Proceedings of the Edinburgh Mathematical Society (2005) **48**, 231–240 ©
DOI:10.1017/S0013091503000737 Printed in the United Kingdom

RADIAL AND TANGENTIAL GROWTH OF CLOSE-TO-CONVEX FUNCTIONS

J. B. TWOMEY

Department of Mathematics, University College Cork, Cork, Ireland (twomeyjb@ucc.ie)

(Received 18 August 2003)

Abstract Some results are presented relating to questions raised in a recent paper by Anderson, Hayman and Pommerenke regarding the size of the set of boundary points of the unit disc at which a univalent function has a prescribed radial growth.

Keywords: radial growth; univalent functions; close-to-convex functions; exceptional sets

2000 *Mathematics subject classification:* Primary 30C55

1. Introduction

Let $\mathcal{S}$ denote the class of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. A familiar distortion theorem [**5**, p. 4] gives the sharp estimates

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2}, \quad |z| = r < 1, \tag{1.2}$$

for such functions, and a theorem of Spencer implies that

$$\liminf_{r\to 1} \frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{\log(1/(1-r))} > 0 \tag{1.3}$$

for at most countably many values of θ [**5**, p. 42]. In a recent paper, Anderson, Hayman and Pommerenke [**1**] considered the set

$$S(f,\psi) = \left\{\theta \in [-\pi,\pi] : \limsup_{r\to 1} \frac{|f(r\mathrm{e}^{\mathrm{i}\theta})|}{\psi(r)} > 0\right\},$$

where ψ is a continuous increasing function on $[0,1)$ for which $\psi(r) \to \infty$ as $r \to 1$ and

$$\liminf_{r\to 1} (1-r)^2 \psi(r) = 0,$$

and showed that there exists a function $f \in \mathcal{S}$ for which $S(f, \psi)$ is residual, and hence uncountably dense in every interval. (This result gives a strongly negative answer to a question of Makarov as to whether the set of θ for which (1.3) holds with '$\liminf$' replaced by '$\limsup$' is also countable for functions in $\mathcal{S}$.) Anderson *et al.* showed further that if ψ satisfies

$$\liminf_{r \to 1} \frac{\log \psi(r)}{\log(1/(1-r))} = 0,$$

then there exists a starlike function g for which $S(g, \psi)$ is residual. A function g is *starlike* if $g \in \mathcal{S}$ and $g(U)$ contains the line segment $[0, w]$ whenever it contains w [**7**, § 2.2]. As noted in [**1**], if $f \in \mathcal{S}$, then a classical theorem of Beurling [**3**, p. 56] implies that

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{r \to 1} f(r\mathrm{e}^{\mathrm{i}\theta})$$

exists as a finite limit outside a set of θ of logarithmic capacity zero, so $S(f, \psi)$ has logarithmic capacity zero for every $f \in \mathcal{S}$. The question is raised as to how the size of $S(f, \psi)$ depends on ψ, and whether, for instance, the size of $S(f, \psi)$ can be measured in terms of some generalized capacity. It is this question which we address here, although our focus is on the growth of $\log |f(z)|$ rather than $|f(z)|$, and we provide some answers for univalent functions which are close-to-convex. Recall that a function f analytic in U, and of the form (1.1), is *close-to-convex* [**7**, § 2.3] if there is a starlike function g such that

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in U. \tag{1.4}$$

This subclass of $\mathcal{S}$ contains the starlike functions and also, for example, contains functions in $\mathcal{S}$ that are *convex in one direction*, that is, functions f for which the intersection of $f(U)$ with each line of a fixed direction is connected or empty. The results we present deal with the radial growth of close-to-convex functions at $\mathrm{e}^{\mathrm{i}\theta} \in \partial U$ and also, more generally, with the growth of such functions as $z \to \mathrm{e}^{\mathrm{i}\theta}$ within certain regions that make tangential contact with ∂U at $\mathrm{e}^{\mathrm{i}\theta}$.

2. Statement of results

To state our results we need the classical notion of *capacity* [**9**, p. 194]. Let K be a kernel, that is, K is a non-negative, even and integrable function on $(-\pi, \pi)$, which is decreasing and unbounded on $(0, \pi)$ and is extended to $\mathbb{R}$ by periodicity. A Borel set $E \subset [-\pi, \pi]$ is said to have positive K-capacity if there exists a positive measure μ of total mass 1 and supported on E for which

$$\sup_{\theta} \int_{-\pi}^{\pi} K(\theta - t)\, \mathrm{d}\mu(t) < \infty.$$

Otherwise E is said to have zero K-capacity. When $K(t) = (\log |t|^{-1})^{\alpha}$, $0 < |t| < \pi$, $0 < \alpha \leqslant 1$, we call the associated capacity '$\log_{\alpha}$-capacity', and note that the case $\alpha = 1$ corresponds to logarithmic capacity. It is clear that if a set E has zero $\log_{\alpha}$-capacity, where $0 < \alpha \leqslant 1$, then E also has zero $\log_{\beta}$-capacity for every $\beta \in (0, \alpha)$.

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

We also need the notion of a *tangential approach region*. Let λ be a decreasing, continuous function on $[0,1]$ with $\lambda(1)=0$, and, for $\theta\in[-\pi,\pi]$, set

$$\Omega_\lambda(\theta)=\{z\in U:|\arg z-\theta|\leqslant\lambda(r)\}.$$

If $\lambda(r)=c(1-r)$ here, c any constant, we have angular regions; if $\lambda(r)/(1-r)\to\infty$ as $r\to 1$, then Ω_λ makes tangential contact with ∂U at $\mathrm{e}^{\mathrm{i}\theta}$.

Theorem 2.1. *Suppose that f is close-to-convex in U. Let $\varphi:[0,1)\to\boldsymbol{R}$ be a positive, continuous function such that both $\varphi(r)$ and $\log(2/(1-r))/\varphi(r)$ increase to ∞ on $[0,1)$. Suppose also that λ and Ω_λ are defined as above and that the continuous, increasing function Φ defined on $[1,\infty)$ by*

$$\Phi\left(\frac{1}{1-r}\right)=\log\left(\frac{2}{1-r}\right)\Big/\varphi(r),\quad 0\leqslant r<1,$$

satisfies the condition

$$\Phi\left(\frac{1}{1-r}\right)=O\left(\Phi\left(\frac{1}{\lambda(r)}\right)\right),\quad r\to 1. \tag{2.1}$$

Set

$$E(f,\varphi,\lambda)=\left\{\theta\in[-\pi,\pi]:\limsup_{z\to\mathrm{e}^{\mathrm{i}\theta}}\frac{\log|f(z)|}{\varphi(r)}>0,\ z\in\Omega_\lambda(\theta)\right\}.$$

Then $E(f,\varphi,\lambda)$ has K_Φ-capacity zero, where

$$K_\Phi(x)=\Phi\left(\frac{1}{|\sin\frac{1}{2}x|}\right),\quad 0<|x|<\pi.$$

If we set $\varphi(r)=(\log(2/(1-r)))^\alpha$, $\alpha\in(0,1)$, so that $\Phi(r)=(\log(2/(1-r)))^{1-\alpha}$, then we can take $\lambda(r)=(1-r)^\gamma$, where γ is any fixed number in $(0,1)$, and we obtain the following special case of Theorem 2.1.

Let f be close-to-convex, let $\alpha\in(0,1)$, and set

$$\Delta_\gamma(\theta)=\{z\in U:|\arg z-\theta|\leqslant(1-r)^\gamma\}$$

for $0<\gamma<1$. Then, for every γ in $(0,1)$,

$$\log|f(z)|=o(\log(1/(1-r)))^\alpha,\quad z\to\mathrm{e}^{\mathrm{i}\theta},\ z\in\Delta_\gamma(\theta),$$

for all $\theta\in[-\pi,\pi]$, except possibly for a set of θ of $\log_{1-\alpha}$-capacity zero. In particular, we have the radial result

$$\log|f(r\mathrm{e}^{\mathrm{i}\theta})|=o(\log(1/(1-r)))^\alpha,\quad r\to 1, \tag{2.2}$$

outside a set of θ of zero $\log_{1-\alpha}$-capacity.

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

We remark that it is known (see [**8**] and also [**6**]) that, if f is univalent, then $f(z) \to f(\mathrm{e}^{\mathrm{i}\theta})$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$ inside the approach regions $\Delta_\gamma(\theta)$, for every $\gamma \in (0,1)$, outside a set of θ of zero logarithmic capacity. This is a strengthening of the result of Beurling referred to in § 1.

The question arises as to the sharpness of the conclusions of Theorem 2.1. In this context we prove a partial result which shows that we cannot replace α in (2.2) by any smaller positive constant, even for starlike functions.

Theorem 2.2. *Suppose that $\alpha \in (0,1)$ and that $0 < \beta < \alpha$. Then there is a set $F = F(\alpha,\beta) \subset [-\pi,\pi]$ of positive $\log_{1-\alpha}$-capacity and a starlike function h such that*

$$\log|h(r\mathrm{e}^{\mathrm{i}\theta})| \neq o((\log(1/(1-r)))^{\beta}), \quad r \to 1, \tag{2.3}$$

for each $\theta \in F$.

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on a number of lemmas.

Lemma 3.1. *Suppose that K is a kernel and that F is an increasing function on $\boldsymbol{R}$ for which $F(x+2\pi) - F(x) = 2\pi$, $x \in \boldsymbol{R}$. If S_K denotes the set of $t \in [-\pi,\pi]$ for which*

$$I_K(t) = \int_{-\pi}^{\pi} K(t-x)\,\mathrm{d}F(x) < \infty, \tag{3.1}$$

then $[-\pi,\pi] \setminus S_K$ has zero K-capacity.

Proof of Lemma 3.1 (cf. the proof of Lemma 1 in [8]). Assume the result is false and that $S = [-\pi,\pi] \setminus S_K$ has positive K-capacity. Then, by the definition of K-capacity, there is a positive measure μ, supported on S, for which

$$\sup_x \int_{-\pi}^{\pi} K(x-t)\,\mathrm{d}\mu(t) < \infty.$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}\int_S I_K(t)\,\mathrm{d}\mu(t) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K(x-t)\,\mathrm{d}F(x)\,\mathrm{d}\mu(t)\\ &= \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} K(x-t)\,\mathrm{d}\mu(t)\right)\mathrm{d}F(x) < \infty,\end{aligned}$$

which is impossible since $I_K(t) = \infty$ on S and μ is supported on S. Consequently, the set of $t \in [-\pi,\pi]$ for which (3.1) fails to hold has zero K-capacity. □

Lemma 3.2. *Let the analytic function*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in U,$$

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

have positive real part in U. Set

$$P(z)=\int_0^z \frac{p(\zeta)-1}{\zeta}\,\mathrm{d}\zeta,\quad z\in U,$$

and

$$P^*(r\mathrm{e}^{\mathrm{i}\theta})\equiv \sup_{0\leqslant\rho\leqslant r}|P(\rho\mathrm{e}^{\mathrm{i}\theta})|$$

for $0<r<1$ and $\theta\in[-\pi,\pi]$. Then, with φ, Φ, K_Φ, λ and Ω_λ defined as in Theorem 2.1,

$$P^*(r\mathrm{e}^{\mathrm{i}\theta})=o(\varphi(r)),\quad z\to\mathrm{e}^{\mathrm{i}\theta},\ z\in\Omega_\lambda(\theta),\tag{3.2}$$

where $r=|z|$, for all $\theta\in[-\pi,\pi]$ except possibly for a set of θ of K_Φ-capacity zero.

Proof of Lemma 3.2. By a standard representation formula [**7**, p. 40], we have

$$p(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+\mathrm{e}^{-\mathrm{i}x}z}{1-\mathrm{e}^{-\mathrm{i}x}z}\,\mathrm{d}F(x),\quad z\in U,$$

where F is an increasing function as defined in Lemma 3.1. Then, by a simple calculation,

$$P(z)=\int_0^z\frac{p(\zeta)-1}{\zeta}\,\mathrm{d}\zeta=\frac{1}{\pi}\int_{-\pi}^{\pi}\log\left(\frac{1}{1-\mathrm{e}^{-\mathrm{i}x}z}\right)\mathrm{d}F(x).$$

We note next, since $r/|1-r\mathrm{e}^{\mathrm{i}t}|^2$ is an increasing function of r in $(0,1)$ for each fixed t, that

$$\operatorname{Re}P(z)+\log r=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{r}{|1-\mathrm{e}^{-\mathrm{i}x}z|^2}\right)\mathrm{d}F(x)$$

increases with $r=|z|$ in $(0,1)$ for each fixed $\arg z$. As an easy consequence of this, and the boundedness of $\operatorname{Im}P(z)$ in U, we deduce that (3.2) holds if

$$\begin{aligned}u(z)&=\operatorname{Re}P(z)+2\log 2\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}\log\left(\frac{2}{|1-\mathrm{e}^{-\mathrm{i}x}z|}\right)\mathrm{d}F(x)=o(\varphi(r)),\quad r=|z|,\end{aligned}$$

as $z\to\mathrm{e}^{\mathrm{i}\theta}$, $z\in\Omega_\lambda(\theta)$.

We next let S_Φ denote the set of $t\in[-\pi,\pi]$ for which

$$I_\Phi(t)=\int_{-\pi}^{\pi}\Phi\left(\frac{1}{|\sin((t-x)/2)|}\right)\mathrm{d}F(x)=\int_{-\pi}^{\pi}K_\Phi(t-x)\,\mathrm{d}F(x)<\infty\tag{3.3}$$

and note that, by Lemma 3.1, $[-\pi,\pi]\setminus S_\Phi$ has zero K_Φ-capacity. We complete the proof of Lemma 3.2 by showing that if $I_\Phi(\theta)<\infty$, then

$$u(z)=o(\varphi(r))\quad\text{as } z\to\mathrm{e}^{\mathrm{i}\theta},\ z\in\Omega_\lambda(\theta).$$

Without loss of generality we take $\theta=0$, so that

$$\int_{-\pi}^{\pi}K_\Phi(x)\,\mathrm{d}F(x)<\infty,\tag{3.4}$$

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

and we need to show that

$$u(re^{i\theta_r}) = o(\varphi(r)) \quad \text{as } r \to 1,\ |\theta_r| \leqslant \lambda(r). \tag{3.5}$$

To this end, let $\varepsilon > 0$ be given and choose $\delta \in (0,1)$ such that

$$\int_{|x|\leqslant\delta} K_\Phi(x)\,dF(x) \leqslant \varepsilon. \tag{3.6}$$

We also choose $r_0 \in (0,1)$ such that $2|\theta_r| \leqslant \delta$ for $r_0 \leqslant r < 1$, and, for such r, we write

$$\begin{aligned}\pi u(re^{i\theta_r}) &= \int_{-\pi}^{\pi} \log\left(\frac{2}{|1-e^{i(\theta_r-x)}|}\right) dF(x)\\ &= \int_{|x|\geqslant\delta} + \int_{2|\theta_r|\leqslant|x|\leqslant\delta} + \int_{|x|\leqslant 2|\theta_r|} = I_1 + I_2 + I_3.\end{aligned}$$

First, if $|x| \geqslant \delta$, then $|x-\theta_r| \geqslant \frac{1}{2}\delta$, so

$$I_1 \leqslant A\log\left(\frac{1}{\delta}\right), \tag{3.7}$$

where (here and below) A is an absolute constant. Next we write

$$G = G(x,r,\delta) = \{x : 2|\theta_r| \leqslant |x| \leqslant \delta,\ |1-re^{i(\theta_r-x)}| \leqslant 1\}.$$

Then

$$\begin{aligned} I_2 &\leqslant A + \int_G \frac{\log(2/(|1-re^{i(\theta_r-x)}|))}{\Phi(1/|1-re^{i(\theta_r-x)}|)}\Phi\left(\frac{1}{|1-re^{i(\theta_r-x)}|}\right) dF(x)\\ &\leqslant A + \varphi(r)\int_G \Phi\left(\frac{1}{|1-re^{i(\theta_r-x)}|}\right) dF(x)\\ &\leqslant A + \varphi(r)\int_{|x|\leqslant\delta} \Phi\left(\frac{1}{\sin(|x|/2|)}\right) dF(x) \leqslant A + \varepsilon\varphi(r), \end{aligned} \tag{3.8}$$

where we have used the monotonicity of ϕ and Φ, the inequalities $|1-re^{i(\theta_r-x)}| \geqslant |\sin(\theta_r - x)| \geqslant \sin(|x|/2)$, and (3.6). Finally,

$$I_3 \leqslant \log\left(\frac{2}{1-r}\right)\int_{|x|\leqslant 2\lambda(r)} dF(x)$$

and, since

$$\Phi\left(\frac{1}{\lambda(r)}\right)\int_{|x|\leqslant 2\lambda(r)} dF(x) \leqslant \int_{|x|\leqslant 2\lambda(r)} \Phi\left(\frac{1}{|\sin x/2|}\right) dF(x) = o(1)$$

as $r \to 1$, by (3.4) and the monotonicity of Φ, it follows that

$$I_3 = o\left(\log\left(\frac{2}{1-r}\right)\Big/\Phi\left(\frac{1}{\lambda(r)}\right)\right) = o(\varphi(r)), \tag{3.9}$$

where we have used (2.1). Combining (3.7), (3.8) and (3.9) we obtain (3.5), and the proof of Lemma 3.2 is complete. □

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

Lemma 3.3 (cf. Theorem 1 in [1]). *Suppose that f is close-to-convex in U, so that, by (1.4), $zf'(z) = g(z)h(z)$, where g is starlike, h is analytic with positive real part in U, and $h(0) = 1$. Set*

$$H(z) = \int_0^z \frac{h(\zeta) - 1}{\zeta}\,\mathrm{d}\zeta, \quad z \in U,$$

and let H^* be defined analogously to P^* in Lemma 3.2. Then, for $r\mathrm{e}^{\mathrm{i}\theta} \in U$ and $r > \frac{1}{2}$,

$$|f(r\mathrm{e}^{\mathrm{i}\theta})| \leqslant A|g(r\mathrm{e}^{\mathrm{i}\theta})|[H^*(r\mathrm{e}^{\mathrm{i}\theta}) + 1]. \tag{3.10}$$

Proof of Lemma 3.3. For $z = r\mathrm{e}^{\mathrm{i}\theta} \in U$,

$$\begin{aligned} f(z) = \int_0^z f'(\zeta)\,\mathrm{d}\zeta &= \int_0^z \frac{g(\zeta)h(\zeta)}{\zeta}\,\mathrm{d}\zeta \\ &= \int_0^z g(\zeta)\frac{h(\zeta)-1}{\zeta}\,\mathrm{d}\zeta + \int_0^z \frac{g(\zeta)}{\zeta}\,\mathrm{d}\zeta. \end{aligned} \tag{3.11}$$

Note that, by (1.2), $|g(\zeta)/\zeta| \leqslant 4$ for $|\zeta| \leqslant \frac{1}{2}$, and so, for $|z| = r > \frac{1}{2}$,

$$\begin{aligned} \left|\int_0^z \frac{g(\zeta)}{\zeta}\,\mathrm{d}\zeta\right| &\leqslant \int_0^r \frac{|g(\rho\mathrm{e}^{\mathrm{i}\theta})|}{\rho}\,\mathrm{d}\rho \\ &\leqslant 2 + 2\int_{1/2}^r |g(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho \\ &\leqslant 2 + 2|g(r\mathrm{e}^{\mathrm{i}\theta})| \leqslant 18|g(r\mathrm{e}^{\mathrm{i}\theta})|, \end{aligned} \tag{3.12}$$

since $|g(\rho\mathrm{e}^{\mathrm{i}\theta})|$ increases with ρ for each fixed θ, as g is starlike, and, using (1.2) again, $|g(r\mathrm{e}^{\mathrm{i}\theta})| \geqslant \frac{1}{4}r \geqslant \frac{1}{8}$ for $r \geqslant \frac{1}{2}$. Next,

$$\begin{aligned} \left|\int_0^z g(\zeta)\frac{h(\zeta)-1}{\zeta}\,\mathrm{d}\zeta\right| &= \left|g(z)H(z) - \int_0^z H(\zeta)g'(\zeta)\,\mathrm{d}\zeta\right| \\ &\leqslant |g(z)H(z)| + \left|\int_0^r H(\rho\mathrm{e}^{\mathrm{i}\theta})g'(\rho\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\rho\right| \\ &\leqslant H^*(r\mathrm{e}^{\mathrm{i}\theta})\left[|g(r\mathrm{e}^{\mathrm{i}\theta})| + \int_0^r |g'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho\right] \\ &\leqslant 3H^*(r\mathrm{e}^{\mathrm{i}\theta})|g(r\mathrm{e}^{\mathrm{i}\theta})|, \end{aligned} \tag{3.13}$$

where we have used the fact [**4**] that

$$\int_0^r |g'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho \leqslant 2|g(r\mathrm{e}^{\mathrm{i}\theta})|$$

for starlike functions. Combining (3.11), (3.12) and (3.13), we obtain (3.10), and the proof of Lemma 3.3 is complete. □

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

The proof of Theorem 2.1 is now easy. If f is close-to-convex, then, in the notation of Lemma 3.3, we have

$$\log|f(re^{i\theta})| \leqslant \log A + \log|g(re^{i\theta})| + H^*(re^{i\theta})$$

for $r > \frac{1}{2}$, by (3.10). Next, since $zg'(z) = g(z)p(z)$, where $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in U,

$$P(z) = \int_0^z \frac{p(\zeta) - 1}{\zeta}\, d\zeta = \int_0^z \left(\frac{g'(\zeta)}{g(\zeta)} - \frac{1}{\zeta}\right) d\zeta = \log\left(\frac{g(z)}{z}\right),$$

and we thus have

$$\log|f(re^{i\theta})| \leqslant \log 2A + P^*(re^{i\theta}) + H^*(re^{i\theta})$$

for $r > \frac{1}{2}$. Applying Lemma 3.2 to P^* and H^*, it is clear, since the union of two sets of zero K_Φ-capacity is again of zero K_Φ-capacity, that

$$\log|f(re^{i\theta})| = o(\varphi(r)), \quad z \to e^{i\theta},\ z \in \Omega_\lambda(\theta),$$

for all $\theta \in [-\pi, \pi]$ except possibly for a set of θ of K_Φ-capacity zero. This completes the proof of Theorem 2.1. □

4. Examples

The examples we construct to prove Theorem 2.2 are similar to examples used by the author in [**8**].

We begin with the definition of a standard Cantor-type set. Let (δ_n) denote a decreasing sequence of positive numbers for which $2\pi = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_n > \cdots$ and $\delta_n \to 0$ as $n \to \infty$. Set $F_0 = [-\pi, \pi]$, and, for $n \geqslant 1$, let F_n be constructed so that F_n is the union of 2^n disjoint closed intervals, each of length $2^{-n}\delta_n$. Delete an open segment in the centre of each of these 2^n intervals, so that each of the remaining 2^{n+1} intervals has length $2^{-n-1}\delta_{n+1}$ and let F_{n+1} be the union of these 2^{n+1} intervals. We define

$$F = \bigcap_{n=0}^{\infty} F_n,$$

and note, by a result of Carleson [**2**, p. 31], that if

$$\sum_{n=1}^{\infty} \frac{(\log(2^n/\delta_n))^\gamma}{2^n} < \infty,$$

then F has positive $\log_\gamma$-capacity, where $0 \leqslant \gamma < 1$.

Suppose next that $0 < \beta < \alpha < 1$, set $\eta = \frac{1}{4}(\alpha - \beta)$, and let (r_n) denote the increasing sequence of numbers in $(0, 1)$ defined by

$$2^n = \left(\log\left(\frac{1}{1 - r_n}\right)\right)^{1-\alpha+\eta}, \quad n \geqslant 1. \tag{4.1}$$

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

Set

$$\delta_n = (1-r_n)\biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{1-\alpha+2\eta}, \quad n \geqslant n_0,$$

where n_0 is a positive integer chosen so that δ_n decreases for $n \geqslant n_0$. We assume that the definition of δ_n is completed in such a way that $\delta_0 = 2\pi$ and $(\delta_n)_{n=0}^{\infty}$ is decreasing. We note that, by (4.1),

$$\sum_{n=n_0}^{\infty} \frac{(\log(2^n/\delta_n))^{1-\alpha}}{2^n} \leqslant \sum_{n=n_0}^{\infty} \biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{-\eta} < \infty.$$

Hence, by the Carleson criterion stated above, the Cantor-type set F with the sequence (δ_n) as just defined has positive $\log_{1-\alpha}$-capacity.

We next define the starlike function h. We begin by partitioning each of the 2^n intervals of F_n of length

$$\frac{\delta_n}{2^n} = (1-r_n)\biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{\eta}, \quad n \geqslant n_0,$$

into k_n subintervals of equal length $(\delta_n/2^n)/k_n$, where

$$k_n = \biggl[\biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{\eta}\biggr].$$

This generates $2^n(k_n+1) = K_n$ partition points, and we denote the ordered sequence of these points by (θ_{mn}), $1 \leqslant m \leqslant K_n$. We now define the function h by

$$h(z) = z \prod_{n=n_0}^{\infty} \prod_{m=1}^{K_n} (1-\bar{z}_{mn}z)^{-2\mu_n}, \quad z \in U,$$

where

$$z_{mn} = r_n \mathrm{e}^{\mathrm{i}\theta_{mn}}, \quad 1 \leqslant m \leqslant K_n,\ n \geqslant n_0,$$

and

$$\mu_n = c\biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{-(1-\alpha)-4\eta},$$

with c chosen so that

$$\sum_{n=n_0}^{\infty} K_n \mu_n = 1.$$

The convergence of the last series is a consequence of (4.1) and the inequality

$$K_n \mu_n \leqslant 2c\biggl(\log\biggl(\frac{1}{1-r_n}\biggr)\biggr)^{-2\eta}.$$

Then

$$\operatorname{Re} \frac{zh'(z)}{h(z)} = \operatorname{Re} \sum_{n=n_0}^{\infty} \mu_n \sum_{m=1}^{K_n} \frac{1+\bar{z}_{mn}z}{1-\bar{z}_{mn}z} > 0$$

https://doi.org/10.1017/S0013091503000737 Published online by Cambridge University Press

for $z \in U$, so h is starlike [**7**, p. 42]. Note that, for $n \geqslant n_0$,

$$\log\left(\frac{4|h(z_{mn})|}{r_n}\right) \geqslant 2\mu_n \log\left(\frac{1}{1-r_n}\right)$$
$$= 2c\left(\log\left(\frac{1}{1-r_n}\right)\right)^{\alpha-4\eta} = 2c\left(\log\left(\frac{1}{1-r_n}\right)\right)^{\beta}. \tag{4.2}$$

Suppose now that $\theta \in F$. Then $\theta \in F_n$ for each $n \geqslant n_0$, and, by the definition of the sequence (θ_{mn}), there is an element θ_{pn}, for some $p (1 \leqslant p \leqslant K_n)$ depending on n, such that

$$|\theta - \theta_{pn}| \leqslant \frac{\delta_n}{2^n k_n} \leqslant 2(1-r_n).$$

Hence, for the corresponding point z_{pn}, since $|zh'(z)/h(z)| \leqslant (1+r)/(1-r)$,

$$\log\left(\frac{|h(z_{pn})|}{r_n}\right) - \log\left(\frac{|h(r_n e^{i\theta})|}{r_n}\right) = \operatorname{Re}\int_{\theta}^{\theta_{pn}}\left(\frac{h'(w)}{h(w)} - \frac{1}{w}\right) i r_n e^{it}\,dt \quad (w = r_n e^{it})$$
$$= O(1), \quad n \to \infty,$$

and it follows from (4.2) that

$$\log\left(\frac{|h(r_n e^{i\theta})|}{r_n}\right) \geqslant 2c\left(\log\left(\frac{1}{1-r_n}\right)\right)^{\beta} + O(1), \quad n \to \infty.$$

This proves Theorem 2.2.

References

1. J. M. Anderson, W. K. Hayman and Ch. Pommerenke, The radial growth of univalent functions, *J. Computat. Appl. Math.* **171** (2004), 27–37.
2. L. Carleson, *Selected problems on exceptional sets* (van Nostrand, Princeton, NJ, 1967).
3. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets* (Cambridge Univerity Press, 1966).
4. R. R. Hall, The length of ray-images under starlike mappings, *Mathematika* **2** (1976), 147–150.
5. W. K. Hayman, *Multivalent functions*, 2nd edn (Cambridge University Press, 1994).
6. Y. Mizuta, On the boundary limits of harmonic functions with gradient in L^p, *Annls Inst. Fourier* **34** (1984), 99–109.
7. Ch. Pommerenke, *Univalent functions* (Vandenhoeckk and Ruprecht, Göttingen, 1975).
8. J. B. Twomey, Tangential boundary behaviour of harmonic and holomorphic functions, *J. Lond. Math. Soc.* **65** (2002), 68–84.
9. A. Zygmund, *Trigonometric series*, Vol. II, 2nd edn (Cambridge University Press, 1959).