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ON THE KOBAYASHI PSEUDOMETRIC REDUCTION OF HOMOGENEOUS SPACES

BY

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ABSTRACT. Given any homogeneous complex manifold $X = G/H$, there exists a natural coset map $\pi:G/H \rightarrow G/K$ satisfying $\pi(x_1) = \pi(x_2)$ if and only if $d_X(x_1, x_2) = 0$, where d_X denotes the Kobayashi pseudometric on X. Its typical fiber $Z := K/H$ is a connected complex submanifold of X. Also G/K has a G-invariant complex structure, provided K satisfies a certain technical assumption (see Theorem 3). If Z is compact as well, then G/K is biholomorphic to a homogeneous bounded domain.

1. **Introduction.** On any connected complex manifold X one has the pseudometric d_X introduced by S. Kobayashi [9]. Thus one also has the equivalence relation $\sim$ defined by $x_1 \sim x_2$ for $x_1, x_2 \in X$ exactly if $d_X(x_1, x_2) = 0$. In general settings it is difficult to say very much about this. An instructive example, due to D. Eisenman and L. Taylor, see [11, p. 130], is the following. Let

$$X = \{ (z, w) \in \mathbf{C}^2 | \, |z| < 1, |zw| < 1 \quad \text{and} \quad |w| < 1 \text{ if } z = 0 \}.$$

Then the disk over $z = 0$ is one equivalence class, while all other equivalence classes consist of single points. This implies that the set of equivalence classes $X/\sim$ need not be locally compact, and thus may not have any complex structure. But the situation in the homogeneous setting is better behaved. For, if $X = G/H$, where G is a real Lie group acting holomorphically on X (for G complex, $d_X = 0$), then there is a closed subgroup K of G such that $X/\sim \, = G/K$ (see Section 2). We call the corresponding fibration $\pi:X = G/H \rightarrow X/\sim \, = G/K$ the *Kobayashi pseudometric reduction* or the *d-reduction* of the homogeneous space X.

In this short note we point out some of the properties of the d-reduction. The first remark, given in the second section, is that the equivalence classes are connected complex submanifolds of X. As well we note that if X has a G-invariant Hermitian metric, then the equivalence classes are not hyperbolic.

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Also we cite an example which shows that the equivalence classes do not always have trivial Kobayashi pseudometric. In the third section we prove that the set of equivalence classes $X/\sim\ = G/K$ has a left-invariant complex structure under the additional assumptions that K satisfies a certain technical condition.

In the fourth section we consider some instances of pushing down the pseudometric to the base of the d-reduction. The most important is when its fiber is compact and where, in order to show that this base is biholomorphic to a homogeneous bounded domain, we use the recent result of K. Nakajima [14] that for homogeneous spaces this is equivalent to being hyperbolic.

It is our pleasant duty to thank A. T. Huckleberry and W. Richthofer for helpful conversations concerning this note and J. Winkelmann for sending us his results. Some of the results in this paper were announced in the survey article [5].

2. **The fiber of the d-reduction.** Throughout we assume that $X = G/H$, where G is a connected real Lie group and H is a closed subgroup such that the coset space has a left-invariant complex structure. Since the Kobayashi pseudometric is distance decreasing with respect to holomorphic maps, biholomorphic maps act as isometries. Hence the partition of X given by the equivalence classes of its d-equivalence relation (defined above in Section 1) is G-invariant and so the stabilizer subgroup K of the equivalence class of eH is a closed subgroup of G containing H such that $X/\sim\ = G/K$, see [16, p. 144]. Thus the fibers of the real equivariant fibration $\pi: X = G/H \to X/\sim\ = G/K$ are exactly the equivalence classes of the d-equivalence relation. We make the following observation about these fibers.

THEOREM 1. *Suppose $X = G/H$ is a homogeneous complex manifold. Let $\pi: G/H \to G/K$ be its d-reduction. Then its fibers are connected complex submanifolds of X which are all biholomorphic to each other. Moreover, the subgroup K is itself connected.*

PROOF. In order to prove that K/H is a complex submanifold of X it suffices to show that the tangent space of K/H at any point x is a complex vector subspace of the tangent space $T_x(X)$. Recall that if v is a tangent vector to X at x, then the infinitesimal Kobayashi pseudometric $F_X(x, v)$ is defined to be the infimum of all positive numbers c such that there exists a holomorphic map f from the unit disk D into X with $f(0) = x$ and $f'(0) = v/c$, see [10, (2.16)]. Since the pseudodistance d_X is the integrated form of F_X, see [17], the unit disk D is homogeneous and f' is complex linear, it is clear that $v \in T_x(K/H)$ implies $iv \in T_x(K/H)$. Hence K/H is a complex submanifold of X and all fibers of π are biholomorphic to each other.

Next we note that d induces a metric on the base $Y := G/K$. If $y_1, y_2 \in Y$, then pick $x_i \in X$ with $\pi(x_i) = y_i$ for $i = 1, 2$. Then because of the triangle inequality and the fact that d_X is identically zero along the fibers of π, setting

$\hat{d}(y_1, y_2) := d_X(x_1, x_2)$ gives a well-defined G-invariant metric on Y.

We now use this metric to show that K/H is connected. Suppose not and let $\tilde{K}$ be the subgroup of K consisting of those connected components which meet H. Set $\tilde{Y} := G/\tilde{K}$. Then

$$X = G/H \xrightarrow{\tilde{\pi}} \tilde{Y} = G/\tilde{K} \xrightarrow{p} Y = G/K$$

is the Stein factorization of the map π, i.e., $\tilde{\pi}$ has connected fibers and p has discrete ones. With this set-up we can just repeat the proof in [9, Theorem 3.4] or [11, Theorem 4.7, p. 58], but for the metric $\hat{d}$. Namely, suppose $x_1, x_2 \in X$ with $\hat{d}(\pi(x_1), \pi(x_2)) = 0$. Then $\pi(x_1) = \pi(x_2)$, i.e., $d_X(x_1, x_2) = 0$. For any $\epsilon > 0$, but sufficiently small, there exists a neighborhood $\tilde{U}$ of $\tilde{\pi}(x_1)$ in $\tilde{Y}$ such that $p:\tilde{U} \to p(\tilde{U})$ is a diffeomorphism and $p(\tilde{U})$ is an ϵ-neighborhood of $\pi(x_1)$ with respect to $\hat{d}$. Since $d_X(x_1, x_2) = 0$, there exist points $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ of D and holomorphic maps $f_1, \ldots, f_k$ of D into X such that $x_1 = f_1(a_1)$, $f_i(b_i) = f_{i+1}(a_{i+1})$ for $i = 1, \ldots, k - 1$ and $f_k(b_k) = x_2$ and such that the sum of the Poincaré distances from a_i to b_i with i running from 1 to k is less than ϵ. Let $\overline{a_ib_i}$ denote the geodesic arc from a_i to b_i in D. Joining the curves $f_1(\overline{a_1b_1}), \ldots, f_k(\overline{a_kb_k})$ in X gives a curve C from x_1 to x_2 in X. Now the maps $\pi \circ f_1, \ldots, \pi \circ f_k$ are distance-decreasing; this depends only on the definition of $\hat{d}$ and not on the existence of a complex structure on Y or $\tilde{Y}$! This together with the fact that $\overline{a_1b_1}, \ldots, \overline{a_kb_k}$ are geodesics in D implies every point of the curve $\pi(C)$ must remain in the ϵ-neighborhood $p(\tilde{U})$ of $\pi(x_1)$. Hence $\tilde{\pi}(x_1) = \tilde{\pi}(x_2)$, i.e., the fibers of π are connected.

Finally consider the covering $G/H^0 \to G/H$, where H^0 denotes the connected component of the identity of H. Since two points in G/H^0 lie in the same equivalence class if and only if their images in G/H lie in the same equivalence class [10, Theorem 2.5], it is clear that $G/H^0 \to G/K$ is the d-reduction of G/H^0. Since *its* fiber is connected, it is clear that K itself must be connected. □

The next problem is to determine the Kobayashi pseudometric on the equivalence classes. As is pointed out by the example given in Section 1, this does not have to be identically zero. A similar situation occurs in the homogeneous setting, but first we note the following.

THEOREM 2. *Suppose $X = G/H$ is a homogeneous complex manifold having a G-invariant Hermitian metric. Let $G/H \to G/K$ be the d-reduction of X. Then its fiber K/H is not hyperbolic.*

PROOF. Assume X itself is not hyperbolic. It follows from the result of A. Kodama [12, Theorem 2] that there is a nonconstant holomorphic map $f:\mathbf{C} \to X$. Now if $x_1, x_2 \in X$ are in different fibers of the d-reduction, one has $d_X(x_1, x_2) > 0$. But this implies that the image of f lies in exactly one fiber of the d-reduction, i.e., K/H is not hyperbolic. $\square$

EXAMPLE (J. Winkelmann [19]). The group

$$G = \{(\mu, \lambda, s, t, u, v) \in \mathbf{R}^6 : \mu > 0, \lambda > 0\}$$

acts on $\mathbf{C}^3$ by the action

$$(x, y, z) \to (\lambda x + t, \lambda^{-1}\mu y + \lambda s x + u, \mu z + \lambda^2 x^2 s + 2u\lambda x + v).$$

There are four open orbits and for definiteness we consider

$$X := \{(x, y, z) \in \mathbf{C}^3 : \mathrm{Im}(f_1) > 0, \mathrm{Im}(f_2) > 0\},$$

where $f_1(x, y, z) := x$ and $f_2(x, y, z) := 2iy\,\mathrm{Im}(x) - z$.

The Levi form of $\mathrm{Im}(f_2)$ has one positive and one negative eigenvalue and the G-action extends to the envelope of holomorphy of X. Thus the bounded holomorphic reduction (see [4]) of X is the projection onto the first factor $(x, y, z) \to x$. As the fiber of this projection restricted to X is $\mathscr{H}^+ \times \mathbf{C}$, where $\mathscr{H}^+$ denotes the upper half plane, it follows that this fiber has nonconstant bounded holomorphic functions.

Now through each point $(x, y, z) \in X$ there is precisely one complex line $\{(x, y + \alpha, z + 2i\alpha\,\mathrm{Im}(x)) : \alpha \in \mathbf{C}\}$ which lies in X. Thus there is a G-equivariant fibration $(x, y, z) \to (x, f_2(x, y, z))$. By a Lie algebra calculation one can see that the base of this "complex line reduction" has no G-invariant complex structure.

Next we note that $p_1 = (i, 0, -i)$ and

$$p_2 = \left(i, \frac{1}{2\epsilon}, \frac{i}{\epsilon} - i\right)$$

are joined by a complex line in X and

$$f(\tau) := \left(i + \tau, \frac{1}{2\epsilon}, \frac{1}{\epsilon}(i + \tau) - i\right)$$

joins p_2 to

$$p_3 = \left(i + \epsilon, \frac{1}{2\epsilon}, \frac{i}{\epsilon} + 1 - i\right).$$

Another complex line joins p_3 to $p_4 = (i + \epsilon, 0, 1 - i)$ and a disk joins p_4 to $p_5 = (i, 0, 1 - i)$. This implies $d_X(p_1, p_5) = 0$ and hence the Kobayashi reduction and the bounded holomorphic reduction of X coincide. Thus $d_{K/H}$ need not be identically zero, even if X admits a G-invariant Hermitian metric! In passing we note that $K/H = \{d_{U_\epsilon} = 0\}$, where $U_\epsilon := \{y \in X : d_X(x, y) < \epsilon\}$, see [10, p. 401].

3. **The base of the d-reduction.** As is pointed out by the "complex line reduction" in the example of Winkelmann, there may not be any invariant complex structure on a quotient obtained by factoring by a j-invariant subalgebra, not even when that subalgebra arises from seemingly natural geometric considerations. However, for subalgebras of a certain form, there does exist an invariant complex structure on the quotient.

We now consider the following set-up. Suppose G/H has a G-invariant complex structure and U is a normal subgroup in G. Letting German letters denote the corresponding Lie algebras, we assume that $\mathfrak{w} := \mathfrak{u} + j\mathfrak{u}$ contains $\mathfrak{h}$, where j denotes the almost complex structure on $\mathfrak{g}$. We denote by W the corresponding Lie group of G which, in general, need not be closed. If W is closed, then we have the following.

LEMMA. *In the above set-up, suppose the subgroup W is closed. Then G/W has a G-invariant complex structure such that the natural map $\pi: G/H \to G/W$ is holomorphic.*

PROOF. We start with the G-invariant complex structure on G/H and note that this induces an endomorphism j of $\mathfrak{g}$ satisfying

1) $j^2 \equiv -1 \bmod \mathfrak{h}$.

2) $[\xi, j\eta] \equiv j[\xi, \eta] \bmod \mathfrak{h}$ for all $\xi \in \mathfrak{h}, \eta \in \mathfrak{g}$.

3) $[j\xi, j\eta] \equiv [\xi, \eta] + j[j\xi, \eta] + j[\xi, j\eta] \bmod \mathfrak{h}$, for all $\xi, \eta \in \mathfrak{g}$.

In order to show that j induces a G-invariant complex structure on G/W one has to check that

2′) $[\xi, j\eta] \equiv j[\xi, \eta] \bmod \mathfrak{w}$ for all $\xi \in \mathfrak{w}, \eta \in \mathfrak{g}$.

To do this we suppose $\mathfrak{u} = \langle \xi_1, \ldots, \xi_r \rangle$ and $\eta, j\eta$ are arbitrary in $\mathfrak{g}$. The table of the Lie algebra for $\mathfrak{g}$ contains the following

$$\begin{array}{c|cc} & \eta & j\eta \\ \hline \xi i & A_i & B_i \\ j\xi_i & C_i & D_i \end{array},$$

where $D_i \equiv A_i + jB_i + jC_i \bmod \mathfrak{h}$ by 3) above. Since $\mathfrak{u}$ is an ideal in $\mathfrak{g}$, we have that $A_i, B_i \in \mathfrak{u}$. Now for $j\xi_i$ equation 2′) becomes

$$\begin{aligned}[j\xi_i, j\eta] - j[j\xi_i, \eta] &= D_i - jC_i \\ &\equiv A_i + jB_i \bmod \mathfrak{h} \\ &\equiv 0 \bmod \mathfrak{w}.\end{aligned}$$

Since $\mathfrak{w} = \langle \xi_1, \ldots, \xi_r, j\xi_1, \ldots, j\xi_r \rangle$, the result follows. □

REMARK. If G is solvable and simply connected, then W is closed by Chevalley's Theorem [2].

Since the subgroup K is closed, we can apply the Lemma to G/K, whenever $\mathfrak{k} = \mathfrak{u} + j\mathfrak{u}$.

THEOREM 3. *Suppose $X = G/H$ is a homogeneous complex manifold. Let $\pi: G/H \to G/K$ be its d-reduction. If $\mathfrak{k} = \mathfrak{u} + j\mathfrak{u}$, where $\mathfrak{u}$ is an ideal in $\mathfrak{g}$, then G/K has a G-invariant complex structure such that the map π is holomorphic.*

REMARK. The condition $\mathfrak{k} = \mathfrak{u} + j\mathfrak{u}$ is difficult to verify and is not directly related to the d-reduction. We include it as an instance where the base has a G-invariant complex structure.

4. **Pushing down the pseudometric.** In general it seems to be a difficult problem to find conditions under which one can "push down" the Kobayashi pseudometric. The aim of this section is to point out a few cases where it is possible to do this.

First we mention in passing that M. Nag [13] has noted that if the total space of a holomorphic fiber bundle is hyperbolic, then so is its base. One has only to observe that the pull-back of the bundle to the universal covering of its base is trivial by a result of H. Royden [18]. Of more use to us at the moment is the following result of M. A. Illarionov [8]. He has pointed out that if $p: E \to B$ is a holomorphic fiber bundle with fiber F satisfying $d_F = 0$ and with a complex Lie group as structure group, then $d_E(x_1, x_2) = d_B(p(x_1), p(x_2))$ for all $x_1, x_2 \in E$. The key point is to note that if $f: D \to B$ is any holomorphic map of the unit disk D into B, then the pull-back f^*E is trivial by Grauert's Oka principle [6].

Now we present an easy consequence of Illarionov's remark which is of interest in the present setting.

THEOREM 4. *Suppose $X = G/H \to G/K$ is the d-reduction of the homogeneous complex manifold X, and suppose G/K has a G-invariant complex structure. If the fiber K/H is compact, then the base G/K is biholomorphic to a homogeneous bounded domain. Indeed, X is biholomorphic to the product $K/H \times G/K$.*

PROOF. For K/H compact the identity component $\mathrm{Aut}^0(K/H)$ is a complex Lie group [1] and acts transitively on K/H. Thus $d_{K/H} = 0$. By a theorem of Fischer-Grauert [3], the d-reduction is a holomorphic bundle. Hence by the above-mentioned result of Illarionov, the base G/K is hyperbolic and thus biholomorphic to a homogeneous bounded domain [14]. The holomorphic triviality of this bundle follows from Grauert's Oka principle [6] and the fact that every homogeneous bounded domain is contractible. □

REMARKS. In the classification of holomorphic real Lie group actions on complex manifolds there are several fibrations which play a central role. These include the $\mathfrak{g}$-anticanonical fibration and the $\mathfrak{m}'$-fibration, see [7] and [15]. Because of Illarionov's result it is clear that the Kobayashi pseudometric pushes down these fibrations too.

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