where $C(A)$ denotes the complementary set of A, and so $A \subset C(K)$. The object of the present note is to investigate the structure of the set A and the nature of derivates of $f(x)$ at its points.

W. H. Young (**21**, Th. 1) proved in 1908 that the set K is residual, the set A being therefore of the first category. He further proved (**21**, Th. 2) that A is necessarily non-empty and is everywhere dense, and then enquired (**21**, p. 69) whether it is a most general set of the first category or it is only enumerable. In response to this query, A. N. Singh (**15**, pp. 5, 6) proved in 1930 that the set $C(K)$ is unenumerable in every interval, and A. P. Morse (**10**, p. 129, remark 2) proved in 1954 that this set has the power of the continuum. The unenumerability of A, however, cannot be inferred from these results. Singh also stated, without proof, in his monograph on non-differentiable functions (**17**, pp. 76 and 96) that it can be shown that $f(x)$ has knot points almost everywhere, implying thereby that A is always of measure zero.

In the present paper we first prove in § 2 the existence of non-differentiable functions, considering Besicovitch's function (**1**) in particular, which have asymmetrical derivates almost everywhere. The knot points of such a function form a set of measure zero, and Singh's statement is thus found to be false.

In § 3 we study in detail the asymmetrical derivates of general non-differentiable functions, and in particular of those of the Besicovitch type (viz. those which have nowhere a unilateral derivative, finite or infinite) and the Weierstrass type (viz. those which have knot points almost everywhere). For any non-differentiable function the set A is found to have the power of the continuum in every interval (see Proposition 1). The property of Besicovitch's function is found to hold to a certain degree for all the non-differentiable

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functions of its type, viz. the set A has a positive measure in every interval (see corollary of Proposition 2). Propositions 2 and 3 also establish the author's two earlier statements.[1]

Lastly we study, in §4, the derivates of Weierstrass's non-differentiable function in continuation with the earlier investigations of G. C. Young **(20)**. Theorem 2 provides the nature of the distribution, viz. the Lebesgue measure, the Baire category, and the cardinal number, of the sets of points where the four derivates of Weierstrass's function take any possible set of values.

2. Besicovitch's function. A. S. Besicovitch **(1)** gave in 1925 the first example[2] of a continuous function which has nowhere a unilateral derivative. We prove in this section that his function has asymmetrical derivates almost everywhere. We shall, however, utilize the following arithmetical definition of his function given by Singh **(18)** in 1941.

Any point x in $[0, 1]$ can be represented in the form

$$x = X_{1,n_1} + \frac{2}{4^{n_1+1}}\Big(X_{2,n_2} + \ldots \Big(\ldots + \frac{2}{4^{n_{r-1}+1}}\Big(X_{r,n_r} + \ldots\Big(\ldots, \tag{1}$$

where the number of brackets may be finite or infinite, the n_r $(r = 1, 2, \ldots)$ have positive integral values except when the number of brackets is finite (say m), in which case the last n_r (viz. n_{m+1}) is ∞, and for each $r = 1, 2, \ldots,$

(i) if $n_r = 1$, $X_{r,1} = 3/4^2$ or $11/4^2$,

(ii) if $n_r > 1$,

$$X_{r,n_r} = \frac{a_{r,1}}{4^2} + \frac{a_{r,2}}{4^3} + \ldots + \frac{a_{r,n_r-1}}{4^{n_r}} + \frac{2^{n_r}+1}{4^{n_r+1}},$$

where

$$a_{r,1} = 0, 5, 8, \text{ or } 13,$$

and, for $1 < m < n_r$,

$$a_{r,m} = 0 \text{ or } 2^m + 3.$$

Besicovitch's function $f(x)$ $(0 \leqslant x \leqslant 1)$ is defined as

$$f(x) = Y_{1,n_1} - \frac{1}{2^{n_1}}\Big(Y_{2,n_2} - \ldots \Big(\ldots - \frac{1}{2^{n_{r-1}}}\,(Y_{r,n_r} - \ldots\Big(\ldots, \tag{2}$$

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[1]See **(3, pp. 10, 11)**, statements (iii) and (iv). These statements were made by the author to demonstrate that the Denjoy analogue, established in that paper, is the best possible in its direction like the classical Denjoy theorem **(2, p. 105)**. What is needed there is that the non-differentiable functions of the Besicovitch and the Weierstrass types map respectively points where $(a.2f)$ and $(a.2\infty)$ (or $(a.3f)$ and $(a.3\infty)$) **(3)** hold into sets of non-zero measure. It has been stated there that these points are mapped into sets of positive measure, whereas what we prove in the present paper is that these points are mapped into sets of positive interior measure, the measurability of these image sets remaining unknown. It would, however, suffice to replace the words "maps points" in both the statements by "maps some of the points," for an image set with positive interior measure does contain a measurable subset with positive measure. For proof of the statement (v) in **(3, p. 11)** see author's paper **(6, p. 193, Cor. 6)**.

[2]For other expositions of Besicovitch's function, see E. D. Pepper **(11)**, A. N. Singh **(18)**, and R. L. Jeffery **(8, pp. 172–181)**. The treatments of Besicovitch, Pepper, and Jeffery are geometrical whereas that of Singh is arithmetical.

where to each X_{r,n_r} $(r = 1, 2, \ldots)$ in any representation of x there corresponds a Y_{r,n_r} given by

(i) if $n_r = 1$, $Y_{r,1} = \frac{1}{2}$,

(ii) if $n_r > 1$,

$$Y_{r,n_r} = \frac{c_{r,1}}{2} + \frac{c_{r,2}}{2^2} + \ldots + \frac{c_{r,n_r-1}}{2^{n_r-1}} + \frac{1}{2^{n_r}},$$

where

$$\begin{aligned} c_{r,1} &= 0 \qquad \text{when } a_{r,1} = 0 \text{ or } 13, \\ &= 1 \qquad \text{when } a_{r,1} = 5 \text{ or } 8, \end{aligned}$$

and, for $1 < m < n_r$,

(a) in case $a_{r,1} = 0$ or 5, then $c_{r,m} = 0$ or 1 according as $a_{r,m} = 0$ or $2^m + 3$,

(b) in case $a_{r,1} = 8$ or 13, then $c_{r,m} = 1$ or 0 according as $a_{r,m} = 0$ or $2^m + 3$.

Let us first prove the following

LEMMA 1. *For any point x in $[0, 1]$ and for each $r = 1, 2, \ldots$, we have, in any representation of $f(x)$,*

$$0 \leqslant Y_{r,n_r} - \frac{1}{2^{n_r}}\left(Y_{r+1,n_{r+1}} - \frac{1}{2^{n_{r+1}}}\left(Y_{r+2,n_{r+2}} - \ldots\left(\ldots \leqslant 1.\right.\right.\right. \tag{3}$$

Proof. In case $n_r > 1$, we have

$$Y_{r,n_r} = \frac{c_{r,1}}{2} + \frac{c_{r,2}}{2^2} + \ldots + \frac{c_{r,n_r-1}}{2^{n_r-1}} + \frac{1}{2^{n_r}},$$

and as the coefficients $c_{r,m}$ $(m = 1, 2, \ldots, n_r - 1)$ take only the values 0 and 1, we clearly have

$$1/2^{n_r} \leqslant Y_{r,n_r} \leqslant 1 - 1/2^{n_r}. \tag{4}$$

This holds again in case $n_r = 1$, for then $Y_{r,1} = \frac{1}{2}$. Relation (4) thus holds for each value of r, whatever be the value of n_r. Repeated application of this relation gives

$$\begin{aligned} &Y_{r,n_r} - \frac{1}{2^{n_r}}\left(Y_{r+1,n_{r+1}} - \frac{1}{2^{n_{r+1}}}\left(Y_{r+2,n_{r+2}} - \ldots\left(\ldots\right.\right.\right. \\ &\quad \geqslant \frac{1}{2^{n_r}} - \frac{1}{2^{n_r}}\left(1 - \frac{1}{2^{n_{r+1}}} - \frac{1}{2^{n_{r+1}}}\left(\frac{1}{2^{n_{r+2}}} - \frac{1}{2^{n_{r+2}}}\left(1 - \frac{1}{2^{n_{r+3}}} - \ldots\left(\ldots\right.\right.\right.\right. \\ &\quad = \frac{1}{2^{n_r+n_{r+1}}}\left(1 + \frac{1}{2^{n_{r+2}+n_{r+3}}}\left(1 + \frac{1}{2^{n_{r+4}+n_{r+5}}}\left(1 + \ldots\left(\ldots\right.\right.\right.\right. \\ &\quad \geqslant 0, \end{aligned}$$

and similarly the other inequality of (3).

We next prove the following

LEMMA 2. *The knot points of Besicovitch's function form a set of measure zero.*

Proof. Let S denote the set of points x of $[0, 1]$ in the representation (1) of which there occur only a finite number of brackets. It is known **(18, § 2)** that *mes* $S = 1$. Since the points with double representation are only enumerable, denoting by S' the set of those points of S which have a unique representation, we have $mes\, S' = 1$. It will, therefore, suffice to show that no point of S' is a knot point of $f(x)$.

A point $x \in S'$ has the representation

$$x = X_{1,n_1} + \frac{2}{4^{n_1+1}}\left(X_{2,n_2} + \ldots \left(\ldots + \frac{2}{4^{n_{r-1}+1}}(X_{r,\infty})\right)\ldots\right)$$

$$= X_{1,n_1} + \frac{2}{4^{n_1+1}}\left(X_{2,n_2} + \ldots \left(\ldots + \frac{2}{4^{n_{r-1}+1}}\left(\frac{a_{r,1}}{4^2} + \ldots + \frac{a_{r,m}}{4^{m+1}} + \ldots\right)\right)\ldots\right),$$

where $a_{r,m}$ takes each of the two values 0 and $2^m + 3$ for infinitely many values of m. $a_{r,1}$ being the first coefficient of the last bracket in the representation of x, let $x \in S_1$ if $a_{r,1} = 0$ or 5 and let $x \in S_2$ if $a_{r,1} = 8$ or 13. Then $S' = S_1 \cup S_2$, and we shall investigate the two sets separately.

Let us first assume that $x \in S_1$. Let $\{m_i\}$ be the infinite sequence of values of m for which $a_{r,m} = 2^m + 3$. Then for any point $x' < x$ which is sufficiently close to x there will exist a smallest index i such that

$$x' = X_{1,n_1} + \frac{2}{4^{n_1+1}}\left(X_{2,n_2} + \ldots \left(\ldots + \frac{2}{4^{n_{r-1}+1}}\left(\frac{a_{r,1}}{4^2} + \ldots + \right.\right.\right.$$
$$\left.\left.\left. + \frac{a_{r,m_i-1}}{4^{m_i}} + \frac{a'_{r,m_i}}{4^{m_i+1}} + \ldots\left(\ldots\right.\right.\right.\right.,$$

where $a'_{r,m_i} = 0$ or $2^{m_i} + 1$. Clearly,

$$f(x) - f(x') = \frac{(-1)^{r-1}}{2^{n_1+\ldots+n_{r-1}}}\left[\left\{\frac{1}{2^{m_i}} + \frac{c_{r,m_i+1}}{2^{m_i+1}} + \ldots\right\} - \left\{\frac{c'_{r,m_i}}{2^{m_i}} + \ldots\left(\ldots\right\}\right],$$

so that

$$\begin{aligned}
&(-1)^{r-1}2^{n_1+\cdots+n_{r-1}}\{f(x) - f(x')\}\\
&\quad > \frac{1}{2^{m_i}} - \left\{\frac{c'_{r,m_i}}{2^{m_i}} + \ldots\left(\ldots\right\}\right.\\
&\quad = \frac{1}{2^{m_i}} - \left\{\frac{0}{2^{m_i}} + \frac{c'_{r,m_i+1}}{2^{m_i+1}} + \ldots + \frac{c'_{r,m_i+\lambda}}{2^{m_i+\lambda}} + \ldots\right\}\\
&\quad \text{or } \frac{1}{2^{m_i}} - \left\{\frac{0}{2^{m_i}} + \ldots + \frac{1}{2^{n'_r}} - \frac{1}{2^{n'_r}}\left(Y'_{r+1,n'_{r+1}} - \ldots\left(\ldots\right\}\right.\right.\\
&\quad \text{or } \frac{1}{2^{m_i}} - \left\{\frac{1}{2^{m_i}} - \frac{1}{2^{m_i}}\left(Y'_{r+1,n'_{r+1}} - \ldots\left(\ldots\right\}\right.\right.\\
&\quad \geqslant \min\left[\left\{\frac{1}{2^{m_i}} - \left(\frac{1}{2^{m_i+1}} + \frac{1}{2^{m_i+2}} + \ldots + \frac{1}{2^{m_i+\lambda}} + \ldots\right)\right\},\right.\\
&\qquad \left\{\frac{1}{2^{m_i}} - \left(\frac{1}{2^{m_i+1}} + \ldots + \frac{1}{2^{n'_r}} - \frac{1}{2^{n'_r}}(0)\right)\right\},\\
&\qquad \left.\left\{\frac{1}{2^{m_i}}(0)\right\}\right], \quad \text{by Lemma 1,}\\
&\quad = 0.
\end{aligned}$$

Thus for every x' in some immediate left neighbourhood of x,

$$\frac{f(x) - f(x')}{x - x'} \leqslant 0 \quad \text{or} \quad \geqslant 0$$

according as r is even or odd. Hence, when r is even, $D_-f(x) < D^-f(x) \leqslant 0$, and when r is odd, $D^-f(x) > D_-f(x) \geqslant 0$, so that in either case x cannot be a knot point of $f(x)$.

A similar argument proves that at every point $x \in S_2$, either

$$D^+f(x) > D_+f(x) \geqslant 0, \qquad \text{or} \qquad D_+f(x) < D^+f(x) \leqslant 0,$$

and again x cannot be a knot point of $f(x)$. Lemma 2 is thus proved.

Now, as $f(x)$ is non-differentiable, it follows from Lemma 2 and the classical Denjoy theorem (**2**, p. 105) that except for a set of measure zero, at each point of [0, 1] the derivates of $f(x)$ satisfy one of the following two relations:

(i) $D^+f = D_-f \neq \infty, \qquad D_+f = -\infty, \qquad D^-f = +\infty,$
(ii) $D_+f = D^-f \neq \infty, \qquad D^+f = +\infty, \qquad D_-f = -\infty.$

As in neither case the derivates of $f(x)$ are symmetrical, it follows that $f(x)$ has asymmetrical derivates almost everywhere in [0, 1].

We have thus proved the following

THEOREM 1. *There exist non-differentiable functions which have asymmetrical derivates almost everywhere, the knot points of such a function forming a residual set of measure zero.*

3. General non-differentiable functions. Let $f(x)$ be a real function defined in an interval I and let, for every real number $r \geqslant 0$,

$$\begin{aligned}
S_{1r}[S_{2r}] &= \{x; D^+f = D_-f = \text{finite} > r\ [< -r],\ D_+f = -\infty,\ D^-f = +\infty\},\\
S_{3r}[S_{4r}] &= \{x; D_+f = D^-f = \text{finite} > r\ [< -r],\ D^+f = +\infty,\ D_-f = -\infty\},\\
S_{1\infty}[S_{2\infty}] &= \{x; D^+f = D_-f = +\infty\ [-\infty],\ D_+f = -\infty,\ D^-f = +\infty\},\\
S_{3\infty}[S_{4\infty}] &= \{x; D_+f = D^-f = +\infty\ [-\infty],\ D^+f = +\infty,\ D_-f = -\infty\}.
\end{aligned}$$

The following two results on non-differentiable functions were established by the author in the course of his earlier investigations on nowhere monotone functions (**4**, p. 666, Cor. 2; **5**, p. 86, Cor. 1), and the results of the present section are rather direct consequences of these results:

I. "If a continuous function $f(x)$ is non-differentiable in I, then each interval $J \subset I$ contains, for each of $i = 1$ to 4, either a subset of S_{i0} of positive measure, or a subset of $S_{i\infty}$ of power c, which is mapped by f into a set of positive interior measure."

II. "If a continuous function $f(x)$ is non-differentiable in I, then for every real number $r > 0$ and each of $i = 1$ to 4, each interval $J \subset I$ contains either a subset of S_{ir} of positive measure or a subset of $S_{i\infty}$ of power c."

For any $r \geqslant 0$ and for each of $i = 1$ to 4, at a point x belonging to any of the sets S_{ir} and $S_{i\infty}$ the derivates of $f(x)$ are asymmetrical and, moreover, at least one derivate of $f(x)$ is $+\infty$ whereas at least one derivate is $-\infty$. We therefore have, from I,

PROPOSITION 1. *If $f(x)$ is a non-differentiable function in I, there exists a set E which has the power of the continuum in every subinterval of I and is mapped by f into a set of positive interior measure, such that at each point of E the derivates of $f(x)$ are asymmetrical and two of them are $+\infty$ and $-\infty$ respectively.*

Remark. Thus for any non-differentiable function $f(x)$ the set A $(\supset E)$ has the power c in every interval. W. H. Young proved (**21**, Th. 3) that the derivates of $f(x)$ are unbounded from above and below at the points of A. The above proposition proves a stronger result, viz. that there exist points of A having power c in every interval where two of the four derivates of $f(x)$ actually assume the values $+\infty$ and $-\infty$. This property was observed by Young (**21**, p. 69) as a special characteristic of Weierstrass's function. We shall see in the corollary of Proposition 2 that this subset of A can even have a positive measure in certain cases.

Now let $f(x)$ be a continuous function which has nowhere a unilateral derivative, finite or infinite.[3] As in this case the sets $S_{i\infty}$ $(i = 1$ to $4)$ are all void, the results I and II give the following

PROPOSITION 2. *If $f(x)$ is a continuous function in I which has nowhere a unilateral derivative, then for every real number $r > 0$, each of the sets S_{ir} $(i = 1$ to $4)$ has a positive measure in every subinterval of I, and for $r = 0$ each of these sets is mapped by f into a set of positive interior measure.*

Remark. Singh concluded (**17**, p. 76) from his statement quoted in the introduction that the derivates of a non-differentiable function are almost everywhere infinite. This proposition proves the existence of non-differentiable functions for which any of the four derivates is finite at a set of points which has a positive measure in every interval. Besicovitch's function has, in fact, almost everywhere two of the four derivates finite.

Proposition 2 yields, as above,

COROLLARY. *For a non-differentiable function $f(x)$ of the Besicovitch type, there exists a set E which has a positive measure in every interval and is mapped by f into a set of positive interior measure, such that at each point of E the derivates of $f(x)$ are asymmetrical and two of them are $+\infty$ and $-\infty$ respectively.*

[3]Apart from Besicovitch's function (**1**), other examples of such functions have been given by A. P. Morse (**9**) and A. N. Singh (**19**). Whereas non-differentiable functions form a residual set in the space of continuous functions (see Saks (**13**, p. 211)), the non-differentiable functions of this type form only a set of the first category (see Saks (**13**, p. 212)).

Next, let $f(x)$ be a non-differentiable function which has almost everywhere a knot point.[4] As in this case the sets S_{i0} ($i = 1$ to 4) are all of measure zero, it follows from I that

PROPOSITION 3. *If a non-differentiable function $f(x)$ has knot points almost everywhere in I, then every interval $J \subset I$ contains a subset with power of the continuum of each of the sets $S_{i\infty}$ ($i = 1$ to 4) which is mapped by f into a set of positive interior measure.*

This at once yields

COROLLARY. *A non-differentiable function $f(x)$ of the Weierstrass type has progressive derivatives $+\infty$ and $-\infty$ at sets of points which have the power of the continuum in every interval, and also has regressive derivatives $+\infty$ and $-\infty$ at sets of points which have the power of the continuum in every interval.*[5]

Singh attempted to prove this result for general non-differentiable functions (**16**, p. 55, Th. II; **14**, p. 108, cor. of Th. III), which is obviously not true as can be seen from Besicovitch's function.

4. Weierstrass's function. The derivates of Weierstrass's non-differentiable function were studied in detail by G. C. Young in the year 1916. She proved (**20**, pp. 165, 167, 171) that

III. "The Weierstrass function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \qquad (0 \leqslant x \leqslant 1),$$

where $0 < a < 1$, b is an odd integer, and $ab > 1 + 3\pi/2$, has

(1) knot points almost everywhere in $[0, 1]$;

(2) cusps pointing upwards at an enumerable everywhere dense set in $[0, 1]$ and cusps pointing downwards at another enumerable everywhere dense set in $[0, 1]$;

(3) at the rest of the points on one side the upper derivate $+\infty$ and the lower derivate $-\infty$, while on the other side at least one derivate is infinite."

In this section we determine the unknown derivate of $W(x)$ at the points considered in III(3), and obtain thereby the distribution of every possible set of points where the derivates of $W(x)$ assume identical values.

According to III(1) and Theorem 1 of W. H. Young (**21**), the knot points of $W(x)$ form a residual set, say E_k, covering almost all the points of $[0, 1]$.

[4]The existence of such functions has long been known. The non-differentiable functions of Weierstrass (see G. C. Young (**20**, p. 171)), Dini (see M. B. Porter (**12**, pp. 178, 179)), and Denjoy (**2**, p. 222) constitute examples of such functions. In fact, the non-differentiable functions of this type form a residual set in the space of continuous functions (see V. Jarník (**7**, p. 49, Th. I)).

[5]Thus the Denjoy non-differentiable function, shown by him (**2**, pp. 222, 223) not to possess anywhere unilateral infinite derivatives on both sides, does possess a unilateral infinite derivative on one side at a set of points which has the power of the continuum in every interval.

Hence the complementary set $C(E_k)$, or any subset of it, is a set of the first category with its measure equal to zero.

It follows from Proposition 3 that each of the sets $S_{i\infty}$ $(i = 1 \text{ to } 4)$, with f replaced by W, has the power c in every subinterval of $[0, 1]$. These sets cover all possible points considered in III(3) where all of the four derivates of $W(x)$ are infinite.

At the remaining points of the open interval $(0, 1)$, on one side the two derivates are $+\infty$ and $-\infty$, and on the other side one derivate is infinite whereas the other one is finite. But, $W(x)$ being a continuous nowhere monotone function of the second species (**5**, p. 83), it follows from Theorem 4 of the author (**5**, p. 86) that each of the four derivates of $W(x)$ assumes any prescribed real value at a set of points which has the power of the continuum in every subinterval of $[0, 1]$.

Thus, if

$$
\begin{aligned}
E_k &= \{x; D^+W = D^-W = +\infty,\quad D_+W = D_-W = -\infty\},\\
E_{c1}\,[E_{c2}] &= \{x; W'_+ = +\infty\ [-\infty],\ W'_- = -\infty\ [+\infty]\},\\
E_1\,[E_2] &= \{x; W'_+ = +\infty\ [-\infty], D^-W = +\infty,\quad D_-W = -\infty\},\\
E_3\,[E_4] &= \{x; W'_- = +\infty\ [-\infty], D^+W = +\infty,\quad D_+W = -\infty\},
\end{aligned}
$$

and if, for every $r \in R \equiv$ the set of all real numbers,

$$
\begin{aligned}
E_{1r} &= \{x; D^+W = r, D_+W = -\infty,\quad D^-W = +\infty,\quad D_-W = -\infty\},\\
E_{2r} &= \{x; D^+W = +\infty,\quad D_+W = r, D^-W = +\infty,\quad D_-W = -\infty\},\\
E_{3r} &= \{x; D^+W = +\infty,\quad D_+W = -\infty,\quad D^-W = r,\quad D_-W = -\infty\},\\
E_{4r} &= \{x; D^+W = +\infty,\quad D_+W = -\infty,\quad D^-W = +\infty,\quad D_-W = r\},
\end{aligned}
$$

the result III of Young may be completed into the following form.

THEOREM 2. *For the Weierstrass function $W(x)$ the sets E_k, E_{ci} $(i = 1, 2)$, E_i $(i = 1$ to $4)$, and E_{ir} $(i = 1$ to 4, $r \in R)$ cover all the points of $(0, 1)$, and the points of these sets are distributed in the interval in the following manner:*

(i) *E_k is residual in $(0, 1)$ with its measure equal to 1,*

(ii) *E_{ci} $(i = 1, 2)$ are both enumerable and everywhere dense in $(0, 1)$, and*

(iii) *each of the sets E_i $(i = 1$ to $4)$ and E_{ir} $(i = 1$ to 4, $r \in R)$ is of the first category with its measure equal to zero and has the power of the continuum in every subinterval of $(0, 1)$.*

Remark. Let α and β be respectively the greatest lower and the least upper bounds of the values assumed by $W(x)$ in $[0, 1]$. As the sets E_k, E_{ci} $(i = 1, 2)$, E_i $(i = 1$ to $4)$, and E_{ir} $(i = 1$ to 4, $r \in R)$ are all everywhere dense in $[0, 1]$, the image of each of these is everywhere dense in $[\alpha, \beta]$. It can further be observed that

(i) the image set $W(E_k)$ covers almost all the points of $[\alpha, \beta]$ (see author's paper (**6**, p. 193, Cor. 6)),

(ii) the image sets $W(E_{c1})$ and $W(E_{c2})$ are both enumerable,

(iii) the image sets $W(E_i)$ ($i = 1$ to 4) have a positive interior measure in every subinterval of $[\alpha, \beta]$ (this follows from Proposition 3 on taking into account the fact that the properties of $W(x)$ hold as well in any subinterval of $[0, 1]$), and

(iv) the image sets $W(E_{ir})$ ($i = 1$ to 4, $r \in R$) are all of measure zero (see author's Denjoy analogue (**3**, p. 9)).

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