RELATIONS BETWEEN THE DIGITS OF NUMBERS AND EQUAL SUMS OF LIKE POWERS

J. B. ROBERTS

It is straightforward, but tedious, to write down the integers whose representations in a given base do not have particular digits in certain positions. In the first section of this paper we give a computational scheme that enables us to carry out such operations in a rapid and simple fashion.

In the second section of the paper we derive a general identity involving the digits of integers in arbitrary Cantor systems of notation.

In the third section we apply this identity and deduce a number of results concerned with the splitting of integers into classes with equal power sums. The computational scheme of the first section leads us to an algorithm for the determination of such splittings.

In the last section we again apply the identity of the second section and derive several further consequences.

1. Every finite array of numbers of the form

$$
\begin{array}{ccccc}
F_0(0) & F_1(0) & & F_{m-1}(0) \\
F_0(1) & F_1(1) & \cdots & F_{m-1}(1) \\
\vdots & \vdots & & \vdots \\
F_0(n_1-1) & F_1(n_2-1) & & F_{m-1}(n_m-1)
\end{array}
\tag{1}
$$

where the $F_j(n)$ are arbitrary complex numbers and the n_j are integers all $\geqslant 2$, uniquely determines a sequence of $n_1 \ldots n_m$ numbers by a method based on what might well be called "the speedometer principle." The sequence is constructed in m steps as follows.

The first step consists in writing

$$F_0(0), \quad F_0(1), \quad \ldots, \quad F_0(n_1-1).$$

The second step consists in adding to these the numbers $F_1(0)$, $F_1(1), \ldots,$ $F_1(n_2-1)$, one by one, to obtain

$$
\begin{gathered}
F_0(0)+F_1(0), \quad F_0(1)+F_1(0), \quad \ldots, \quad F_0(n_1-1)+F_1(0), \\
F_0(0)+F_1(1), \quad \ldots, \quad F_0(n_1-1)+F_1(n_2-1).
\end{gathered}
$$

We continue in the same manner, adding at the third step the numbers $F_2(0)$, $F_2(1), \ldots, F_2(n_3-1)$ to the sequence just obtained. After m steps we arrive at the desired sequence.

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If we put

$$p_0 = 1, \quad p_1 = n_1, \quad p_2 = n_1 n_2, \quad \ldots, \quad p_m = n_1 \ldots n_m,$$

it is not difficult to verify that if the Cantor expansion of n relative to the p_j is

$$(2) \qquad n = a_0 + a_1 p_1 + \ldots + a_{m-1} p_{m-1}, \qquad 0 \leqslant a_j < n_{j+1},$$

then, denoting the sequence associated with (1) by S and its $(n + 1)$st term by S_n,

$$(3) \qquad S_n = F_0(a_0) + F_1(a_1) + \ldots + F_{m-1}(a_{m-1}).$$

In the special case where

$$(4) \qquad F_j(k) = kp_j, \qquad j = 0, \ldots, m - 1,$$

it is clear that $S_n = n$ and therefore S is just the sequence of non-negative integers. When (4) is the case, the array (1) becomes:

$$(5) \qquad \begin{array}{ccccc} 0 & 0 & 0 & \ldots & 0 \\ 1 & p_1 & p_2 & \ldots & p_{m-1} \\ 2 & 2p_1 & 2p_2 & \ldots & 2p_{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ n_1 - 1 & (n_2 - 1)p_1 & (n_3 - 1)p_2 & \ldots & (n_m - 1)p_{m-1} \end{array}$$

Suppose now that we replace rp_s in (5) by qp_s, where r and q $(r \neq q)$ are fixed numbers from 0 to $n_{s+1} - 1$ inclusive. Denote the sequences associated with (5), before and after the replacement, by S and S'. Then $S_n = S_m'$ whenever n and m are of the form

$$n = a_0 + a_1 p_1 + \ldots + a_{s-1} p_{s-1} + qp_s + a_{s+1} p_{s+1} + \ldots + a_{m-1} p_{m-1},$$

$$m = n_1 + rp_s - qp_s.$$

Further, all numbers of the form m are missing from S'. Thus the sequence S' does not contain any number whose Cantor expansion (relative to the p_j) has digit r in the $(s + 1)$st place from the right.

If we now delete the new qp_s, the sequence associated with the resulting array contains all the distinct terms of S' but has no repetitions. Thus, *deletion of rp_s from* (5) *leads to the sequence of integers* (*from* 0 *to* $p_m - 1$ *inclusive*) *whose Cantor expansions do not have digit r in the* $(s + 1)$*st position from the right.*

A further deletion of tp_u leads to the elimination from the remaining terms of all those having a t in the $(u + 1)$st position from the right. This process may be continued in the same manner.

Example 1. Calculate those integers from 0 to 100, inclusive, whose base 3 expansions do not have units digit 2 or "hundreds" digit 0 or 2.

Since $3^5 > 100$, we take $m = 5$ and write the array (5) where $p_1 = 3$, $p_2 = 3^2$, $p_3 = 3^3$, $p_4 = 3^4$, and all $n_i = 3$.

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 9 & 27 & 81 \\ 2 & 6 & 18 & 54 & 162 \end{array}$$

We now delete 2 from the first column and 0 and 2.3^2 from the third column, obtaining

$$\begin{array}{ccccc} 0 & 0 & 9 & 0 & 0 \\ 1 & 3 & & 27 & 81 \\ & 6 & & 54 & 162 \end{array}$$

This leads to:

0, 1

0, 1, 3, 4, 6, 7

9, 10, 12, 13, 15, 16

9, 10, 12, 13, 15, 16, 36, 37, 39, 40, 42, 43, 63, 64, 66, 67, 69, 70

9, 10, 12, . . . , 69, 70, 90, 91, 93, 94, 96, 97.

The last row is the desired set of numbers.

Noting that whenever a column starts with a 0 the new stage begins with the previous sequence, the process shortens as is indicated below (we use semicolons at each spot where a repetition is avoided):

0, 1; 3, 4, 6, 7

9, 10, 12, 13, 15, 16; 36, 37, 39, 40, 42, 43, 63, 64, 66, 67, 69, 70; 90, 91, 93, 94, 96, 97.

This last observation assures us that if the array in (1) is continued indefinitely to the right and all except a finite number of the $F_j(0)$ are 0, then the array determines a unique *infinite* sequence.

Example 2. Give an array that leads to the sequence of integers whose base d expansions have units digit one of the distinct numbers $i_0, \ldots, i_{b-1}$, where $b \leqslant d$ and each i_j satisfies $0 \leqslant i_j < d$.

We give two distinct arrays leading to this sequence.

(a) By our earlier remarks the following array clearly leads to the desired sequence:

$$(6) \qquad \begin{array}{ccccc} i_0 & 0 & 0 & 0 & \\ \vdots & d & d^2 & d^3 & \\ \vdots & 2d & 2d^2 & 2d^3 & \\ i_{b-1} & \vdots & \vdots & \vdots & \cdots \\ & (d-1)d & (d-1)d^2 & (d-1)d^3 & \end{array}$$

(b) Consider now the array

$$
(7) \qquad \begin{array}{ccccc}
i_0 & 0 & 0 & 0 & \\
\vdots & d & db & db^2 & \\
\vdots & 2d & 2db & 2db^2 & \\
i_{b-1} & \vdots & \vdots & \vdots & \cdots \\
 & (b-1)d & (b-1)db & (b-1)db^2 &
\end{array}
$$

If S is the associated sequence and

$$n = c_0 + c_1 b + c_2 b^2 + \ldots + c_m b^m, \qquad 0 \leqslant c_j < b,$$

then

$$S_n = i_{c_0} + \sum_{j=0}^{m-1} b^j c_{j+1} d = i_{c_0} + d \sum_{j=0}^{m-1} c_{j+1} b^j.$$

Thus, S_n always differs from a multiple of d by one of $i_0, \ldots, i_{b-1}$. On the other hand, since

$$\sum_{j=0}^{m-1} c_{j+1} b^j$$

runs over all non-negative integers, every number whose base d expansion has units digit one of $i_0, \ldots, i_{b-1}$ appears in the sequence S.

2. Throughout this section n and S_n are as given in (2) and (3) of Section 1. Using the multinomial theorem we find that

$$S_n^{\,t} = \left(\sum_{j=0}^{m-1} F_j(a_j) \right)^t = \sum \frac{t!}{u_1! \ldots u_m!} F_0^{u_1}(a_0) \ldots F_{m-1}^{u_m}(a_{m-1}),$$

where the sum is over $u_1 + \ldots + u_m = t$, $0 \leqslant u_1, \ldots, 0 \leqslant u_m$. Thus, for arbitrary functions $f_1, \ldots, f_m$ we have:

$$
(8) \qquad \begin{aligned}
&\sum_{a_0=0}^{n_1-1} \ldots \sum_{a_{m-1}=0}^{n_m-1} f_1(a_0) \ldots f_m(a_{m-1}) S_n^{\,t} \\
&= \sum \frac{t!}{u_1! \ldots u_m!} \left(\sum_{a_0=0}^{n_1-1} f_1(a_0) F_0^{u_1}(a_0) \right) \ldots \left(\sum_{a_{m-1}=0}^{n_m-1} f_m(a_{m-1}) F_{m-1}^{u_m}(a_{m-1}) \right),
\end{aligned}
$$

where the sum is again over the same range as above.

We shall be interested primarily in those special cases of (8) in which, for each $j = 1, \ldots, m$,

$$(9) \qquad \sum_{n=0}^{n_j-1} f_j(n) F_{j-1}^u(n) = 0, \qquad 0 \leqslant u \leqslant \alpha_j,$$

where $\alpha_j \geqslant -1$. (The case $\alpha_j = -1$ is interpreted as meaning that the left side of (9) does not vanish for $u = 0$.)

Assuming (9), for all j, we shall see that the right side of (8) vanishes for all t satisfying

$$(10) \qquad 0 \leqslant t < \alpha_1 + \ldots + \alpha_m + m.$$

This conclusion will follow if we can be sure that in each term at least one u_j is less than or equal to the corresponding α_j. But, suppose that $u_j > \alpha_j$ for all j. Then

$$t = u_1 + \ldots + u_m \geqslant \alpha_1 + \ldots + \alpha_m + m,$$

which is contrary to (10). Thus the right side, and therefore also the left side, of (8) vanishes for those t satisfying (10). We state this conclusion as a theorem.

THEOREM 1. *If* (9) *is true and* n, S_n *are given by* (2) *and* (3) (i.e. S_n is the $(n+1)$st term of the sequence determined by (1)), *then*

$$\sum_{a_0=0}^{n_1-1} \ldots \sum_{a_{m-1}=0}^{n_m-1} f_1(a_0) \ldots f_m(a_{m-1}) S_n{}^t = 0 \tag{11}$$

for $0 \leqslant t < \alpha_1 + \ldots + \alpha_m + m$.

It should be noted that by use of the binomial theorem we may replace (11) by the equivalent identity:

$$\sum_{a_0=0}^{n_1-1} \ldots \sum_{a_{m-1}=0}^{n_m-1} f_1(a_0) \ldots f_m(a_{m-1}) P(x + S_n) = 0, \tag{12}$$

for P any polynomial of degree less than $\alpha_1 + \ldots + \alpha_m + m$.

When we take $F_j(n) = n\beta_j$, then

$$S_n = a_0\beta_0 + \ldots + a_{m-1}\beta_{m-1}$$

and (12) reduces to (**5**, Theorem 3).

3. For $j = 1, \ldots, m$ let h_{j-1} be a (mod n_j) projection map (i.e. a map of period n_j taking all integers *onto* the integers $0, 1, \ldots, n_j - 1$), let L be the least common multiple of $n_1, \ldots, n_m$ and put C_r for the set of n, given by (2), satisfying

$$Lh_0(a_0)/n_1 + \ldots + Lh_{m-1}(a_{m-1})/n_m \equiv r \pmod{L}.$$

Then putting

$$f_j(n) = \exp(2\pi s i h_{j-1}(n)/n_j), \qquad j = 1, \ldots, m,$$

we see that (9) holds for $\alpha_j = -1$ or 0 depending on whether n_j divides s or not. Substituting these f_j into (11), we obtain

$$\sum_{r=0}^{L-1} \left(\sum_{n \in C_r} S_n{}^t \right) \exp(2\pi s r i/L) = 0 \tag{13}$$

for $0 \leqslant t < \alpha_1 + \ldots + \alpha_m + m$. For those t satisfying $0 \leqslant t < \alpha_1 + \ldots + \alpha_m + m$ for each of the s, $0 < s < L$, we may equate the coefficients in (13). We have therefore proved the following theorem.

THEOREM 2. $\sum_{n \in C_r} S_n{}^t$ *is independent of* r, $0 \leqslant r < L$, *for each* t, $0 \leqslant t < m - \bar{\nu}$, *where* $\bar{\nu}$ *is the maximum number of* $n_1, \ldots, n_m$ *dividing an* s, $0 < s < L$.

The quantity $S_n{}^t$ may be replaced by $P(x + S_n)$ for P any polynomial of degree smaller than $m - \bar{\nu}$ by using (12) rather than (11).

This theorem generalizes (**5**, Theorem 7) and (**6**, Theorem 3). Information concerning the size of $\bar{\nu}$ may be found in (**6**).

Consider now the array (1) in which, for $j = 0, \ldots, m-1$,

$$\begin{cases} F_j(n) = Lh_j(n)/n_{j+1}, \\ h_j(0) = 0. \end{cases} \tag{14}$$

Then putting S' for the sequence arising from this array we have, using (3),

$$S_n' = \sum_{j=0}^{m-1} F_j(a_j) = \sum_{j=0}^{m-1} Lh_j(a_j)/n_{j+1}.$$

This means that

$$n \in C_r \textit{ if and only if } S_n' \equiv r \pmod{L}. \tag{15}$$

We now introduce an array similar to (1) except that the entries are no longer *numbers* but are ordered pairs of numbers. The array of numbers appearing in the first spots of the ordered pairs is an array (1) with arbitrary $F_j(n)$ while the array of numbers in the second spots of the ordered pairs is an array (1) in which the $F_j(n)$ satisfy (14). We use $F_j(n)$ for the typical first element and $H_j(n)$ for the typical second element. Then our array of ordered pairs looks as follows:

$$\begin{matrix} (F_0(0), 0) & \ldots & (F_{m-1}(0), 0) \\ (F_0(1), H_0(1)) & \ldots & (F_{m-1}(1), H_{m-1}(1)) \\ \vdots & & \vdots \\ (F_0(n_1-1), H_0(n_1-1)) & \ldots & (F_{m-1}(n_m-1), H_{m-1}(n_m-1)) \end{matrix} \tag{16}$$

(We have replaced the $H_j(0)$ by 0 for each j since the $H_j(n)$ satisfy (14)—with F replaced by H in (14).)

We "add" two elements of the array (16) by first adding respective components and then replacing the second component by its least non-negative residue (mod L). With this addition (16) determines a sequence of ordered pairs just as (1) determines an ordinary sequence of numbers. Putting $\mathfrak{S}$ for the sequence of ordered pairs and S, S', respectively, for the sequences determined by the $F_j(n)$, $H_j(n)$ we have, for $n \in C_r$,

$$\mathfrak{S}_n = (S_n, S_n') = \left(\sum_{j=0}^{m-1} F_j(n), r\right). \tag{17}$$

This gives us a mechanical method for the determination of the numbers in the various classes C_r: The numbers in C_r are simply the first components of those elements of $\mathfrak{S}$ whose second component is r.

Example. Consider the array

$$\begin{matrix} (1, 0) & (0, 0) & (0, 0) \\ (2, 1) & (8, 1) & (16, 1) \\ (3, 2) & (12, 2) & (48, 2) \end{matrix} \tag{18}$$

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The sequence it determines is:

$$(1, 0), (2, 1), (3, 2); (9, 1), (10, 2), (11, 0), (13, 2), (14, 0), (15, 1); (17, 1), (18, 2), (19, 0), (25, 2), (26, 0), (27, 1), (29, 0), (30, 1), (31, 2), (49, 2), (50, 0), (51, 1), (57, 0), (58, 1), (59, 2), (61, 1), (62, 2), (63, 0).$$

Thus

$$C_0 = \{1, 11, 14, 19, 26, 29, 50, 57, 63\},$$
$$C_1 = \{2, 9, 15, 17, 27, 30, 51, 58, 61\},$$
$$C_2 = \{3, 10, 13, 18, 25, 31, 49, 59, 62\}.$$

By Theorem 2, since $n_1 = n_2 = n_3 = 3$ and $\bar{\nu} = 0$, we see that these classes have equal tth power sums for $0 \leqslant t \leqslant 2$.

In this example we used canonical (mod 3) projection maps (i.e. ones that mapped each of $0, 1, 2$ onto itself). We would get a similar splitting of the same numbers by using any of the other seven possible combinations of (mod 3) projection maps (with the $h_j(0)$ always 0). For a proof that among these eight splittings there are at least four distinct ones see **(6)**.

Noting that the array (5) with $m = 3$, $n_1 = n_2 = n_3 = 4$, is:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 4 & 16 \\ 2 & 8 & 32 \\ 3 & 12 & 48 \end{array}$$

we see that the array of first elements in (18) yields a sequence of numbers whose base 4 expansion has no 0 in the right-hand place, no 1 in the second place from the right, and no 2 in the third place from the right, i.e. the splitting derived from (18) is a splitting of all numbers from 0 to 63 whose base 4 expansions agree in no digit with 210.

This result is a special case of the following theorem.

THEOREM 3. *The integers from* 0 *to* $b^m - 1$, *inclusive, whose base* b *expansions agree in no digit with* $c_{m-1}\, c_{m-2} \ldots c_0$ *may be split into* b *classes such that the sum of the* tth *powers of the elements in a given class is the same for all classes for all* t *from* 0 *to* $m - 1$ *inclusive, and the splitting may be accomplished in* $(b - 1)!^{m-1}$ *ways by making use of* (16).

Except for the number of ways the splitting may be accomplished, the proof in no way differs from that given in the special case. The number of such splittings for arbitrary $F_j(n)$ is greater than or equal to the number of distinct ways one can choose the $H_j(n)$ in (16) so as to obtain distinct splittings of the $F_j(n)$ when the $F_j(n)$ are chosen in such a way as to lead to the sequence of natural numbers from 0 to $b^m - 1$. This was shown to be greater than or equal to $(b - 1)!^{m-1}$ in **(6)**.

A similar result holds when the n_j are not equal.

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Using the first $m + 1$ columns of the array (7) for the first elements in our pairs we prove the following result in exactly the same way.

THEOREM 4. *The integers from* 0 *to* $db^m - 1$, *inclusive, whose base* d *expansions do not have a units digit equal to any one of a fixed, but arbitrary, set of* $d - b$ *of the integers* $0, 1, \ldots, d - 1$ *may be split into* b *classes such that the sum of the t-th powers of the elements in a given class is the same for all classes for all* t *from* 0 *to* $m - 1$, *inclusive, and the splitting may be accomplished in* $(b - 1)!^{m-1}$ *ways by making use of* (16).

As may be seen from the more specialized discussion in **(6)**, we cannot expect the number $(b - 1)!^{m-1}$ appearing in the last two results to be best possible.

We derive one more consequence of Theorem 2. It will not involve the array (16).

It is well known that

$$\sum_{s=0}^{m} (-1)^s \binom{m}{s} s^t = 0 \qquad \text{for } 0 \leqslant t < m. \tag{19}$$

We generalize this formula to the multinomial coefficients. Taking all $n_j = b \geqslant 2$ and $F_j(n) = n$ in (13) we obtain:

For each t, $\sum_{n \in C_r} (a_0 + \ldots + a_{m-1})^t$ *is independent of* r; $0 \leqslant r < b$, $0 \leqslant t < m$.

The a_j are, of course, the coefficients in the base b expansion of n. We now let $\left\{ m \atop s \right\}_b$ denote the number of numbers 0 to $b^m - 1$, inclusive, whose base b digit sum is s. Using this notation the above result becomes:

(20) *For each* t, $\sum \left\{ m \atop s \right\}_b s^t$ *is independent of* r; $0 \leqslant r < b$, $0 \leqslant t < m$. *Here the sum is over* $s \equiv r \pmod b$.

Now $\left\{ m \atop s \right\}_b$ is merely the number of ways we can write s as the sum of j_0 ones, j_1 twos, j_2 threes, $\ldots, j_{b-2}$ $(b - 1)$'s, where

$$j_0 + 2j_1 + \ldots + (b - 1)j_{b-2} = s. \tag{21}$$

That is

$$\left\{ m \atop s \right\}_b = \sum m!/j_0! \ldots j_{b-1}!, \tag{22}$$

where $j_{b-1} = m - j_0 - \ldots - j_{b-2}$ and the sum is taken over all non-negative j_i satisfying (21).

Substituting into (20) we obtain for the sum

$$\sum \sum (m!/j_0! \ldots j_{b-1}!) s^t$$

where the inside sum is over $j_0 + 2j_1 + \ldots + (b-1)j_{b-2} = s$ and the outside sum is over $s \equiv r \pmod{b}$.

Finally we have the following theorem.

THEOREM 5. *For each t,*

$$\sum \frac{m!}{j_0! \ldots j_{b-1}!} (j_0 + 2j_1 + \ldots + (b-1)j_{b-2})^t$$

is independent of r; $0 \leqslant r < b$, $0 \leqslant t < m$. Here the sum is over

$$j_0 + 2j_1 + \ldots + (b-1)\, j_{b-2} = r \pmod{b}.$$

When b is taken to be 2 we obtain a proposition equivalent to (19) and therefore Theorem 5 is an actual extension of (19).

(Because of (22) one immediately sees the truth of the identity

$$(23) \qquad (1 + x + x^2 + \ldots + x^{b-1})^m \quad \sum_{s=0}^{m(b-1)} \left\{ \begin{matrix} m \\ s \end{matrix} \right\}_b x^s,$$

which provides us with a generating function for the $\left\{ \begin{matrix} m \\ s \end{matrix} \right\}_b$.)

4. In this section we derive several more consequences of Theorem 1. These will all be independent of the derivations in Section **3**.

1. Put

$$f_j(n) = (-1)^n \binom{n_j - 1}{n}, \qquad j = 1, \ldots, m.$$

Then for arbitrary $F_j(n)$ we know that (9) holds at least for $\alpha_j = 0$. Therefore, substituting into (11) yields

$$(24) \quad \sum_{a_0=0}^{n_1-1} \ldots \sum_{a_{m-1}=0}^{n_m-1} (-1)^{a_0 + \ldots + a_{m-1}} \binom{n_1 - 1}{a_0} \ldots \binom{n_m - 1}{a_m - 1} S_n^{\,t} = 0$$

for $0 \leqslant t < \alpha_1 + \ldots + \alpha_m + m$.

In particular, if $F_j(n) = n\beta_j$, $j = 1, \ldots, m$, the α_j of (9) are equal to $n_j - 2$ (as can be seen by using (19)). Thus

$$(25) \quad \sum_{a_0=0}^{n_1-1} \ldots \sum_{a_{m-1}=0}^{n_m-1} (-1)^{a_0 + \ldots + a_{m-1}} \binom{n_1 - 1}{a_0} \ldots \binom{n_m - 1}{a_m - 1}$$
$$\times (a_0 \beta_0 + \ldots + a_{m-1} \beta_{m-1})^t = 0 \qquad \text{for } 0 \leqslant t < n_1 + \ldots + n_m - m.$$

This special case of (24) is equivalent to (**5**, Theorem 5) and when all n_j are equal to b and $\beta_j = b^j$ is equivalent to (**3**, equation (7)).

2. Put

$$f_j(n) = n - (n_j - 1)/2, \qquad j = 1, \ldots, m.$$

Then for arbitrary $F_j(n)$, equation (9) is valid for $\alpha_j = 0$ for all j. Hence, from (11) we obtain

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$$\text{(26)}\quad \sum_{a_0=0}^{n_1-1} \cdots \sum_{a_{m-1}=0}^{n_m-1} \left(a_0 - \frac{n_1-1}{2}\right) \cdots \left(a_{m-1} - \frac{n_m-1}{2}\right) S_n{}^t = 0, \qquad 0 \leqslant t < m.$$

(*a*) When all $n_j = 2$,

$$a_j - \tfrac{1}{2} = \begin{cases} \tfrac{1}{2} & \text{for } a_j = 1, \\ -\tfrac{1}{2} & \text{for } a_j = 0, \end{cases}$$

and, putting $v_2(n)$ for the sum of the digits in the base 2 expansion of n, from (26) (suppressing a factor of $1/2^m$), we find:

$$\text{(27)}\qquad \sum_{n=0}^{2^m-1} (-1)^{v_2(n)} S_n{}^t = 0, \qquad 0 \leqslant t < m.$$

When $F_j(n) = np_j$, then $S_n = n$ and (27) is a special case of (25). This very special case gives another proof of the following well-known theorem.

THEOREM 6. *For each t, $0 \leqslant t < m$, the sum of the t-th powers of the integers from* 0 *to* $2^m - 1$, *inclusive, with an odd number of ones in their base* 2 *expansions equals the sum of the t-th powers of those integers in the same range with an even number of ones in their base* 2 *expansions.*

A numerical example for $m = 3$ is:

$$0^t + 3^t + 5^t + 6^t = 1^t + 2^t + 4^t + 7^t, \qquad t = 0, 1, 2.$$

(*b*) When all $n_j = 3$,

$$a_j - 1 = \begin{cases} -1 & \text{for } a_j = 0, \\ 0 & \text{for } a_j = 1, \\ 1 & \text{for } a_j = 2; \end{cases}$$

so, from (26) we find that

$$\text{(28)}\qquad \sum_{n=0}^{3^m-1} \prod_{j=0}^{m-1} (a_j - 1) S_n{}^t = 0, \qquad 0 \leqslant t < m.$$

The only non-zero terms in (28) arise from those n whose base 3 expansions have no digit 1.

In the special case $F_j(n) = np_j$, then $S_n = n$ and the above result yields the following theorem.

THEOREM 7. *For each t, $0 \leqslant t < m$, the sum of the t-th powers of the integers from* 0 *to* $3^m - 1$, *inclusive, having no ones and an odd number of twos in their base* 3 *expansions equals the sum of the t-th powers of those integers in the same range having no ones and an even number of twos.*

A numerical example for $m = 3$ is:

$$0^t + 8^t + 20^t + 24^t = 2^t + 6^t + 18^t + 26^t, \qquad t = 0, 1, 2.$$

3. Put

$$f_j(n) = (-1)^n - 1/n_j, \qquad j = 1, \ldots, m,$$

and suppose that all n_j are odd and equal to $2k + 1$. When $F_j(n) = n\beta_j$, then

$$\sum_{n=0}^{n_j-1} f_j(n)F_{j-1}^u(n) = \beta_{j-1}^u \sum_{n=0}^{2k} \left((-1)^n - \frac{1}{2k+1}\right)n^u$$

$$= \beta_{j-1}^u \frac{2k}{2k+1} \left((0^u + 2^u + 4^u + \ldots + (2k)^u) - \frac{k+1}{k}(1^u + 3^u + \ldots + (2k-1)^u)\right)$$

$$\begin{aligned} &= 0 \qquad \text{for } u = 0, 1, \\ &\neq 0 \qquad \text{for } u = 2. \end{aligned}$$

Thus, each α_j in (9) is equal to 1. Denoting the number of odd digits in the base $2k + 1$ expansion of n by $\theta_{2k+1}(n)$, we find from (11) (suppressing a factor of $2k/(2k + 1)$):

$$\sum_{n=0}^{(2k+1)^m-1} (-(k+1)/k)^{\theta_{2k+1}(n)} S_n{}^t = 0, \qquad 0 \leqslant t < 2m. \tag{29}$$

In the very special case $\beta_j = p_j$, $k = 1$ we find that

$$(-2)^{\theta_3(n)} = (-1)^{v_3(n)} (a_0{}^2) \ldots (a_{m-1}^2),$$

where $v_3(n)$ is the sum of the base 3 digits of n. In this particular case (29) is equivalent to (25) when all n_j are taken equal to 3. It should be noted, however, that when $\beta_j = p_j$, $k = 2$, equation (29) is *not* contained in (25).

References

1. D. H. Lehmer, *The Tarry-Escott problem*, Scripta Math., *13* (1947), 37–41.
2. M. E. Prouhet, *Mémoire sur quelques relations entre les puissances des nombres*, C. R. Acad. Sci., *33* (1851), 225.
3. J. B. Roberts, *A curious sequence of signs*, Amer. Math. Monthly, *64* (1957), 317–322.
4. ——— *A new proof of a theorem of Lehmer*, Can. J. Math., *10* (1958), 191–194.
5. ——— *Polynomial identities*, Proc. Amer. Math. Soc., *11* (1960), 723–730.
6. ——— *Splitting consecutive integers into classes with equal power sums*, Amer. Math. Monthly, *71* (1964), 25–37.
7. E. M. Wright, *Equal sums of like powers*, Proc. Edinb. Math. Soc., *8* (1949), 138–142.
8. ——— *Equal sums of like powers*, Bull. Amer. Math. Soc., *54* (1948), 755–757.
9. ——— *Prouhet's 1851 solution of the Tarry-Escott problem of 1910*, Amer. Math. Monthly, *66* (1959), 199–201.

University of London
and
Reed College