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RIGHT-DEFINITE MULTIPARAMETER STURM–LIOUVILLE PROBLEMS WITH EIGENPARAMETER-DEPENDENT BOUNDARY CONDITIONS

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Abstract We study a system of ordinary differential equations linked by parameters and subject to boundary conditions depending on parameters. We assume certain definiteness conditions on the coefficient functions and on the boundary conditions that yield, in the corresponding abstract setting, a right-definite case. We give results on location of the eigenvalues and oscillation of the eigenfunctions.

Keywords: multiparameter Sturm–Liouville boundary-value problems; eigenparameter-dependent boundary conditions; eigenvalues; oscillation counts; Minkowski definiteness conditions

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1. Introduction

The one-parameter Sturm–Liouville differential equation

$$-(py')' + qy = \lambda ry \tag{1.1}$$

subject to boundary conditions

$$b_0 y(0) = d_0 (py')(0) \tag{1.2}$$

and

$$b_1 y(1) = d_1 (py')(1), \tag{1.3}$$

where p, p', q and r are continuous functions on $[0,1]$ with p and r positive, and $(b_s, d_s) \in \mathbb{R}^2 \backslash \{0\}$, $s = 0, 1$, has countably many real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$, accumulating at infinity, each with (up to a sign) unique eigenfunction y_m

with $\|y_m\|_2 = 1$. The eigenfunctions $\{y_m\}_{m=0}^{\infty}$ are complete in $L_2[0,1]$, and y_m possesses exactly m roots in $(0,1)$, i.e. y_m has an *oscillation count* equal to m (see [**9**, Chapter 8] or [**15**] for all of these).

Among the many generalizations of these results [**8**, **10**, **18**], Binding, Browne and Seddighi [**8**] were interested in the case in which the boundary conditions (1.2) and/or (1.3) are replaced by eigenparameter-dependent boundary conditions

$$(a_0\lambda + b_0)y(0) = (c_0\lambda + d_0)(py')(0) \tag{1.4}$$

and

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)(py')(1), \tag{1.5}$$

where

$$a_0d_0 - b_0c_0 < 0, \quad c_0 \neq 0 \qquad \text{and} \qquad a_1d_1 - b_1c_1 > 0, \quad c_1 \neq 0. \tag{1.6}$$

There are still countably many eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$, accumulating at infinity, each with (up to a sign) unique eigenfunction y_m with $\|y_m\|_2 = 1$, but the oscillation pattern changes. All the oscillation counts occur. However, there is a repeated oscillation count if either boundary conditions (1.2) and (1.5) or boundary conditions (1.3) and (1.4) are assumed, and there are two double-oscillation counts or a triple-oscillation count if (1.4) and (1.5) are assumed. Here conditions (1.4) or (1.5) are always assumed together with the corresponding conditions in (1.6). We refer to [**8**] for details on all of these. There exists an orthonormal basis of eigenvectors of the induced self-adjoint operator on $L_2[0,1] \oplus \mathbb{C}^2$ (or on $L_2[0,1] \oplus \mathbb{C}$ if only one boundary condition is replaced) (see [**10**]). We remark that these results may fail if the sign conditions in (1.6) are omitted. Then non-real and non-semisimple eigenvalues may occur [**6**, **7**]. However, if sign conditions are kept but $c_i = 0$, $i = 0,1$, then the situation is simpler: Sturm's Theorem holds and there is no repetition of the oscillation counts [**8**, Corollary 5.2].

In the multiparameter generalizations of the theory, one considers the equations

$$-(p_jy_j')' + q_jy_j = \left(\sum_{k=1}^{n} \lambda_k r_{jk}\right) y_j, \quad j = 1,2,\ldots,n, \tag{1.7}$$

where the p_j, p_j', q_j and r_{jk} are real and continuous functions on $[0,1]$ and the p_j are positive on $[0,1]$.

Assume for a moment that $n = 2$. Under the separated end conditions

$$y_j(0)\cos\alpha_j = (p_jy_j')(0)\sin\alpha_j$$

and

$$y_j(1)\cos\beta_j = (p_jy_j')(1)\sin\beta_j, \quad j = 1,2,$$

with $\det[r_{jk}]_{j,k=1}^2 > 0$, which is known as right definiteness, Klein's Oscillation Theorem states that for each non-negative integer pair (n_1, n_2) there is a unique eigenvalue

$\boldsymbol{\lambda}^{(n_1,n_2)} \in \mathbb{R}^2$ and (up to scalar multiples) a unique pair of eigenfunctions $y_j^{(n_1,n_2)}$ with n_j zeros in $(0,1)$. We refer to Ince [**14**] for the oscillation theorem and to [**17**] for oscillation results under weaker conditions on the coefficients and under alternative definiteness conditions.

In this paper we study the existence and location of eigenvalues, and oscillation of eigenfunctions for the equations (1.7) subject to boundary conditions

$$(a_{j0}\lambda_j + b_{j0})y_j(0) = (c_{j0}\lambda_j + d_{j0})(p_j y_j')(0) \tag{1.8}$$

and

$$(a_{j1}\lambda_j + b_{j1})y_j(1) = (c_{j1}\lambda_j + d_{j1})(p_j y_j')(1). \tag{1.9}$$

The oscillation theory in this case has been studied only recently. The two-parameter problem with eigenparameter-dependent boundary conditions was considered by Bhattacharyya, Binding and Seddighi in [**2**], where it was shown that there can be at most four eigenvalues corresponding to the same oscillation count. Our results are a multiparameter generalization of [**2**].

The paper is organized as follows. In § 2 we formulate the assumptions which we shall work with. We assume the so-called Minkowski definiteness conditions on the functions r_{jk}, $j,k = 1,2,\dots,n$, together with $c_{js} \neq 0$ and certain sign conditions on numbers:

$$\omega_{js} = a_{js}d_{js} - b_{js}c_{js}, \quad s = 0,1.$$

We also find a lower bound for the singular values of a Minkowski matrix. In § 3 we consider the special case when the boundary conditions depend on the parameters only at one end. The existence and the oscillation theorems depend on the behaviour of the eigensurfaces. Using the results of § 3 we consider the general case in § 4.

2. Preliminaries

By a transformation of the independent variable, we can assume without loss of generality that the p_j, $j = 1,2,\dots,n$, are identically equal to 1 (see [**8**, Appendix]). Then differential equations (1.7) become

$$-y_j'' + q_j y_j = \left(\sum_{k=1}^{n} \lambda_k r_{jk}\right) y_j, \quad j = 1,2,\dots,n, \tag{2.1}$$

and the boundary conditions (1.8) and (1.9) become

$$(a_{j0}\lambda_j + b_{j0})y_j(0) = (c_{j0}\lambda_j + d_{j0})y_j'(0) \tag{2.2}$$

and

$$(a_{j1}\lambda_j + b_{j1})y_j(1) = (c_{j1}\lambda_j + d_{j1})y_j'(1), \tag{2.3}$$

respectively.

To begin with, we fix some notation. For a function y in $L^2[0,1]$, we denote by $\bar{r}_{jk}(y)$ the integral $\int_0^1 r_{jk}|y|^2$. If $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ is an n-tuple of functions in $L^2[0,1]$, then we denote by $\rho_0(\boldsymbol{y})$ the determinant $\det[\bar{r}_{jk}(y_j)]_{j,k=1}^n$. We write B_1 for the unit ball of $L^2[0,1]$.

In what follows we use the following assumptions.

(I) q_j and r_{jk}, $j,k = 1,2,\dots,n$, are real and continuous functions on $[0,1]$.

(II) (a) $a_{j0} = c_{j0} = 0$, $(b_{j0}, d_{j0}) \neq (0,0)$, $j = 1,2,\dots,n$; or

(b) $\omega_{j0} < 0$ and $c_{j0} \neq 0$ for $j = 1,2,\dots,n$.

(III) $\omega_{j1} > 0$ and $c_{j1} \neq 0$ for $j = 1,2,\dots,n$.

(IV) $\bar{r}_{jk}(y) \leqslant 0$ for $j,k = 1,2,\dots,n$, $j \neq k$, and for all $y \in L^2[0,1]$, $y \neq 0$.

(V) $\sum_{k=1}^n \bar{r}_{jk}(y) > 0$ for $j = 1,2,\dots,n$ and for all $y \in L^2[0,1]$, $y \neq 0$.

By scaling the constants a_{js}, b_{js}, c_{js} and d_{js} we can replace the inequalities in assumptions (II) (b) and (III) by $\omega_{j0} = -1$ and $\omega_{j1} = 1$, respectively. We assume that these simplifications are made.

Following [**5**] we call assumptions (IV) and (V) the *Minkowski conditions*. Since we assume (I), i.e. r_{jk} are continuous functions, it follows that the Minkowski condition (V) is *uniform*, i.e. there exists a constant $\gamma > 0$ such that for all $y \in B_1$ and $j = 1,2,\dots,n$,

(V$'$) $\sum_{k=1}^n \bar{r}_{jk}(y) > \gamma$.

After an invertible transformation of parameters is performed, the uniform Minkowski conditions follow from uniform right definiteness and uniform ellipticity conditions [**5**, pp. 19, 23]. The latter conditions are more familiar in the literature on multiparameter spectral theory [**1**, **4**, **16**, **17**]. A system of equations (2.1) (or, more generally, a system of equations (1.7)) is called *uniformly right definite* if there exists a constant $\gamma > 0$ such that $\rho_0(\boldsymbol{y}) > \gamma$ for all $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in B_1^n$, and it is called *uniformly elliptic* if there exist $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ and $\gamma' > 0$ such that $\sum_{k=1}^n \alpha_k \rho_{0jk}(\boldsymbol{y}) > \gamma'$ for all $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in B_1^n$ and all j. Here $\rho_{0jk}(\boldsymbol{y})$ is the cofactor of $\rho_0(\boldsymbol{y})$ corresponding to $\bar{r}_{jk}(y_j)$. Note that if we assumed uniform right definiteness and uniform ellipticity it would not be possible, in general, to obtain the Minkowski conditions by an invertible linear transformation of parameters without losing the form of boundary conditions (2.2) and (2.3) and assumptions (II) (b) and (III). After such a transformation of parameters, more general boundary conditions are obtained from (2.2) and (2.3); namely, each λ_j is replaced by a linear combination of all the eigenparameters λ_k, $k = 1,2,\dots,n$. (Multiparameter Sturm–Liouville problems with these general boundary conditions are studied in [**3**].) However, before eigenvalue and oscillation theory for such multiparameter problems can be discussed, some further analysis of one-parameter Sturm–Liouville differential equations with eigenparameter-dependent boundary conditions would be required. Here we follow in the path of [**2**]; we assume the stronger conditions and apply the available one-parameter analysis of Binding, Browne and Seddighi [**8**].

At the end of this section we introduce the notion of a Minkowski matrix and give a bound for its minimal singular value.

A real matrix $A = [a_{jk}]_{j,k=1}^n$ is called a *Minkowski matrix* if the following conditions hold:

(1) $a_{jk} \leqslant 0$ for $j, k = 1, 2, \dots, n$, $j \neq k$; and

(2) $\sum_{k=1}^n a_{jk} \geqslant \gamma > 0$ for $j = 1, 2, \dots, n$.

The constant γ above is called *a bound* of the Minkowski matrix A. Note that conditions (1) and (2) imply that $a_{jj} > 0$ for $j = 1, 2, \dots, n$.

Lemma 2.1. *If A is a Minkowski matrix with a bound γ and $\sigma_n(A)$ is its minimal singular value, then*

$$\sigma_n(A) \geqslant (\gamma/\sqrt{n}).$$

Proof. The minimal singular value satisfies a relation $\sigma_n(A) = \min_{\|x\|_2=1} \|Ax\|_2$ (see, for example, [**11**, p. 428]). We choose a vector $x = [x_j]_{j=1}^n$ with a norm $\|x\|_2 = 1$. Suppose that k is such that $|x_k| \geqslant |x_j|$ for $j = 1, 2, \dots, n$. Then we have

$$\begin{aligned} \left|\sum_{j=1}^n a_{kj} x_j\right| \geqslant |a_{kk} x_k| - \left|\sum_{j=1,\ j\neq k}^n a_{kj} x_j\right| &\geqslant a_{kk}|x_k| + \sum_{j=1,\ j\neq k}^n a_{kj}|x_j| \\ &\geqslant \left(\sum_{j=1}^n a_{kj}\right)|x_k| \geqslant \gamma|x_k|. \end{aligned}$$

Because we assume that $\|x\|_2 = 1$ it follows that $|x_k| \geqslant (1/\sqrt{n})$. The above inequality implies that $\|Ax\|_2 \geqslant (\gamma/\sqrt{n})$. □

3. Eigenvalues in the case in which boundary conditions at one end depend on eigenparameter

We first consider in detail the problem (2.1), (2.2) and (2.3) under assumptions (I), (II) (a) and (III)–(V) and study the properties of the corresponding eigenvalue hypersurfaces. This is a generalization of two-parameter results proved in [**2**]. The proofs here are similar and depend on results in [**8**]. A crucial new step is an application of Hadamard's Inverse Function Theorem [**12**, Theorem A].

Let us now fix j and consider Sturm–Liouville problem (2.1), (2.2) and (2.3) under assumptions (II) (a) and (III)–(V). We write $\boldsymbol{\lambda}_j$ for the set of parameters λ_l, $l \neq j$.

Lemma 3.1. *There exists an infinite sequence $\lambda_j = \lambda_j^{(m)}(\boldsymbol{\lambda}_j)$, $m = 0, 1, 2, \dots$, of real eigenvalue hypersurfaces. Each of the functions $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ depends continuously on all $\lambda_l \in \boldsymbol{\lambda}_j$, and for each value $\boldsymbol{\lambda}_j \in \mathbb{R}^{n-1}$ the sequence of eigenvalues $\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\}_{m=0}^\infty$ is strictly increasing.*

Proof. We fix $j = 1$ for simplicity. We view the boundary-value problem

$$-y_1'' + \left(q_1 - \sum_{k=2}^{n} \lambda_k r_{1k}\right) y_1 = \lambda_j r_{11} y_1$$

together with (2.2) and (2.3) as a parametrized one-parameter Sturm–Liouville boundary-value problem with eigenparameter-dependent boundary condition. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with required properties follows by [**8**, Theorems 3.1 and 3.2]. □

Lemma 3.2. *For each eigenvalue $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ there exists a real eigenfunction $y_j^{(m)} = y_j^{(m)}(x, \boldsymbol{\lambda}_j^{(m)})$ with $\|y_j^{(m)}\| = 1$ for all $\boldsymbol{\lambda}_j$ and such that for each $x \in [0, 1]$ and each compact set $K_j \subset \mathbb{R}^{n-1}$ the eigenfunction $y_j^{(m)}$ and its derivative with respect to x depend continuously on $\boldsymbol{\lambda}_j \in K_j$. Furthermore, there exists a sequence of natural numbers $N_j^{(m)} = N_j^{(m)}(\boldsymbol{\lambda}_j)$, $m = 0, 1, 2, \dots,$ such that $y_j^{(m)}$ has m zeros on the interval $(0, 1)$ for $m \leqslant N_j^{(m)}$ and $m - 1$ zeros on $(0, 1)$ for $m > N_j^{(m)}$.*

Proof. The proof is similar to the proof of [**2**, Lemma 2.2]. For simplicity we fix $j = 1$ and suppress it. Let

$$\boldsymbol{y} = \begin{pmatrix} y \\ \dfrac{\mathrm{d}}{\mathrm{d}x} y \end{pmatrix} \quad \text{and} \quad A(x, \boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 \\ q - \lambda(\boldsymbol{\lambda}^{(m)}) r_1 - \displaystyle\sum_{l=2}^{n} \lambda_l r_l & 0 \end{pmatrix}.$$

Then $\boldsymbol{y}$ is a solution of

$$\boldsymbol{y}' = A(x, \boldsymbol{\lambda}) \boldsymbol{y}.$$

Observe that A is a continuous function of x and $\boldsymbol{\lambda}$. Then for $\boldsymbol{\lambda}$ lying in a compact subset K the operator norm $\|A(x, \boldsymbol{\lambda})\|$ on $L^2[0, 1] \oplus L^2[0, 1]$ has an upper bound which may depend on x. Then the function $f_{\boldsymbol{\lambda}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f_{\boldsymbol{\lambda}}(x, \alpha) = A(x, \boldsymbol{\lambda})\alpha$, for $x \in [0, 1]$ and $\alpha \in \mathbb{R}^2$ is Lipschitz. The continuity of $y^{(m)}(x, \boldsymbol{\lambda})$ and $(\mathrm{d}/\mathrm{d}x) y^{(m)}(x, \boldsymbol{\lambda})$ then follows by [**13**, Theorem 3.2] using the same arguments as in the proof of [**2**, Lemma 2.2]. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda})$ with required properties follows by [**8**, Theorem 3.1]. □

Theorem 3.3. *The partial derivative of $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ with respect to $\lambda_l \in \boldsymbol{\lambda}_j$ exists and is equal to*

$$\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l}(\boldsymbol{\lambda}_j) = -\left(\bar{r}_{jj}(y_j^{(m)}) + \frac{y_j^{(m)}(1)^2}{(c_{j1}\lambda_j^{(m)} + d_{j1})^2}\right)^{-1} \bar{r}_{jl}(y_j^{(m)}), \tag{3.1}$$

where $y_j^{(m)}(1) = y_j^{(m)}(1, \boldsymbol{\lambda}_j)$. Moreover, the derivative

$$\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l}(\boldsymbol{\lambda}_j)$$

is continuous, positive and bounded on the entire $\mathbb{R}^{n-1}$.

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Proof. For simplicity we assume that $j = 1$ and $l = 2$. We write $\boldsymbol{\lambda}'$ for the set of remaining parameters λ_r, $r = 3, 4, \dots, n$, and fix $\boldsymbol{\lambda}' \in \mathbb{R}^{n-2}$ and a non-negative integer m. Since m is fixed we suppress it.

Let $y_1 = y_1(x, \lambda_2, \boldsymbol{\lambda}')$ be the eigenfunction corresponding to $\lambda_1(\lambda_2, \boldsymbol{\lambda}')$ and $z_1 = z_1(x, \lambda_2 + \epsilon, \boldsymbol{\lambda}')$ be the eigenfunction corresponding to $\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}')$ for some $\epsilon > 0$. So we have

$$-y_1'' + q_1 y_1 = \left(\lambda_1(\lambda_2, \boldsymbol{\lambda}') r_{11} + \lambda_2 r_{12} + \sum_{t=2}^{n} \lambda_t r_{1t}\right) y_1 \tag{3.2}$$

and

$$-z_1'' + q_1 z_1 = \left(\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}') r_{11} + (\lambda_2 + \epsilon) r_{12} + \sum_{t=2}^{n} \lambda_t r_{1t}\right) z_1. \tag{3.3}$$

Multiplying the first equation by z_1 and the second by y_1, subtracting and integrating, we obtain

$$(y_1' z_1 - y_1 z_1')|_0^1 = (\lambda_1(\lambda_2 + \epsilon, \boldsymbol{\lambda}') - \lambda_1(\lambda_2, \boldsymbol{\lambda}')) \int_0^1 r_{11} y_1 z_1 + \epsilon \int_0^1 r_{12} y_1 z_1. \tag{3.4}$$

Dividing by ϵ and using the continuity established in Lemmas 3.1 and 3.2, we have

$$-\left(\frac{y_1(1)^2 \omega_{11}}{(c_{11}\lambda_1 + d_{11})^2}\right) \frac{\partial \lambda_1}{\partial \lambda_2} = \bar{r}_{11}(y_1) \frac{\partial \lambda_1}{\partial \lambda_2} + \bar{r}_{12}(y_1).$$

Then

$$\frac{\partial \lambda_1}{\partial \lambda_2} = -\left(\bar{r}_{11}(y_1) + \frac{y_1(1)^2}{(c_{11}\lambda_1 + d_{11})^2}\right)^{-1} \bar{r}_{12}(y_1). \tag{3.5}$$

Since y_1 and $\bar{r}_{jk}$ are continuous it follows that $\partial \lambda_1 / \partial \lambda_2$ is continuous. Note that $\|y_1\| = 1$ by Lemma 3.2. Then the Minkowski condition (IV) and identity (3.5) imply that $(\partial \lambda_1 / \partial \lambda_2) > 0$ for all $(\lambda_2, \boldsymbol{\lambda}') \in \mathbb{R}^{n-1}$. By the continuity of r_{12} it follows that $M_{12} = \max\{r_{12}(x);\ 0 \leqslant x \leqslant 1\}$ is finite. The uniform Minkowski conditions imply that $\bar{r}_{11}(y_1) > n\gamma$. Using these and identity (3.5) it follows that

$$\frac{\partial \lambda_1}{\partial \lambda_2}(\boldsymbol{\lambda}') < \frac{M_{12}}{n\gamma}$$

for all $\boldsymbol{\lambda}' \in \mathbb{R}^{n-1}$.

For other derivatives, one carries out the same calculation with 1 and 2 replaced by j and l, respectively. □

For each n-tuple $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ of non-negative integers we consider the set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \dots, n$. We fix $\boldsymbol{m}$ and, for brevity of notation, suppress it. Consider next the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) =$

$(\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$. Assume that $y_j = y_j(x, \boldsymbol{\lambda}_j)$ is the eigenfunction corresponding to $\lambda_j(\boldsymbol{\lambda}_j)$ and write

$$f_{j1}(y_j) = -\frac{y_j(1)}{c_{j1}\lambda_j + d_{j1}}.$$

By Theorem 3.3 it follows that function F is a C^1-function. Its Jacobian matrix is equal to

$$J(F) = \begin{pmatrix} 1 & -\dfrac{\partial\lambda_1}{\partial\lambda_2} & \cdots & -\dfrac{\partial\lambda_1}{\partial\lambda_n} \\ -\dfrac{\partial\lambda_2}{\partial\lambda_1} & 1 & \cdots & -\dfrac{\partial\lambda_2}{\partial\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\partial\lambda_n}{\partial\lambda_1} & -\dfrac{\partial\lambda_n}{\partial\lambda_2} & \cdots & 1 \end{pmatrix}.$$

Lemma 3.4. *The determinant of the Jacobian matrix $J(F)$ is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$.*

Proof. Recall that the uniform Minkowski condition (V′) holds. Then $\bar{r}_{jk}(y_j) \leqslant 0$ for $j \neq k$ and $\sum_{k=1}^n \bar{r}_{jk}(y_j) \geqslant \gamma > 0$. Let s_j be the sum of the entries of the jth row of the Jacobian matrix $J(F)$. Take $j = 1$ and apply Theorem 3.3 to show that

$$s_1 = 1 - \sum_{k=2}^n \frac{\partial\lambda_1}{\partial\lambda_k} = 1 + \sum_{k=2}^n \frac{\bar{r}_{1k}(y_1)}{\bar{r}_{11} + f_{11}(y_1)^2} \geqslant 1 + \sum_{k=2}^n \frac{\bar{r}_{1k}(y_1)}{\bar{r}_{11}} \geqslant \frac{\gamma}{R} > 0,$$

where $R = \max\{\bar{r}_{kk}(y_k);\ k = 1, 2, \dots, n\}$. In a similar way we see that $s_j \geqslant (\gamma/R) > 0$ for $j = 2, 3, \dots, n$. The Gershgorin Circle Theorem (see, for example, [**11**, p. 341]) then implies that there is a constant $\beta > 0$ such that real parts of all the eigenvalues of $J(F)$ are greater than β. Since non-real eigenvalues, if any, occur in conjugate pairs, it follows that the determinant $\det J(F)$ is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$. □

Lemma 3.5. *The function F is* proper *[**12**], i.e. $\|\boldsymbol{\lambda}\|_2 \to \infty$ implies $\|F(\boldsymbol{\lambda})\|_2 \to \infty$.*

Proof. We write $F = (F_j)_{j=1}^n$. The inner product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ is denoted by $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$, and the p-norm of a vector $\boldsymbol{a} \in \mathbb{R}^n$ is denoted by $\|\boldsymbol{a}\|_p$. By the Mean Value Theorem applied to $F_j : \mathbb{R}^n \to \mathbb{R}$ and vectors $\boldsymbol{a}, \boldsymbol{\lambda} \in \mathbb{R}^n$ there is a vector $\boldsymbol{b}_j$ in the convex hull of the set $\{\boldsymbol{a}, \boldsymbol{\lambda}\}$ such that

$$(F_j(\boldsymbol{\lambda}) - F_j(\boldsymbol{a}))^2 = \langle \operatorname{grad} F_j(\boldsymbol{b}_j), \boldsymbol{\lambda} - \boldsymbol{a} \rangle, \quad j = 1, 2, \dots, n, \tag{3.6}$$

where $\operatorname{grad} F_j(\boldsymbol{b}_j)$ is the gradient of F_j at $\boldsymbol{b}_j$. By the definition of F it follows that

$$\operatorname{grad} F_j(\boldsymbol{b}_j) = \left(-\frac{\partial\lambda_j}{\partial\lambda_1}(\boldsymbol{b}_j) \quad -\frac{\partial\lambda_j}{\partial\lambda_2}(\boldsymbol{b}_j) \quad \cdots \quad -\frac{\partial\lambda_j}{\partial\lambda_n}(\boldsymbol{b}_j) \right).$$

Next we consider the matrix

$$G = \begin{pmatrix} 1 & -\dfrac{\partial\lambda_1}{\partial\lambda_2}(\boldsymbol{b}_1) & \cdots & -\dfrac{\partial\lambda_1}{\partial\lambda_n}(\boldsymbol{b}_1) \\ -\dfrac{\partial\lambda_2}{\partial\lambda_1}(\boldsymbol{b}_2) & 1 & \cdots & -\dfrac{\partial\lambda_2}{\partial\lambda_n}(\boldsymbol{b}_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\partial\lambda_n}{\partial\lambda_1}(\boldsymbol{b}_n) & -\dfrac{\partial\lambda_n}{\partial\lambda_2}(\boldsymbol{b}_n) & \cdots & 1 \end{pmatrix}.$$

We apply Theorem 3.3 and use the uniform Minkowski conditions to prove that G is a Minkowski matrix with a bound γ. Calculations are similar to those in the proof of Lemma 3.4 and we omit them. Next it follows by relations (3.6) and Lemma 2.1 that

$$\|F(\boldsymbol{\lambda}) - F(\boldsymbol{a})\|_4^2 = \|G(\boldsymbol{\lambda} - \boldsymbol{a})\|_2 \geqslant (\gamma/\sqrt{n})\|\boldsymbol{\lambda} - \boldsymbol{a}\|_2.$$

Finally, if $\|\boldsymbol{\lambda}\|_2 \to \infty$, then $\|F(\boldsymbol{\lambda})\|_2 \to \infty$, since the 2-norm and the 4-norm on $\mathbb{R}^n$ are equivalent. Hence F is a proper function. □

Theorem 3.6. *For each n-tuple $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ of non-negative integers, the set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \dots, n$, has exactly one point of intersection in $\mathbb{R}^n$.*

Proof. We fix $\boldsymbol{m}$ and suppress it. We consider the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) = (\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$. Lemmas 3.4 and 3.5 tell us that F is a proper function and that the determinant of its Jacobian is positive for all $\boldsymbol{\lambda} \in \mathbb{R}^n$. By Hadamard's Inverse Function Theorem [**12**, Theorem A] it follows that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Hence the inverse image $F^{-1}(0)$, which is the intersection of the eigenvalue hypersurfaces $\lambda_j = \lambda_j(\boldsymbol{\lambda}_j)$, $j = 1, 2, \dots, n$, is a single point. □

Next we describe the limiting behaviour of the eigenvalue hypersurfaces.

Proposition 3.7. *The eigenvalue hypersurfaces have the following properties:*

(1) $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ *is an increasing function in each parameter* $\lambda_l \in \boldsymbol{\lambda}_j$;

(2) $\lambda_j^{(0)}(\boldsymbol{\lambda}_j) < \min\{0, -d_{j1}/c_{j1}\}$ *for all* j;

(3) $\lim_{\lambda_k \to \infty} \lambda_j^{(0)}(\boldsymbol{\lambda}_j) = \min\{0, -d_{j1}/c_{j1}\}$ *for all* j *and* $k \neq j$;

(4) $\lim_{\lambda_k \to \infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = \infty$ *for* $m > 0$, $j, k = 1, 2, \dots, n$, $j \neq k$; *and*

(5) $\lim_{\lambda_k \to -\infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geqslant 0$, $j, k = 1, 2, \dots, n$, $j \neq k$.

Proof. Property (1) is obvious from positivity of all the partial derivatives. We shall only prove Property (2) in detail.

For (2), one has to go back to [**8**, pp. 60–64]. Consider the jth equation as a one-parameter problem by fixing $\boldsymbol{\lambda}_j \in \mathbb{R}^{n-1}$. Let θ be the Prüfer angle. Then θ is a function

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of $x \in [0,1]$, the eigenparameter λ_j and the $n-1$ constants $\boldsymbol{\lambda}_j$. The zeroth eigensurface $\lambda_j^{(0)}$ is the intersection point of $\varphi(\lambda_j) = \cot\theta(1, \lambda_j, \boldsymbol{\lambda}_j)$ with the hyperbola $\psi_j(\lambda_j) = (a_j\lambda_j + b_j)/(c_j\lambda_j + d_j)$. Now, because of the assumptions on a_j, b_j, c_j and d_j, the hyperbola is increasing. On the other hand, the graph of φ has countably many branches. The hyperbola cuts the leftmost branch of φ in the left half-plane. Since the vertical asymptote for the hyperbola is $-d_j/c_j$, the point of intersection has to lie on the left of this vertical line also. Hence (2) is proved.

The proof of (3) depends on the fact that φ, as defined above, is an increasing function in each $\lambda_k \in \boldsymbol{\lambda}_j$. For a proof of this, see [**8**]. Thus $\lambda_j^{(0)}$, which is the intersection of φ and ψ_j, will exceed any constant $c < \min\{0, -d_j/c_j\}$ for sufficiently large λ_k.

The proofs of (4) and (5) follow by considering the corresponding asymptotic problems and are similar to the proof of [**2**, Lemma 3.4]. □

Suppose that $\boldsymbol{\lambda} \in \mathbb{R}^n$ is an eigenvalue of the problem (2.1), (2.2) and (2.3) under assumptions (I), (II) (a) and (III)–(V), and that $y_j(\boldsymbol{\lambda})$, $j = 1, 2, \dots, n$, are the corresponding eigenfunctions. Let h_j be the number of zeros of $y_j(\boldsymbol{\lambda})$ on the interval $(0,1)$. The n-tuple of non-negative integers $\boldsymbol{h} = (h_1, h_2, \dots, h_n)$ is called the *oscillation count* of $\boldsymbol{\lambda}$ and h_j is called the jth *oscillation count* of $\boldsymbol{\lambda}$.

By [**8**, Theorem 3.1] and the properties proved in Proposition 3.7 it follows that on each hypersurface $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$ with $m_j > 0$ we have, in general, $2^n - 1$ oscillation counts. The jth oscillation count changes when we cross the hyperplane $\lambda_j = -d_{j1}/c_{j1}$. In the case $n = 2$ the oscillation count changes as the curve crosses either of two lines $\lambda_j = -d_{j1}/c_{j1}$, $j = 1, 2$. If the eigencurve does not cross the intersection of the two lines, we have three oscillation counts, one for each 'quadrant' that the curve intersects. For general n, we get $2^n - 1$ oscillation counts unless the hypersurface $\lambda_j = \lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$ crosses the intersection of all the hyperplanes $\lambda_j = -d_{j1}/c_{j1}$, $j = 1, 2, \dots, n$.

The number $N_j^{(m_j)}$ is determined so that

$$\lambda_j^{N_j^{(m_j)}-1}(\boldsymbol{\lambda}_j) < -\frac{d_{j1}}{c_{j1}} \leqslant \lambda_j^{N_j^{(m_j)}}(\boldsymbol{\lambda}_j).$$

Hence

$$h_j = \begin{cases} m_j, & \text{if } \lambda_j^{(m_j)} < -\dfrac{d_{j1}}{c_{j1}}, \\ m_j - 1, & \text{otherwise.} \end{cases} \tag{3.7}$$

The following result now follows by Proposition 3.7 and relations (3.7) above.

Theorem 3.8. *If there are M eigenvalues with the same oscillation count, then*

(1) $M \leqslant 2^n$,

(2) *there is at most one oscillation count corresponding to $M = 2^n$ eigenvalues,*

(3) *for $M \neq 2^k$, $k = 0, 1, 2, \dots, n-1$, there is only a finite number of oscillation counts that correspond to M eigenvalues, and*

(4) *for $M = 2^k$, $k = 0, 1, 2, \dots, n-1$, there is an infinite number of oscillation counts that correspond to M eigenvalues.*

Remark 3.9. It was pointed out by the referee that one might want to consider differential equations (2.1) with some of the boundary conditions (2.3) either not eigenparameter dependent or eigenparameter dependent with $\omega_{j1} > 0$ *but* $c_{j1} = 0$. Call this latter case the *exceptional* eigenparameter-dependent boundary condition. In either of the two cases, Sturm's Oscillation Theorem holds for $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ [**8**, Corollary 5.2], i.e. there is no repetition for the jth oscillation count. Suppose that t, $0 \leqslant t \leqslant n$, is the number of non-exceptional eigenparameter-dependent boundary conditions (2.3), i.e. the number of j such that $\omega_{j1} > 0$ and $c_{j1} \neq 0$. Then one can modify our arguments to show that Theorem 3.8 remains valid if n is replaced by t throughout.

4. Eigenvalue hypersurfaces in the case in which boundary conditions at both ends are eigenparameter dependent

Now we consider the problem (2.1), (2.2) and (2.3) under assumptions (II) (b) and (III)–(V) and study the properties for the corresponding eigenvalue hypersurfaces. The arguments in the proofs are similar to those above under assumption (II) (a). We specify which results are used in the proofs but do not give all details.

Lemma 4.1. *There exists an infinite sequence $\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\}_{m=0}^{\infty}$ of real eigenvalues. Each of $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ depend continuously on all $\lambda_l \in \boldsymbol{\lambda}_j$, and the sequence of eigenvalues $\{\lambda_j^{(m)}(\boldsymbol{\lambda}_j)\}_{m=0}^{\infty}$ is strictly increasing for each $\boldsymbol{\lambda}_j \in \mathbb{R}^{n-1}$.*

Proof. We fix $j = 1$ for simplicity. We view boundary-value problem

$$-y_1'' + \left(q_1 - \sum_{k=2}^{n} \lambda_k r_{1k}\right) y_1 = \lambda_j r_{11} y_1$$

together with (2.2) and (2.3) as a parametrized one-parameter Sturm–Liouville boundary-value problem with eigenparameter-dependent boundary conditions. The existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with the required properties follows by [**8**, Theorems 4.2 and 4.3]. □

Lemma 4.2. *For each eigenvalue $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ there exists a real eigenfunction $y_j^{(m)}(x, \boldsymbol{\lambda}_j)$ of norm 1 for all $\boldsymbol{\lambda}_j$ such that for each $x \in [0, 1]$ and each compact set $K_j \subset \mathbb{R}^{n-1}$ the eigenfunction $y_j^{(m)}(x, \boldsymbol{\lambda}_j)$ and its derivative with respect to x depend continuously on $\boldsymbol{\lambda}_j \in K_j$. Furthermore, there exists a sequence of natural numbers $\{N_{j1}^{(m)}(\boldsymbol{\lambda}_j)\}_{m=0}^{\infty}$ and $\{N_{j2}^{(m)}(\boldsymbol{\lambda}_j)\}_{m=0}^{\infty}$ such that $y_m(\boldsymbol{\lambda}_j)$ has m zeros on $(0, 1)$ for $m \leqslant N_{j1}^{(m)}(\boldsymbol{\lambda}_j)$, $m-1$ zeros on $(0, 1)$ for $N_{j1}^{(m)}(\boldsymbol{\lambda}_j) < m < N_{j2}^{(m)}(\boldsymbol{\lambda}_j)$, and $m-2$ zeros on $(0, 1)$ for $m \geqslant N_{j2}^{(m)}(\boldsymbol{\lambda}_j)$.*

The proof is the same as the proof of Lemma 3.2 except that at the end of it the existence of $\lambda_1^{(m)}(\boldsymbol{\lambda}_1)$ with the required properties follows by [**8**, Theorem 4.2].

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Proposition 4.3. *Partial derivatives of $\lambda_j^{(m)}(\boldsymbol{\lambda}_j)$ with respect to $\lambda_l \in \boldsymbol{\lambda}_j$ exist and are equal to*

$$\frac{\partial \lambda_j^{(m)}}{\partial \lambda_l} = -\biggl(\bar{r}_{jj}(y_j^{(m)}) + \sum_{s=0}^{1} \frac{y_j^{(m)}(s)^2}{(c_{js}\lambda_j + d_{js})^2}\biggr)^{-1} \bar{r}_{jl}(y_j^{(m)}). \tag{4.1}$$

Proof. For simplicity we assume that $j = 1$ and $l = 2$. We use the notation of the proof of Proposition 3.3. Consider the identity (3.4). Dividing it by ϵ, using the boundary conditions (2.2) and (2.3), and the continuity established in Lemmas 4.1 and 4.2, we obtain

$$-\biggl(\frac{y_1(1)^2\omega_{11}}{(c_{11}\lambda_1 + d_{11})^2} - \frac{y_1(0)^2\omega_{10}}{(c_{10}\lambda_1 + d_{10})^2}\biggr)\frac{\partial \lambda_1}{\partial \lambda_2} = \bar{r}_{11}(y_1)\frac{\partial \lambda_1}{\partial \lambda_2} + \bar{r}_{12}(y_1).$$

For other derivatives, one carries out the same calculation with 1 and 2 replaced by j and l, respectively. □

Theorem 4.4. *The set of eigenvalue hypersurfaces $\lambda_j = \lambda_j^{m_j}(\boldsymbol{\lambda}_j)$, $j = 1, 2, \dots, n$, has exactly one intersection point in $\mathbb{R}^n$ for each n-tuple $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ of non-negative integers.*

Proof. The proof is almost identical to the proof Theorem 3.6. We first prove two lemmas equivalent to Lemmas 3.4 and 3.5. For that we use Proposition 4.3 to show that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\boldsymbol{\lambda}) = (\lambda_j - \lambda_j(\boldsymbol{\lambda}_j))_{j=1}^n$ is a C^1 function and to show that its Jacobian matrix has a positive determinant. Further we show that F is a proper function and hence it is a diffeomorphism by Hadamard's Inverse Function Theorem [**12**, Theorem A]. Then $F^{-1}(0)$ is the intersection point of the eigenvalue hypersurfaces. □

The limiting behaviour of the eigenvalue hypersurfaces follows by [**8**, Theorem 4.4 and Corollary 4.5].

Proposition 4.5. *The eigenvalue hypersurfaces have the following properties:*

(1) $\lambda_j^{(0)}(\boldsymbol{\lambda}_j) < \min\{0, -d_{j0}/c_{j0}, -d_{j1}/c_{j1}\}$ *for all* j;

(2) $\lim_{\lambda_k \to \infty} \lambda_j^{(0)}(\boldsymbol{\lambda}_j) = \min\{0, -d_{j0}/c_{j0}, -d_{j1}/c_{j1}\}$ *for all* j *and* $k \neq j$;

(3) $\lim_{\lambda_k \to -\infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geqslant 0$, $j, k = 1, 2, \dots, n$, $j \neq k$; *and*

(4) $\lim_{\lambda_k \to \infty} \lambda_j^{(m)}(\boldsymbol{\lambda}_j) = -\infty$ *for* $m \geqslant 0$, $j, k = 1, 2, \dots, n$, $j \neq k$.

Suppose that $\boldsymbol{\lambda} \in \mathbb{R}^n$ is an eigenvalue of the problem (2.1), (2.2) and (2.3) under assumptions (I), (II) (b) and (III)–(V), and that $y_j(\boldsymbol{\lambda})$, $j = 1, 2, \dots, n$, are the corresponding eigenfunctions. By [**8**, Theorem 4.2] it follows that on each hypersurface $\lambda_j^{(m_j)}(\boldsymbol{\lambda}_j)$ with $m_j > 0$ we have 3^n oscillation counts. That is, the jth oscillation count changes when we cross the hyperplanes $\lambda_j = -d_{js}/c_{js}$, $s = 0, 1$. Write

$$e_0 = \min\biggl\{-\frac{d_{js}}{c_{js}},\ s = 0, 1\biggr\} \quad \text{and} \quad e_1 = \max\biggl\{-\frac{d_{js}}{c_{js}},\ s = 0, 1\biggr\}.$$

Then the numbers $N_{jk}^{(m_j)}$, $k = 1, 2$, are determined so that

$$\lambda_j^{N_{j1}^{(m_j)}-1}(\boldsymbol{\lambda}_j) < e_0 \leqslant \lambda_j^{N_{j2}^{(m_j)}-1}(\boldsymbol{\lambda}_j) < e_1 \leqslant \lambda_j^{N_{j1}^{(m_j)}}(\boldsymbol{\lambda}_j).$$

It further follows that

$$h_j = \begin{cases} m_j, & \text{if } \lambda_j^{(m_j)} < e_0, \\ m_j - 1, & \text{if } e_0 \leqslant \lambda_j^{(m_j)} < e_1, \\ m_j - 2, & \text{otherwise.} \end{cases} \tag{4.2}$$

Proposition 4.5 and relations (4.2) above are used to obtain the following result.

Theorem 4.6. *If there are M eigenvalues with the same oscillation count, then*

(1) $M \leqslant 3^n$,

(2) *there is at most one oscillation count corresponding to $M = 3^n$ eigenvalues,*

(3) *for $M \neq 3^k$, $k = 0, 1, 2, \dots, n-1$, there is only a finite number of oscillation counts that correspond to M eigenvalues, and*

(4) *for $M = 3^k$, $k = 0, 1, 2, \dots, n-1$, there is an infinite number of oscillation counts that correspond to M eigenvalues.*

Remark 4.7. We want to make a remark similar to Remark 3.9. Our arguments can easily be adapted to also treat the cases when some of the boundary conditions (2.2) or (2.3) either do not depend on the eigenparameter or depend on the eigenparameter but $(-1)^{s+1}\omega_{js} > 0$ and $c_{js} = 0$ for $s = 0$ or $s = 1$. Suppose t_1 is the number of boundary-value problems with both boundary conditions non-exceptionally eigenparameter dependent, i.e. t_1 is the number of j such that $(-1)^{s+1}\omega_{js} > 0$ and $c_{js} \neq 0$ for $s = 0, 1$, and that t_2 is the number of boundary-value problems with only one of the boundary conditions non-exceptionally eigenparameter dependent. Obviously, we have $0 \leqslant t_1 + t_2 \leqslant n$. Then the following modified version of Theorem 4.6 holds.

If there are M eigenvalues with the same oscillation count, then

(1) $M \leqslant 3^{t_1}2^{t_2}$,

(2) there is at most one oscillation count corresponding to $M = 3^{t_1}2^{t_2}$ eigenvalues,

(3) for $M \neq 3^{k_1}2^{k_2}$, $k_j = 0, 1, 2, \dots, t_j - 1$, $j = 1, 2$, there is only a finite number of oscillation counts that correspond to M eigenvalues, and

(4) for $M = 3^{k_1}2^{k_2}$, $k_j = 0, 1, 2, \dots, n-1$, $j = 1, 2$, there is an infinite number of oscillation counts that correspond to M eigenvalues.

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