A SHORT PROOF OF THE CARTWRIGHT-LITTLEWOOD FIXED POINT THEOREM

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The purpose of this paper is to give a short proof of the Cartwright-Littlewood fixed point theorem (**2**, p. 3, Theorem A).

THEOREM A. *If T is a* (1-1) *continuous and orientation preserving transformation of the Euclidean plane E onto itself which leaves a bounded continuum M invariant and if M does not separate E, then some point of M is left fixed by T.*

We shall first prove a lemma suggested by Newman and proved by him independently (in an unpublished paper). We make use of his notation and some of his methods.

LEMMA 1. *If T is a* (1-1) *continuous and orientation preserving transformation of the Euclidean plane E onto itself which leaves a bounded continuum M invariant but leaves no point of M fixed and if M does not separate E, then there is a* (1-1) *continuous and orientation preserving transformation T' of E onto itself which coincides with T on M and leaves no point of E fixed.*

Proof. Since T, by hypothesis, leaves fixed no point of M, there exists a simple closed curve C_1 with inner domain D_1 containing M, such that if $x \epsilon \bar{D}_1$ then $T(x) \neq x$. Let C_2 and D_2 designate $T(C_1)$ and $T(D_1)$ respectively. By the Brouwer fixed point theorem for the 2-cell, neither of the domains D_1 and D_2 can contain the other. Hence $C_1 \cap C_2$ contains at least two points and, by a known theorem (**3**, p. 87; **4**, p. 168) the component G of $D_1 \cap D_2$ containing M has for its boundary a simple closed curve J. (See Fig. 1.) We may suppose J is the unit circle since it can be made so by a suitable topological mapping of the entire plane E.

For $r = 1, 2$ the components D_{ri} of $D_r - \bar{G}$ have each as frontier a simple closed curve composed of an arc L_{ri} of J and an arc of C_r with common endpoints. For each pair of subscripts r and i, let L'_{ri} be a circular arc of radius $1 - \delta$ with the same endpoints as L_{ri}, where $\delta > 0$ is small enough to ensure that no two arcs L'_{ri} meet except in endpoints. This is possible since the arcs L_{ri} of J are disjoint except for endpoints.

Let Δ_{ri} be the inner domain of $L_{ri} \cup L'_{ri}$. By a standard theorem there is a topological map ϕ_{ri} which maps $\bar{D}_{ri}$ onto $\bar{\Delta}_{ri}$ and leaves fixed each point of L_{ri}. Hence if

$$\bar{\Delta}_r = \bar{G} \cup \bigcup_i \bar{\Delta}_{ri} \qquad (r = 1, 2)$$

the functions ϕ_r defined by

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$$\phi_r \mid \bar{G} = 1 \text{ (the identity map)}$$
$$\phi_r \mid \bar{D}_{ri} = \phi_{ri} \qquad (r = 1, 2)$$

are topological maps of $\bar{D}_r$ onto $\bar{\Delta}_r$ for $r = 1, 2$.

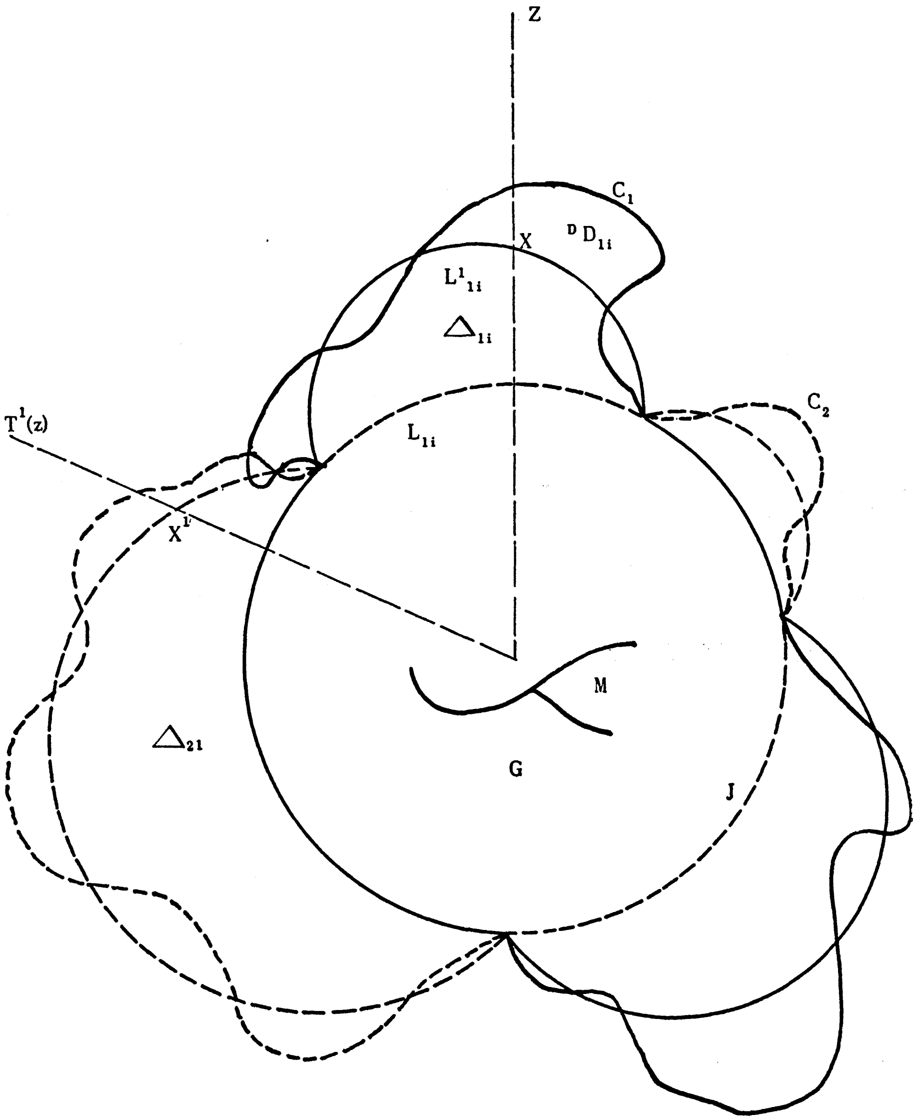

Figure 1

Let $T' : \bar{\Delta}_1 \rightarrow \bar{\Delta}_2$ be defined as $T' = \phi_2 \circ T \circ \phi_1^{-1}$. Then $T'|M = T|M$ since $T = T'$ in G. T' has no fixed point in $\bar{\Delta}_1$. For if $x \in \bar{G}$, $T'(x) = T(x) \neq x$; and if $x \in \bar{\Delta}_1 - \bar{G}$, $x \notin \bar{\Delta}_2 = T'(\bar{\Delta}_1)$.

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Let T' be extended to the whole of E as follows: Let z be a point of $E - \bar{\Delta}_1$. Then z is expressible uniquely as $x + \rho\mu_x$, where $x \in \mathfrak{F} D_1$ and μ_x is the unit vector in the direction Ox, and $\rho > 0$. Let x' designate $T'(x)$ and define $T'(z) = x' + \rho\mu_x{}'$. This a topological mapping of E onto E. Suppose T' has a fixed point $z = T'(z)$. Then the directions from O to $z = x + \rho\mu_x$ and to $T'(z) = x' + \rho\mu_{x'}$ are the same and hence $\mu_x = \mu_{x'}$ and by subtraction $x = x' = T'(x)$ which contradicts the fact that T' has no fixed point in $\bar{\Delta}_1$. Hence $T'(z) \neq z$, and T' is the desired transformation.

Proof of Theorem A. Suppose that under the hypotheses of the theorem T leaves fixed no point of M. Then by Lemma 1 there is an orientation preserving homeomorphism T' of the plane E onto itself which coincides with T on M and leaves no point of E fixed. If p is a point of M then by a theorem of Brouwer (**1**, p. 45, Theorem 8) the set of points in the sequence $T'^n(p)$ $(n = 1, 2, \ldots)$ has no convergent subsequence. This contradicts the fact that M is compact. It follows that the assumption that T leaves no point of M fixed is false.

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