Canad. Math. Bull. Vol. **23** (3), 1980

GAPS BETWEEN SPHERES IN NORMED LINEAR SPACES

BY
ROBERT H. LOHMAN

ABSTRACT. The geometric notions of a gap and gap points between "concentric" spheres in a normed linear space are introduced and studied. The existence of gap points characterizes finite-dimensional spaces. General conditions are given under which an infinite-dimensional normed linear space admits concentric spheres such that both these spheres and their dual spheres fail to have gap points.

In this paper, we introduce the idea of a gap and gap point between two "concentric" spheres in a normed linear space X. It is shown how the existence of gap points in X is related to norm-preserving extensions (with respect to two equivalent norms) of certain continuous linear functionals on subspaces of X [Proposition 2]. In addition, we show how the existence of gap points in X implies the existence of gap points in the dual of X [Proposition 3] and in quotients of X [Proposition 5]. In the main results of the paper, finite-dimensional normed linear spaces are characterized in terms of existence of gap points [Proposition 7] and infinite-dimensional Banach spaces that are weakly compactly generated or have weakly compactly generated dual spaces are shown to admit concentric spheres such that both these spheres and their dual spheres fail to have gap points [Propositions 8, 9].

X denotes a real or complex vector space. If p is a norm on X, then $U(p)=\{x:p(x)\le 1\}$ and $S(p)=\{x:p(x)=1\}$. The dual of X is denoted by X^* and the dual norm by p^* (i.e., $p^*(f)=\sup\{|f(x)|:p(x)\le 1\}$). If $A\subset X$ (respectively, $B\subset X^*$), we let A^0 (respectively, B_0) denote $\{f\in X^*:|f(x)|\le 1$ for all $x\in A\}$ (respectively, $\{x\in X:|f(x)|\le 1$ for all $f\in B\}$). Recall that $U(p)^0=U(p^*)$ and $U(p^*)_0=U(p)$. If A is a convex, circled, absorbing subset of X, the gauge functional p_A of A is defined by $p_A(x)=\inf\{\lambda>0:x\in\lambda A\}$. A subset A of a normed linear space is a convex body provided A is a closed, convex, bounded set with nonempty interior. A Banach space X is said to be weakly compactly generated (WCG) in case there exists a weakly compact subset A of X such that the closed linear span of A is all of X. In particular, all separable or

Received by the editors May 31, 1978 and, in revised form, February 12, 1979.
AMS 1970 Subject Classification. 46B99
Key Words and Phrases: Gap, gap point, reflexive, finite-dimensional, weakly compactly generated, weak*-basic sequence.

reflexive Banach spaces are *WCG*. If X is a Banach space, a sequence $(x_n^*) \subset X^*$ is called a weak*-basic sequence in case: (i) Every element in the weak*-closed linear span of (x_n^*) has a unique expansion of the form $\sum_{n=1}^{\infty} \alpha_n x_n^*$ relative to the weak* topology, and (ii) the coefficient functionals of (x_n^*) are weak*-continuous. Fundamental results on weak*-basic sequences can be found in [5].

Definition 1. Let p and q be equivalent norms on X such that $q \le p$. The number $g(p, q) - 1$, where

$$g(p, q) = \sup\{p(x) : q(x) = 1\},$$

is called the gap between the spheres $S(p)$ and $S(q)$. If $q(x_0) = 1$ and $p(x_0) = g(p, q)$, then x_0 is called a gap point for (p, q).

Of course, if $U(q)$ is a multiple of $U(p)$, every member of $S(q)$ is a gap point. Also, if X is a finite-dimensional vector space, then $S(q)$ is compact for the p-topology. Consequently, a gap point exists. The following examples give other conditions under which gap points exist:

(i) Let V be a convex, circled subset of the normed space (X, p) such that V contains a point v_0 of maximum p-norm; for instance, V might be compact. If q denotes the gauge functional of $\overline{U(p) + V}$, then $(1 + p(v_0)^{-1})v_0$ is a gap point for (p, q).

(ii) Given a normed space (X, q), let f be a continuous linear functional that attains its norm on $U(q)$, say at x_0. If the norm p is defined by $p(x) = q(x) + |f(x)|$, then x_0 is a gap point for (p, q).

(iii) Let (X, q) be a Banach algebra with identity and let x be an invertible element of X such that $q(x^{-1}) \le 1 \le q(x)$. If q_x denotes the norm whose unit ball is $xU(q)$, then x is a gap point for (q, q_x).

Remark. If p, q are as in Definition 1, it is easy to see that

$$g(p, q) = \sup\{p(x) : q(x) \le 1\} = \inf\{\lambda > 0 : U(q) \subset \lambda U(p)\},$$

and $U(q) \subset g(p, q)U(p)$.

Proposition 2. *Let p and q be equivalent norms on X such that $q \le p$. Let x_0 be a gap point for (p, q) and let Y be a linear subspace of X containing x_0. If f is a continuous linear functional on Y that attains its p-norm at $p(x_0)^{-1}x_0$, then*

(a) *f attains its q-norm at x_0 and $q^*(f) = g(p, q)p^*(f)$;*

(b) *Every linear extension of f to X that preserves its p-norm also preserves its q-norm.*

Proof. (a). Let $q(x) = 1$, $x \in Y$. Then

$$|f(x)| \le p(x)p^*(f) \le g(p, q)p^*(f),$$

implying $q^*(f) \le g(p, q)p^*(f)$. On the other hand,

$$q^*(f) \ge |f(x_0)| = g(p, q)\,|f(p(x_0)^{-1}x_0| = g(p, q)p^*(f).$$

Therefore

$$|f(x_0)| = q^*(f) = g(p, q)p^*(f).$$

(b). Let h be a linear extension of f to X such that $p^*(h) = p^*(f)$. Then h attains its p-norm at $p(x_0)^{-1}x_0$. By (a),

$$q^*(h) = g(p, q)p^*(h) = g(p, q)p^*(f) = q^*(f).$$

REMARK. If p and q are equivalent norms on X and $q \le p$, then q^* and p^* are equivalent norms on X^* and $p^* \le q^*$. Moreover,

$$\begin{aligned} g(q^*, p^*) &= \inf\{\lambda > 0 : U(p)^0 \subset \lambda U(q)^0\} \\ &= \inf\{\lambda > 0 : U(p)^0 \subset (\lambda^{-1}U(q))^0\} \\ &= \inf\{\lambda > 0 : \lambda^{-1}U(q) \subset U(p)\} \\ &= g(p, q). \end{aligned}$$

Therefore the gap between $S(q^*)$ and $S(p^*)$ is the same as the gap between $S(p)$ and $S(p)$.

PROPOSITION 3. *Let p and q be equivalent norms on X with $q \le p$. If $S(q)$ contains a gap point for (p, q), then $S(p^*)$ contains a gap point for (q^*, p^*).*

Proof. Let $q(x_0) = 1$ and $p(x_0) = g(p, q)$. Choose $f_0 \in X^*$ such that $p^*(f_0) = 1$ and $f_0(x_0) = p(x_0)$. Then we certainly have $q^*(f_0) \ge g(p, q)$. On the other hand, we must have $q^*(f_0) \le g(q^*, p^*)$. The preceding remark completes the proof.

REMARK. In general, the converse of Proposition 3 fails. For example, let $X = c_0$, let p denote the usual supremum norm on X, and let q denote the norm whose unit ball is

$$U(q) = \{(x_n) : |x_n| \le 2 - 1/n\}.$$

It is easily checked that $S(q)$ (respectivley, $S(p^*)$) does not contain a gap point for (p, q) (respectively, (q^*, p^*)). However, since

$$U(p^{**}) = \{(x_n) \in \ell_\infty : |x_n| \le 1\},$$

and

$$U(q^{**}) = \{(x_n) \in \ell_\infty : |x_n| \le 2 - 1/n\},$$

it is clear that the sequence $(2 - 1/n) \in U(q^{**})$ is a gap point for (p^{**}, q^{**}).

If X is reflexive, Proposition 3 yields the following result.

COROLLARY 4. *Let (X, p) be a reflexive Banach space and let q be an equivalent norm on X such that $q \le p$. Then $S(q)$ contains a gap point for (p, q) if and only if $S(p^*)$ contains a gap point for (q^*, p^*).*

Before proceeding, let us recall a notion of orthogonality introduced in [1] (also, see [3]). Let (X, p) be a normed space, Y a linear subspace of X and $x_0 \in X$. We say x_0 is orthogonal to Y relative to p in case

$$p(x_0+y) \geq p(x_0), \quad \text{for all} \quad y \in Y.$$

Now let p and q be equivalent norms on X such that $q \leq p$. If Y is a closed linear subspace of X and $\pi: X \to X/Y$ denotes the quotient mapping, let p_π, q_π denote the quotient norms on X/Y determined by p and q, respectively. Then p_π and q_π are equivalent norms on X/Y such that $q_\pi \leq p_\pi$.

PROPOSITION 5. *Let p and q be equivalent norms on X such that $q \leq p$. Let x_0 be a gap point for (p, q) and let Y be a closed linear subspace of X such that x_0 is orthogonal to Y relative to p. Then*

(a) *x_0 is orthogonal to Y relative to q;*
(b) $g(p, q) = g(p_\pi, q_\pi)$;
(c) *πx_0 is a gap point for (p_π, q_π).*

Proof. (a). Suppose there exists $y \in Y$ such that $q(x_0+y) < q(x_0) = 1$. Let $z = q(x_0+y)^{-1}(x_0+y)$. Then $q(z) = 1$ and

$$p(z) \geq q(x_0+y)^{-1}p(x_0) = q(x_0+y)^{-1}g(p, q) > g(p, q),$$

a contradiction.

(b). Recall that π maps the interior of $U(p)$ onto the interior of $U(p_\pi)$. Therefore, if $\lambda > 0$ and $U(q) \subset \lambda U(p)$, then $U(q_\pi) \subset \lambda U(p_\pi)$. Taking the infimum over all such λ shows $g(p_\pi, q_\pi) \leq g(p, q)$. On the other hand, since x_0 is orthogonal to Y relative to both p and q, $q_\pi(\pi x_0) = q(x_0) = 1$ and $p_\pi(\pi x_0) = p(x_0) = g(p, q)$. It follows that $g(p_\pi, q_\pi) = g(p, q)$. At the same time we have also shown πx_0 is a gap point for (p_π, q_π), thus establishing (c).

We can now sharpen Proposition 3 as follows.

COROLLARY 6. *Let p and q be equivalent norms on X such that $q \leq p$. Let x_0 be a gap point for (p, q) and let Y be a closed linear subspace of X such that x_0 is orthogonal to Y relative to p. Then a gap point exists in $Y^\perp$ for (q^*, p^*).*

Proof. By Proposition 5, $\pi(x_0)$ is a gap point for (p_π, q_π) in X/Y. By Proposition 3, a gap point exists in $(X/Y)^*$ for (q_π^*, p_π^*). Using the standard identification $(X/Y, p_\pi)^* = (Y^\perp, p^*)$, the result follows.

Our next result characterizes finite-dimensional spaces in terms of the existence of gap points and operators attaining their norm.

PROPOSITION 7. *Let (X, q) be a normed linear space. the following are equivalent:*

(i) *X is of finite dimension;*
(ii) *For every equivalent norm p such that $q \leq p$, a gap point exists for (p, q);*

(iii) *Every bounded linear operator* $T: X \to c_0$ *attains its* q*-norm;*

(iv) *For every equivalent norm* r *such that* $r \le q$, *a gap point exists for* (q, r).

Proof. (i) ⇒ (ii) and (i) ⇒ (iv) are clear.

(ii) ⇒ (iii). Let $T: X \to c_0$ be a bounded linear operator. Define $p(x) = q(x) + \|T(x)\|$ for $x \in X$. Then p is an equivalent norm on X such that $q \le p$. If x_0 is a gap point for (p, q), T attains its q-norm at x_0.

(iii) ⇒ (i). Assume that every bounded linear operator $T: X \to c_0$ attains its q-norm. If X is infinite-dimensional, then by [6] there is a normalized sequence (f_n) in X^* such that $f_n \to 0$ pointwise on X. Let (α_n) be a sequence of positive scalars strictly increasing to 1. Define $T: X \to c_0$ by $T(x) = (\alpha_n f_n(x))$ for all $x \in X$. Then T is a linear operator, T has norm one, yet T does not attain its q-norm. The contradiction shows that X is of finite dimension.

(iv) ⇒ (i). If X is infinite-dimensional, X contains a normalized basic sequence (x_n). Let r denote the norm on X whose unit ball is the closed, convex, circled hull of the set

$$U(q) \cup \{(2 - n^{-1})x_n : n = 1, 2, \ldots\}.$$

It is clear that r is an equivalent norm, $r \le q$ and $g(q, r) = 2$. By hypothesis, there exists $x_0 \in X$ such that $r(x_0) = 1$ and $q(x_0) = 2$.

By the definition of $U(r)$, there is a sequence (u_n) in $U(q)$ and finitely nonzero scalar sequences $(\alpha_{kn})_{n=1}^{\infty}$, $(\beta_{kn})_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} (|\alpha_{kn})| + |\beta_{kn}|) \leq 1$, such that the sequence

$$v_k = \sum_{n=1}^{\infty} \alpha_{kn} u_n + \sum_{n=1}^{\infty} \beta_{kn}(2 - n^{-1})x_n, \quad k = 1, 2, \ldots,$$

converges to x_0. For each k, $q(v_k) \le 2 - \sum_{n=1}^{\infty} |\alpha_{kn}|$ and if m is fixed,

$$\begin{aligned} q(v_k) &\le \sum_{n=1}^{\infty} |\alpha_{kn}| + 2 \sum_{n \ne m} |\beta_{kn}| + |\beta_{km}| (2 - m^{-1}) \\ &\le 1 + \sum_{n=1}^{\infty} |\beta_{kn}| - |\beta_{km}| \, m^{-1} \\ &\le 2 - |\beta_{km}| \, m^{-1}. \end{aligned}$$

Since $q(v_k) \to 2$, it follows that $\lim_k \sum_{n=1}^{\infty} |\alpha_{kn}| = 0$. Therefore, x_0 is in the closed linear span of (x_n), say $x_0 = \sum_{n=1}^{\infty} \delta_n x_n$. The preceding inequality also shows that for each n, $\lim_k \beta_{kn} = 0$. Therefore $\delta_n = 0$ for all n. The contradiction shows that X is of finite dimension.

Remarks. (1) If X is a reflexive Banach space, then every bounded linear operator $T: X \to \ell_1$ attains its norm. Therefore, in statement (iii) of Proposition 7, c_0 cannot be replaced by an unspecified, but fixed, Banach space.

(2) An immediate consequence of Proposition 7 and Corollary 4 is the fact that if (X, q) is an infinite-dimensional, reflexive Banach space, then there are equivalent norms p, r on X with $r \le q \le p$ such that gap points do not exist for (p, q), (q, r), (q^*, p^*), (r^*, q^*). We do not know if this result is true for all infinite-dimensional Banach spaces. The fact, however, that both X and X^* are *WCG* if X is reflexive suggests the preceding observation might be extendable to the case when X and X^* are *WCG*. This is, in fact, true and it results from the content of the next two propositions. Their proofs depend upon properties of *WCG* spaces, weak*-basic sequences, and a modification of part of the proof of Proposition 7.

PROPOSITION 8. *If (X, q) is an infinite-dimensional WCG Banach space, then there is an equivalent norm p on X with $q \le p$ such that gap points do not exist for (p, q), (q^*, p^*).*

Proof. Let (x_n^*) be a normalized sequence in X^* such that $x_n^* \to 0$ weak*. Since X is *WCG*, we may assume by [4, p. 114] that (x_n^*) is a weak*-basic sequence. Let s denote the norm on X^* whose unit ball is the weak*-closed, convex, circled hull of the set

$$V = U(q^*) \cup \{2 - n^{-1})x_n^* : n = 1, 2, \ldots\}.$$

Then s is an equivalent norm, $s \le q^*$ and $g(q^*, s) = 2$. If a gap point $x_0^* \in X^*$ exists for (q^*, s), there is a net (v_γ^*) in V such that $v_\gamma^* \to x_0^*$ weak*. For each γ choose a sequence $(u_{\gamma n}^*)_{n=1}^\infty$ in $U(q^*)$ and finitely nonzero scalar sequences $(\alpha_{\gamma n})_{n=1}^\infty$, $(\beta_{\gamma n})_{n=1}^\infty$, as in Proposition 7, such that

$$v_\gamma^* = \sum_{n=1}^{\infty} \alpha_{\gamma n} u_{\gamma n}^* + \sum_{n=1}^{\infty} \beta_{\gamma n}(2 - n^{-1})x_n^*.$$

We again have $q^*(v_\gamma^*) \to 2$ and, arguing as in Proposition 7, we see that $\lim_\gamma \sum_{n=1}^\infty |\alpha_{\gamma n}| = 0$. Thus, x_0^* is in the weak*-closed linear span of (x_n^*). Since (x_n^*) is weak*-basic, there is a sequence (δ_n) of scalars such that $x_0^* = \sum_{n=1}^\infty \delta_n x_n^*$ relative to weak* topology. Because we still have $\lim_\gamma \beta_{\gamma n} = 0$ for each n, the fact that the coefficient functionals for (x_n^*) are weak*-continuous implies $\delta_n = 0$ for all n. The contradiction shows that no gap points exist for (q^*, s). Let p denote the norm on X whose unit ball is $U(s)_0$. Since, $U(s)$ is weak*-closed, $p^* = s$. By Proposition 3, no gap points exist for (p, q).

PROPOSITION 9. *If (X, q) is an infinite-dimensional Banach space such that X^* is WCG, then there is an equivalent norm r on X with $r \le q$ such that gap points do not exist for (q, r), (r^*, q^*).*

Proof. Every quotient of X^* is *WCG*. Sine ℓ_∞ is not *WCG*, X does not contain an isomorphic copy of ℓ_1. By [7], there is a normalized sequence (x_n) in X such that $x_n \to 0$ weakly. Let $j : X \to X^{**}$ denote the canonical imbedding and let $j(x_n)$ play the role of x_n^* in proof of Proposition 8. By that proof, there is a norm t on X^{**} such that t is equivalent to q^{**}, $t \le q^{**}$, no gap points exist for (q^{**}, t), and $U(t)$ is weak*-closed, convex, circled hull of the set $U(q^{**}) \cup \{(2-n^{-1})j(x_n): n = 1, 2, \ldots\}$. Define $r(x) = t(j(x))$ for all $x \in X$. Then r is a norm on X, equivalent to q, with $r \ge q$. By Goldstine's theorem [2, p. 424], we have

$$U(r^{**}) = U(r^*)^0 = U(r)^{00} = (j(X) \cap U(t))_0{}^0 = U(t).$$

Consequently, $r^{**} = t$. By Proposition 3, gap points do not exist for (q, r) or (r^*, q^*).

If X is a reflexive Banach space, and Y is a closed linear subspace of X of finite codimension, then every convex body in X contains a point that is farthest from Y. As another application of Proposition 7, we prove the converse of this statement. More precisely, we have

Corollary 10. *Let (X, p) be a normed space and let Y be a closed linear subspace of X. If every circled convex body A in X contains a point x_A such that*

$$d(x_A, Y) = \sup\{d(x, Y): x \in A\},$$

then Y is of finite codimension in X.

Proof. Let $\pi: X \to X/Y$ denote the quotient mapping and let q be the quotient norm determined by p. Suppose r is a norm on X/Y that is equivalent to q and $r \le q$. Since $U(r)$ is a bounded subset of X/Y, there is a constant $M > 0$ such that $\pi(MU(p)) \supset U(r)$. Let

$$A = \pi^{-1}(U(r)) \cap MU(p),$$

and choose $x_A \in A$, as in the hypotheses. Then $\pi x_A \in U(r)$ and

$$q(\pi x_A) = d(x_A, Y) = \sup\{d(x, Y): x \in A\} = \sup\{q(\hat{x}): \hat{x} \in U(r)\}.$$

Therefore πx_A is a gap point for (q, r). By Proposition 7, X/Y is of finite dimension.

Remarks. Let p and q be equivalent norms on X such that $q \le p$. The number $g(p, q) - 1$ that we have discussed here is a measure of the largest gap between the spheres $S(p)$ and $S(q)$. Note, too, that $g(p, q) - 1$ is the usual distance in the Hausdorff metric between $U(p)$ and $U(q)$, measured using the norm p. The corresponding gap points are "wide" gap points. In a similar manner, if we let

$$s(p, q) = \inf\{p(x): q(x) = 1\},$$

then $s(p, q) - 1$ is a measure of the smallest gap between the spheres $S(p)$ and $S(q)$ and it then is natural to define a "narrow" gap point as a vector x such that $q(x) = 1$ and $p(x) = s(p, q)$. If $c > 0$ is chosen so that $cp \le q$, then a wide (respectively, narrow) gap point exists for (p, q) if and only if a narrow (respectively, wide) gap point exists for (q, cp). Using the results presented here, narrow gap analogues of Propositions 2, 3, 7, 8, 9 and Corollary 4 are easy to state and prove. For instance, Proposition 8 becomes

PROPOSITION 8′. *If (X, q) is an infinite-dimensional WCG Banach space, then there is an equivalent norm r on X with $r \leq q$ such that narrow gap points do not exist for (q, r), (r^*, q^*).*

REFERENCES

1. G. Birkoff, *Orothogonality in linear metric spaces*, Duke Math. J., **1** (1935), 169–172.
2. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
3. R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J., **12** (1945), 291–302.
4. W. B. Johnson, *On quasi-complements*, Pacific J. Math., **48** (1973), 113–118.
5. W. B. Johnson, and H. P. Rosenthal, *On w^*-basic sequences and their applications to the study of Banach spaces*, Studia Math., **43** (1972), 77–92.
6. A. Nissenzweig, *w^* sequential convergence*, Israel J. Math., **22** (1975), 266–272.
7. H. P. Rosenthal, *A characterization of Banach spaces containing ℓ_1*, Proc. Nat. Acad. Sci. (U.S.A.), **71** (1974), 2411–2413.

DEPARTMENT OF MATHEMATICS
KENT STATE UNIVERSITY
KENT, OHIO 44242