Canad. Math. Bull. Vol. **30** (4), 1987

UNIFORM MAZUR'S INTERSECTION PROPERTY OF BALLS

BY

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ABSTRACT. We give a dual characterization of the following uniformization of the Mazur's intersection property of balls in a Banach space X: for every $\epsilon > 0$ there is a $K > 0$ such that whenever a closed convex set $C \subset X$ and a point $p \in X$ are such that diam $C \leq 1/\epsilon$ and dist$(p, C) \geq \epsilon$, then there is a closed ball B of radius $\leq K$ with $B \supset C$ and dist$(p, B) \geq \epsilon/2$.

The property (I) of a Banach space X that every closed convex bounded subset C of X is an intersection of balls was first studied in the context of Banach spaces in [5]. R. R. Phelps found in [6] necessary and also sufficient conditions for the property (I). Finally the property (I) was characterized in terms of norm density of w^*-denting points of the dual unit ball in the dual unit sphere in [3]. We have been studying a uniformization of the property (I), called here (UI), namely the condition that the radius of a ball separating a convex set from a point depends only on the diameter of C and the distance of the point to a given set (in the sense stated in Abstract). The main purpose of this note is to point out that the property (UI) has an interesting dual characterization (Theorem 1). Propositions 1–3 then relate the property (UI) to other smoothness properties of Banach spaces.

The research in this note is based on that in [6] and [3] and Theorem 1 is a uniform version of Theorem 2.1 in [3] and Lemma 4.1 in [6]. We first encountered the property (UI) in some renorming problems (cf. e.g. [1], [8]).

The spaces in this note are assumed to be real and balls to be closed. The unit ball $\{x \in X; \|x\| \leq 1\}$ is denoted by B_1, the unit sphere $\{x \in X; \|x\| = 1\}$ by S_1. Similarly we define B_1^* or S_1^* in the case of the dual norm of X^*. Generally, a ball with radius r centered at $x \in X$ is denoted by $B(x, r)$. If $x \in S_1 \subset X$, then $D(x) = \{f \in B_1^*; f(x) = 1\}$. If $A \subset S_1$, then $D(A) = \bigcup D(x)$, $x \in A$. Dist(p, C) means the distance of a point p to the set C and diam C stands for the diameter of C. If $\delta > 0$, then a δ-net T in S_1 is a subset of S_1 such that for every $s \in S_1$ there is a $t \in T$ such that $\|s - t\| < \delta$. The set of all positive integers is denoted by N.

DEFINITION 1. *We say that a Banach space X has the uniform Mazur's intersection property* (UI) *if for every* $\epsilon > 0$ *there is a* $K > 0$ *such that whenever a closed convex*

Received by the editors April 13, 1986.

AMS Subject Classification (1980): 46B20.

Key words: ball intersection property, smoothness.

The authors' research supported in part by NSERC (Canada) and University of Alberta grants.

set $C \subset X$ *and a point* $p \in X$ *are such that* $\operatorname{diam} C \leq 1/\epsilon$ *and* $\operatorname{dist}(p, C) \geq \epsilon$, *then there is a ball* B *in* X *of radius* $\leq K$ *such that* $B \supset C$ *and* $\operatorname{dist}(p, B) \geq \epsilon/2$.

We will need the following definition from [3], [7]

DEFINITION 2. *If* $x \in S_1 \subset X$ *and* $\epsilon, \delta > 0$, *then we say that* $x \in M_{\epsilon,\delta}$ *if*

$$\sup_{0<\|y\|<\delta} \frac{\|x + y\| + \|x - y\| - 2}{\|y\|} < \epsilon.$$

The following Lemma 1 is a quantitative version of Lemma 2.1 in [3].

LEMMA 1. *Let* $x \in S_1$, $\epsilon, \delta > 0$, $\delta < \epsilon/2 < 1/2$. *Consider the following statements:*
(i) $x \in M_{\epsilon,\delta}$
(ii) $\operatorname{diam}\{f \in B_1^*, f(x) \geq 1 - \delta^2\} < 2\epsilon$
(iii) $\operatorname{diam}\{\bigcup D(z), z \in S_1, \|z - x\| \leq \delta^2\} < 2\epsilon$
(iv) $x \in M_{2\epsilon,\delta^2/2}$
Then (i) implies (ii), (ii) implies (iii) and (iii) implies (iv).

PROOF. A quantitative version of that of Lemma 2.1 in [3].
(i) $\Rightarrow$ (ii). Assume that (ii) fails. Then for every $n \in N$, there are $f_n, g_n \in B_1^*$ with $f_n(x) \geq 1 - \delta^2$, $g_n(x) \geq 1 - \delta^2$ and $\|f_n - g_n\| > 2\epsilon - 1/n$. Choose $y_n \in S_1$ such that

$$(f_n - g_n)(y_n) > 2\epsilon - 1/n.$$

Then

$$\|x + \delta y_n\| + \|x - \delta y_n\| \geq f_n(x + \delta y_n) + g_n(x - \delta y_n) = f_n(x) + g_n(x) + \delta(f_n - g_n)(y_n)$$
$$\geq 2 - 2\delta^2 + \delta(2\epsilon - 1/n)$$

Therefore $\dfrac{\|x + \delta y_n\| + \|x - \delta y_n\| - 2}{\delta} \geq 2\epsilon - 2\delta - 1/n \geq \epsilon - 1/n$ since $\delta < \epsilon/2$.
This contradicts $x \in M_{\epsilon,\delta}$.

(ii) $\Rightarrow$ (iii). Obvious since $(\bigcup D(z); z \in S_1, \|z - x\| \leq \delta^2) \subset \{f \in B_1^*; f(x) \geq 1 - \delta^2\}$.

(iii) $\Rightarrow$ (iv). It is known (see [3], p. 111) that for $0 < \lambda < 1$, $y \in S_1 \subset X$,

$$\frac{\|x + \lambda y\| + \|x - \lambda y\| - 2}{\lambda} \leq (f - g)(y) \leq \|f - g\|$$

for every

$$f \in D((x + \lambda y)/\|x + \lambda y\|),\ g \in D((x - \lambda y)/\|x - \lambda y\|).$$

Moreover,

$$\|((x + \lambda y)/\|x + \lambda y\|) - x\| \leq 2\lambda$$

Therefore, by (iii),

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$$\sup_{\substack{0<\lambda\leq\delta^2\\ \|y\|=1}} \frac{\|x+\lambda y\|+\|x-\lambda y\|-2}{\lambda} \leq \operatorname{diam}\{f_z \in D(z),\ z \in S_1,\ \|z-x\| \leq \delta^2\} < 2\epsilon$$

Hence

$$x \in M_{2\epsilon,\delta^2/2}.$$

THEOREM 1. *Let X be a Banach space. Then all the statements listed below are equivalent:*

(i) X has the property (UI).

(ii) For every $\epsilon > 0$ there is a $\delta > 0$ such that for every $f \in S_1^* \subset X^*$ there is an $x \in S_1 \subset X$ such that if $z \in S_1$, $\|z - x\| \leq \delta$, then $\|f_z - f\| < \epsilon$ for every $f_z \in D(z)$.

(iii) For every $\epsilon > 0$ there is a $\delta > 0$ such that if T is a δ-net in $S_1 \subset X$ and for every $t \in T$, f_t is chosen element of $D(t)$, then $\{f_t,\ t \in T\}$ is an ϵ-net in $S_1^* \subset X^*$.

(iv) for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $f \in S_1^* \subset X^*$ there is an $x \in M_{\epsilon,\delta}$ such that $\|f - f_x\| < \epsilon$ for every $f_x \in D(x)$.

PROOF. (i) $\Rightarrow$ (ii) A quantitative version of the proof of Lemma 4.1 in [6]. Let $\epsilon \in (0, 1)$ be given. Choose $K > 0$ from (i) for $\epsilon/4$. Then follow the Phelps' proof of Lemma 4.1 in [6]. We only need to show that $\delta = \delta(\epsilon, f)$ obtained for various $f \in S_1^*$ in the Phelps' proof all have (for our fixed $\epsilon > 0$) a lower bound $\delta = \epsilon(8(K + 1))^{-1}$. So, let $f \in S_1^*$ be given. Denote by $D = B_1 \cap f^{-1}(0)$ and pick a $u \in S_1$ such that $f(u) > 1 - \epsilon/2$. Let $u' = (\epsilon/2)u$. Then $\operatorname{dist}(u', f^{-1}(0)) = f(u') > (\epsilon/2)(1 - \epsilon/2) \geq \epsilon/4$. By (i), there is a ball $B(=N_r(z)$ in the Phelps' notation) centered at $z \in X$ with radius r such that $r \leq K$, $B \supset D$ and $\operatorname{dist}(u', B) \geq \epsilon/8$. Let w be the intersection of the line segment $[z, u']$ with the boundary of B and C be the convex hull of $B \cup \{u\}$. Let h be the distance of u' to $B (h \geq \epsilon/8)$. Then simple homothety argument shows that if ℓ is a line connecting u' with a boundary point of B, then $\operatorname{dist}(w, \ell) > rh(r + h)^{-1}$. Since $\epsilon/8 \leq h \leq 1$ we have $rh(r + h)^{-1} \geq \epsilon r(8(r + 1))^{-1}$; δ is then obtained in the Phelps' proof by taking

$$\delta = r^{-1}\cdot rh(r+h)^{-1} \geq \epsilon(8(r+1))^{-1} \geq \epsilon(8(K+1))^{-1}.$$

(ii) $\Rightarrow$ (iii). obvious.

(iii) $\Rightarrow$ (ii). If (ii) does not hold, then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is an $f_\delta \in S_1^*$ such that for every $x \in S_1 \subset X$ there is an $z_x \in S_1$ with $\|z_x - x\| \leq \delta$ and $f_{z_x} \in D(z_x)$ such that $\|f_\delta - f_{z_x}\| \geq \epsilon$. Pick for every $x \in S_1 \subset X$ such a z_x and f_{z_x}. Then $\{z_x;\ x \in S_1\}$ forms a 2δ net in S_1 but $\{f_{z_x};\ x \in S_1\}$ does not form an ϵ-net in S_1^*. So then (iii) does not hold.

(ii) $\Rightarrow$ (iv). Let $\epsilon > 0$ be given. Choose $\delta > 0$, $\delta < \epsilon/32$ for $\epsilon/16$ by (ii). Then for every $f \in S_1^*$ choose again by (ii) an $x \in S_1$ such that if $\|z - x\| \leq \delta$, $z \in S_1$, then $\|f_z - f\| < \epsilon/16$ for every $f_z \in D(z)$. Therefore then $\operatorname{diam}\{D(z);\ \|z - x\| \leq \delta,\ z \in S_1\} < \epsilon/4$. Then, by Lemma 1, $x \in M_{\epsilon/4,\delta/2} \subset M_{\epsilon,\delta/2}$. Hence (iv) holds.

(iv) $\Rightarrow$ (i). Based on the main idea in [5]. See also [6], [3]. Clearly, it is enough to show

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that for every $\epsilon > 0$ there is a $K > 0$ such that if C is a closed convex subset of X such that $\operatorname{dist}(0, C) \geq \epsilon$ and $\operatorname{diam} C \leq 1/\epsilon$, then there is a ball B of radius $\leq K$ with $B \subset C$ and $\operatorname{dist}(0, B) \geq \epsilon/2$. Given $\epsilon > 0$, denote by $L = \epsilon/2 + 1/\epsilon$ and choose a $\delta \in (0, 1)$ by (iv) for $\epsilon/(4L)$. Finally put $K = L/\delta$. We will show that if C is a closed convex set in X with $\operatorname{diam} C \leq 1/\epsilon$ and $\operatorname{dist}(0, C) \geq \epsilon$, then there is a ball B with radius $\leq K$ and $\operatorname{dist}(0, B) \geq \epsilon/2$. If $C \cap (X \backslash B(0, L)) \neq \emptyset$, then pick a $c \in C \cap (X \backslash B(0, L))$ and observe that $B(c, 1/\epsilon) \supset C$, $\operatorname{dist}(0, B(c, 1/\epsilon) \geq \epsilon/2$ and $1/\epsilon \leq K$. If $C \subset B(0, L)$, choose, by a standard separation theorem, an $f \in S_1^*$ such that $\inf f(C) \geq \epsilon$. By (iv), there is an $x \in M_{\epsilon/(4L), \delta}$ such that for every $f_x \in D(x)$, we have

$$\|f - f_x\| < \epsilon/(4L)$$

Choose $f_x \in D(x)$. Consider the family of balls:

$$B_\lambda = B(\lambda(\epsilon/2)x, (\lambda - 1)(\epsilon/2)),\ \lambda > 1$$

Since $\operatorname{dist}(0, B_\lambda) = \epsilon/2$ for every $\lambda > 1$, it is enough to show that if $\lambda_0 = 2L/(\epsilon\delta)$, then $B_{\lambda_0} \supset C$. For, then the radius of $B_{\lambda_0} \leq L/\delta = K$. Suppose the contrary, i.e. that $C \cap (X \backslash B_{\lambda_0}) \neq \emptyset$ and choose a $z \in C \cap (X \backslash B_{\lambda_0})$. Put $y = 2(\lambda_0\epsilon)^{-1}z$.
Then

$$\begin{aligned}
\frac{\|x + y\| + \|x - y\| - 2}{\|y\|} &= \frac{\|x + y\| - 1}{\|y\|} + \frac{\|\lambda_0(\epsilon/2)x - z\| - \lambda_0(\epsilon/2)}{\|z\|} \\
&\geq \frac{\|x + y\| - 1}{\|y\|} + \frac{(\lambda_0 - 1)(\epsilon/2) - \lambda_0(\epsilon/2)}{\|z\|} \\
&\geq f_x(y/\|y\|) - (\epsilon/2)\|z\|^{-1} \\
&= f_x(z/\|z\|) - (\epsilon/2)\|z\|^{-1} \\
&\geq f(z/\|z\|) - (\epsilon/2)\|z\|^{-1} - \|f - f_x\| \\
&\geq \epsilon/\|z\| - (\epsilon/2)\|z\|^{-1} - \epsilon/(4L) \\
&\geq \epsilon/(2L) - \epsilon/(4L) = \epsilon/(4L)
\end{aligned}$$

(the last inequality because $\|z\| \leq L$). However,

$$\|y\| = 2(\lambda_0\epsilon)^{-1}\|z\| \leq 2(\lambda_0\epsilon)^{-1}L = \delta$$

and therefore

$$x \notin M_{\epsilon/(4L), \delta},\quad \text{a contradiction showing that}\quad B_{\lambda_0} \supset C.$$

Theorem 1 is proved.

We finish the paper with two propositions showing the relationship of the property (UI) to other smoothness properties of Banach spaces. First, based on [6], [3] is the following.

PROPOSITION 1. *Let X be a Banach space. Consider the following properties of X*
(i) X^* is uniformly convex.

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(ii) X has the property (UI).

(iii) X has the property (I).

(iv) The set of all extreme points of the dual unit ball B_1^* *of* X^* *is dense in the unit sphere* S_1^* *of* X^*.

Then (i) implies (ii), (ii) implies (iii) and (iii) implies (iv). If X is finite dimensional, then (ii), (iii) and (iv) are all equivalent.

PROOF. (i) $\Rightarrow$ (ii). This statement is contained in [9]. Due to Theorem 1, it now has a straightforward proof by noticing that if X^* is uniformly convex then the differential of the norm of X is uniformly continuous on $S_1 \subset X$ (cf. e.g. [2], p. 36), which fact directly implies that the statement in Theorem 1 (iii) is true

(ii) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (iv) see [6], Theorem 4.3.

If X is finite dimensional, then (iii) and (iv) are equivalent by [6], Theorem 4.4. We will show that (iii) implies (ii) in this case: Suppose that a finite dimensional X does not have the property (UI). Then there is an $\epsilon > 0$ such that for every $n \in N$ there exists an $f_n \in S_1^*$ such that for every $x \in S_1$, there is an $z_x^n \in S_1$ and $f_{z_x^n} \in D(z_x^n)$ with $\|z_x^n - x\| \leq 1/n$ and $\|f_{z_x^n} - f_n\| \geq \epsilon$. Take $\epsilon/2$ and f = a limit point of the sequence $\{f_n\}$. We show that the following statement (*) holds for $\epsilon/2$ and f:

For every $\delta > 0$ and for every $x \in S_1 \subset X$ there is an $z_x \in S_1$ (*) $\|z_x - x\| \leq \delta$ and $f_{z_x} \in D(z_x)$ such that $\|f_{z_x} - f\| \geq \epsilon/2$. For, having $\delta > 0$ and $x \in S_1$ given, fix $n \in N$ so big that

$$\|f_n - f\| < \epsilon/2 \qquad \text{and} \qquad 1/n < \delta.$$

For this n, choose as above in this proof a $z_x^n \in S_1$ and $f_{z_x^n} \in D(z_x^n)$ such that $\|z_x^n - x\| \leq 1/n$ and $\|f_{z_x^n} - f_n\| \geq \epsilon$. Then $\|z_x^n - x\| < \delta$ an $\|f_{z_x^n} - f\| \geq \epsilon/2$. Therefore (*) is true, which fact in turn implies that X does not then have the property (I) ([6], Lemma 4.1).

It follows from Proposition 1 that there are spaces X which have the property (UI) but X^* are not uniformly convex. We now show that there are spaces which have property (I) but fail to have property (UI).

PROPOSITION 2. *For* $n \in N$, *let* X_n *be the 2-dimensional space* $\ell_{p_n}^2$ *where* $p_n > 1$, $\lim p_n = 1$. Let $X = (\Sigma \oplus X_n)_2$, *the Hilbert sum of* X_n. *Then X has the property (I) but fails to have the property (UI).*

PROOF. Since X is reflexive and has Frechet differentiable norm (cf. e.g. [4]), X has the property (I) by a result of S. Mazur ([5]). The fact that X does not have the property (UI) directly follows from the following two observations.

1. Let C be the square in the plane R^2 with vertices $(\pm 2, \pm 2)$ and let $P = (3, 0) \in R^2$. Let B_n be a sequence of balls in $\ell_{p_n}^2$ centered at (s_n^1, s_n^2) with radii r_n and such that $P \notin B_n \supset C$. Then $\lim_n r_n = \infty$. For if not, then there would exist a subsequence

$\{n_k\}$ of $\{n\}$ such that

$$\lim_k s^1_{n_k} = s^1, \qquad \lim_k s^2_{n_k} = s^2, \qquad \lim_k r_{n_k} = r < \infty.$$

Then for every $c = (c^1, c^2) \in C$, we have

$$|c^1 - s^1| + |c^2 + s^2| \leq r,$$

while

$$|3 - s^1| + |s^2| \geq r.$$

This is a contradiction with the elementary fact that P lies in the interior of any ℓ_1^2-ball which contains C.

(2) For $n \in N$, let C_n be a subset of X defined by $C_n = \{(0, 0, \ldots c_n, 0, \ldots), c_n \in C\}$ and P_n be a point in X defined by

$$P_n = (0, 0, \ldots p_n, 0, \ldots).$$

Then diam $C_n \leq 8$ and dist$(P_n, C_n) = 1$ for every $n \in N$. If X had the property (UI), there would exist a sequence B_n of balls in X with radii r_n such that $P_n \notin B_n \supset C_n$ for every $n \in N$ and sup $r_n < \infty$. Then considering intersections of B_n with the subspaces $(0, 0, \ldots, X_n, 0, \ldots) \subset X$ one can easily produce a sequence B_n' of convex bodies in R^2, each B_n' being a ball in X_n with X_n-radius $\leq r_n$, $B_n' \supset C$ and $P \notin B_n'$. This would be a contradiction with (1). Proposition 2 is proved.

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